Rational Numbers Distribution and Resonance

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This study solves a problem on the distribution of rational numbers along the number plane and number line. It is shown that the distribution is linked to resonance phenomena and also to stability of oscillating systems.

"God created numbers, all the rest has been created by Man...". With greatest esteem to Leopold Kronecker, one of the founders of the contemporary theory of numbers, it is impossible to agree with him in both the divine origin of number and Man's creation of mathematics. I propound herein the idea that numbers, their relations, and all mathematics in general are objective realities of our world. A part of science is not only understanding things, but also studying the relations that are objective realities in nature.

In this work I am going to consider a problem concerning the distribution of rational numbers along the number line and also in the number plane, and the relation of this distribution to resonance phenomena and stability of oscillating systems in low linear perturbations.

Any oscillating process involving at least two interacting oscillators is necessarily linked to abstract numbers — ratios between the oscillation periods. This fact displays a close relationship between such sections of science as the physical theory of oscillations and the abstract theory of numbers.

As is well known, the rational numbers are distributed on the number line everywhere compactly, so this problem statement that a function of their distribution exists might be thought false, as the case of prime numbers. But, as we will see below, it is not false — a rational numbers distribution function has an objective reality, manifest in numerous physical phenomena of Nature. This thesis will become clearer if we consider the "number lattice" introduced by Minkowski (Fig. 1). Therein are given all points of coordinates p and q which are related to numerators and denominators, respectively. If we exclude all points of the Minkowski lattice with coordinates have a common divisor different from unity, this plane will contain only "rational points" p/q (the non-cancelled fractions). Their distribution in the plane is defined by a sequence of numbers forming a rational series (Fig. 1).

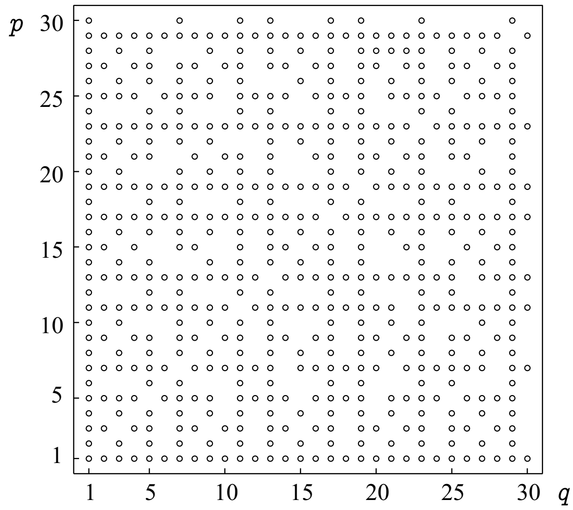

Fig. 1: The lattice of numbers (Minkowski's lattice).

This simplest drawing shows that rational numbers are distributed *inhomogeneously* in the Minkowski number plane. It is easy to see that this distribution is symmetric with respect to the axis $p = q$. Numbers of columns (and rows) in intervals, limited by this axis and one of the coordinate axes, are equal to Euler functions — the numbers less than m and relatively prime with m. Therefore, if we expand the number lattice infinitely, the *average density* of rational numbers in the plane (the ratio between the number of rational numbers and the number of all possible pairs of natural numbers the points of the lattice) approaches the limit

$$\lim_{N\to\infty} \frac{\mathrm{Ra}\left(N^2\right)}{N^2} = \lim_{N\to\infty} \frac{2}{N^2} \sum_{m=1}^{\infty} \varphi\left(m\right) = $$

$$= \frac{1}{\zeta\left(2\right)} = \frac{6}{\pi^2} = \left(\sum_{m=1}^{\infty} \frac{1}{n^2}\right)^{-1},$$

where N is the number of rational numbers, $\mathrm{Ra}\left(N^2\right)$ is the number of rational numbers located inside the square whose elements are of length equal to N, $\varphi\left(m\right)$ is Euler's function, $\zeta\left(n\right)$ is Riemann's zeta function, m and n are natural numbers. In particular, we can conclude from this that when $N < \infty$ the average density of rational numbers located in the plane is restricted to a very narrow interval of numerical values. It is possible to this verify by very simple calculations.

To study the problem of what is common to the rational number distribution and resonance phenomena it is necessary to have a one-dimensional picture of the function $\mathrm{Ra}\left(x\right)$ on the number line. In this problem, because the set of rational numbers is infinitely dense, we need to give a criterion for selecting a finite number of rational numbers which could give an objective picture of their distribution on the number line. We can do this in two ways. First, we can study, for instance, the distribution of rational number rays, drawn from the origin of coordinates in the Minkowski lattice. This

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is a contemporary development of the method created by Klein [1]. Second, we can employ continued fractions, taking into account Khinchin's remark that "continued fractions... in their pure form display properties of the numbers they represent" [2]. So we can employ the mathematical apparatus of continued fractions as a systematic ground in order to find an analogous result that had been previously obtained by a purely arithmetical way.

We will use the second option because it is easier (although it is more difficult to imagine). So, let us plot points by writing a single-term continued fraction $1/n$ (so these are the numbers $1/1$, $1/2$, $1/3$, ...) inside an interval of unit length. We obtain thereby the best approximations of these numbers. This could be done inside every interval $1/(n+1) < x < 1/n$ by plotting points which are numerical values of a two-term continued fraction

$$\frac{1}{m+\frac{1}{n}} = \frac{n}{mn+1}\,.$$

These points, according to the theory of continued fractions, are the best approximations of the numbers $1/n$ from the left side.We then get the best approximations of the numbers $1/n$ from the right side, expressed by the fractions

$$\frac{1}{l+\frac{1}{1+\frac{1}{n}}} = \frac{n+1}{l\,(n+1)+n}\,, \qquad l, n, m = 1, 2, 3\ldots\,.$$

We will call the approximation obtained the *first order approximation* (the second and third rank approximation in Khinchin's terminology). It is evident that every rational point of k-th order obtained in this way has analogous sequences of the $(k+1)$-th rank and higher. Such sequences fill the whole set of rational numbers.

To consider the simplest cases of resonance it would be enough to take the first order approximation, but to consider numerous processes such as colour vision, musical harmony, or Bohr's orbit distribution in atoms, requires a high order distribution function for rational numbers.

To obtain the function $\mathrm{Ra}\,(x)$ as a regular diagram we define this function (meaning the finite approximation order, the first order in this case) as a quantity in reverse to the interval between the neighbouring rational points located on the number line, where the points are plotted in the fashion of Khinchin, mentioned above. If the numerical values of the numbers l, m, n are limited, this interval is finite (see Fig. 2a). Such a drawing gives a possibility for estimating the structure of rational number distribution along the number line. In Fig. 2a we consider the distribution structure of rational numbers derived from a three-component continued fraction. For the purpose of comparison, Fig. 2b depicts a voltage function dependent on the stimulating frequency in an oscillating contour (drawn in the same scale as that in Fig. 2a). In this case an alternating signal frequency at a constant voltage was applied to the input of a resonance amplifier (an active LC-filter having a frequency of 1 kHz and the quality $Q=17$). The frequency at the input was varied within the interval 200–1000 Hz through steps of 25 Hz. The average numerical value of the outgoing voltage was measured for two different voltages of the incoming signal — 0.75V and 1.25V.

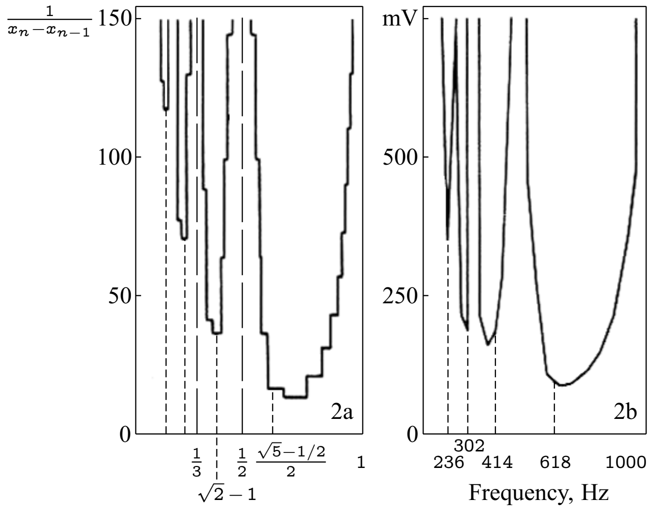

Fig. 2: The rational numbers distribution.

The apexes of both functions shown in the diagrams are located at the points plotted by the fractions $1/n$. This fact is trivial, because both apexes are actually analogous to Fourier-series expansions of white noise. Such experimental diagrams could be obtained in a purely theoretical way.

Much more interesting is the problem of the minimum numerical values of both functions. The classical theory of oscillations predicts that the minimum points should coincide with the minimum amplitude of forced oscillations, while according to the theory of continued fractions the minimum points should coincide with irrational numbers which, being the roots of the equation $x^2 \pm px - 1 = 0$ for all p, are approximated by rational numbers less accurately than by other numbers [2].

Direct calculations give the following numbers

$$M_1 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}} = \frac{\sqrt{5}\mp 1}{2} = (0.6180339\ldots)^{\pm 1},$$

$$M_2 = \frac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}} = \frac{\sqrt{8}\mp 1}{2} = (0.4142135\ldots)^{\pm 1},$$

....

$$M_n = \frac{1}{n+\frac{1}{n+\frac{1}{n+\ldots}}} = \frac{\sqrt{n^2+4}\mp n}{2}\,.$$

In other words, the first conclusion is that the distribution of rational numbers, represented by continued fractions with

Table 1: Orbital radii of planets in the solar system in comparison with the calculated values of the radii $R_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real R_k	Calculated R_k	n	$\frac{R_{k(\text{calc})}}{R_{k(\text{real})}}$
Mercury	0.0744	0.0765	-13	1.0282
Venus	0.1390	0.1401	-7	1.0079
Earth	0.1922	0.1926	-5	1.0021
Mars	0.2929	0.3028	-3	1.0338
Asteroids	0.6180	0.6180	-1	1.0000
Jupiter	1.0000	1.0000	0	1.0000
Saturn	1.8334	1.6180	1	0.8825
Uranus	3.6883	3.3028	3	0.8955
Neptune	5.7774	5.1926	5	0.8988
Pluto	7.6398	7.1401	7	0.9346

Table 2: Orbital periods of planets in the solar system in comparison with the calculated values of the periods $T_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real T_k	Calculated T_k	n	$\frac{T_{k(\text{calc})}}{T_{k(\text{real})}}$
Mercury	0.0203	0.0203	-49	1.0000
Venus	0.0519	0.0524	-19	1.0096
Earth	0.0843	0.0828	-12	0.9822
Mars	0.1586	0.1623	-6	1.0233
Asteroids	0.4877	0.4142	-2	0.8493
Jupiter	1.0000	1.0000	0	1.0000
Saturn	2.4834	2.4142	2	0.9721
Uranus	7.0827	7.1378	7	1.0077
Neptune	13.8922	14.0711	13	1.0129
Pluto	21.1166	21.0475	21	0.9967

Note: Here the measurement units are the orbital radius and period of Jupiter. For asteroids the overall average orbit is taken, its radius 3.215 astronomical units and period 5.75 years are the average values between asteroids.

a limited number of elements, takes its minimum density at the points of a unit interval on number line as shown by the aforementioned numbers. The second conclusion is that if these numbers express ratios between interacting frequencies, the amplitude of the forced oscillations takes its minimum numerical value.

It is evident that an oscillating system, where the oscillation parameters undergo changes due to interactions inside the system, will be maximally stable in that case where the forced oscillation amplitude will be a minimum.

The simplest verification of this thesis is given by the solar system. As we know it Laplace's classic works, the whole solar system (the planet orbits on the average) are stable under periodic gravitational perturbations only if the ratios between the orbital parameters are expressed by irrational numbers. If we will take this problem forward, proceeding from the viewpoint proposed above, the ratios between the orbital periods T_k/T_0 or, alternatively, the ratios between their functions (the average orbital radii R_k/R_0) will be close to those numbers that correspond to the minima of the rational numbers density on number line

$$\frac{T_k}{T_0};\frac{R_k}{R_0} \approx M_n = \frac{\left(\sqrt{n^2+4} \mp n\right)}{2}.$$

The truth or falsity of this can be decided by using Table 1 and Table 2.

As a matter of fact, all that has been said on the distribution of rational numbers on a unit interval could be extrapolated for the entire number line (proceeding from the above mentioned concept).

All that has been said gives a possibility to formulate the next conclusions:

1. Rational numbers having limited numerator and denominator are distributed inhomogeneously along the number line;
2. Oscillating systems, having a peculiarity to change their own parameters because of interactions inside the systems, have a tendency to reach a stable state where the separate oscillators frequencies are interrelated by specific numbers — minima of the rational number density on number line.

References

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