Gravitational Waves and Gravitational Inertial Waves in the General Theory of Relativity: A Theory and Experiments

Larissa Borissova
E-mail: lborissova@yahoo.com

This research shows that gravitational waves and gravitational inertial waves are linked to a special structure of the Riemann-Christoffel curvature tensor. Proceeding from this a classification of the waves is given, according to Petrov's classification of Einstein spaces and gravitational fields located therein. The world-lines deviation equation for two free particles (the Synge equation) is deduced and that for two force-interacting particles (the Synge-Weber equation) in the terms of chronometric invariants — physical observable quantities in the General Theory of Relativity. The main result drawn from the deduced equations is that in the field of a falling gravitational wave there are not only spatial deviations between the particles but also deviations in the time flow. Therefore an effect from a falling gravitational wave can manifest only if the particles located on the neighbouring world-lines (both geodesics and non-geodesics) are in motion at the initial moment of time: gravitational waves can act only on moving neighbouring particles. This effect is purely parametric, not of a resonance kind. Neither free-mass detectors nor solid-body detectors (the Weber pigs) used in current experiments can register gravitational waves, because the experimental statement (freezing the pigs etc.) forces the particles of which they consist to be at rest. In aiming to detect gravitational waves other devices should be employed, where neighbouring particles are in relative motion at high speeds. Such a device could, for instance, consist of two parallel laser beams.

Contents

1 Introduction and advanced results

The fact that gravitational waves have not yet been discovered has attracted the attention of experimental physicists over the last decade. Initial interest in gravitational waves arose in 1968–1971 when Joseph Weber, professor at Maryland University (USA), carried out his first experiments with gravitational antennae. He registered weak signals, in common with all his independent antennae, which were separated by up to 1000 km [1]. He supposed that some processes at the centre of the Galaxy were the origin of the registered signals. However such an interpretation had a significant drawback: the frequency of the observed signals (more than 5 per month) meant that the energy spent by the signal's source, located at the centre of the Galaxy, should be more than $M_{\odot}c^2{\times}10^3$ per annum ($M_{\odot}$ is the mass of the Sun, c is the velocity of light). This energy expenditure is a fantastic value, if we accept today's bounds on the age of the Galaxy [2, 3, 4].

In 1972 the experiments were approbated by the a common group of researchers working at Moscow University and the Institute of Space Research (Moscow, Russia). Their antennae were similar to Weber's antennae, but they were separated by 20 km. The registering system in their antennae was better than that for the Weber detectors, making the whole system more sensitive. But. . . 20 days of observations gave no signals that would be more than noise [5].

The experiments were continued in 1973–1974 at laboratories in Rochester University, Bell Company, and IBM in USA [6, 7], Frascati, Münich, Meudon (Italy, Germany, France) [8], Glasgow University (Scotland) [9] and other laboratories around the world. The experimental systems used in these attempts were more sensitive than those of the Weber detectors, but none registered the Weber effect.

Because theoretical considerations showed that huge gravitational waves should be accompanied by other radiations, the researchers conducted a search for radio outbreaks [10] and neutron outbreaks [11]. The result was negative. At the same time it was found that Weber's registered effects were related to solar and geomagnetic activities, and also to outbreaks of space beams [12, 13].

The search for gravitational waves has continued. Higher precision and more sensitive modifications of the Weber antennae (solid detectors of the resonance kind) are used in this search. But even the second generation of Weber detectors have not led scientists to the expected results. Besides gravitational antennae of the Weber kind, there are antennae based on free masses. Such detectors consist of two freely suspended masses located far from one another, within the visibility of a laser range-finder. Supposed deviations of the masses, derived from a gravitational wave, should be registered by the laser beam.

So gravitational waves have not been discovered in experiments. Nonetheless it is accepted by most physicists that the discovery of gravitational waves should be one of the main verifications of the General Theory of Relativity. The main arguments in support of this thesis are:

1. Gravitational fields bear an energy described by the energy-momentum pseudotensor [14, 15];
2. A linearized form of the equations of Einstein's equations permits a solution describing weak plane gravitational waves, which are transverse;
3. An energy flux, radiated by gravitational waves, can be calculated through the energy-momentum pseudotensor of the field [14, 15];
4. Such waves, because of their physical nature, are derived from instability of components of the fundamental metric tensor (this tensor plays the part of the four-dimensional gravitational potential).

These theoretical considerations were placed into the foreground of the theory for detecting gravitational waves, the main part in the theory being played by the theoretical works of Joseph Weber, the pioneer and famous expert in the detection of gravitational waves [16]. His main theoretical claim was that he deduced equations of deviation of world-lines — equations that describe relative oscillations of *two non-free particles* in a gravitational field, particles which are connected by a force of non-gravitational nature. Equations of deviation of geodesic lines, describing relative oscillations of *two free particles*, was obtained earlier by Synge [17]. In general, relative oscillations of test-particles, both free particles and linked (interacting) particles, are derived from the space curvature*, given by the Riemann-Christoffel four-dimensional tensor. Equality to zero of all its components in an area is the necessary and sufficient condition for the four-dimensional space (space-time) to be flat in the area under consideration, so no gravitational fields exist in the area.

Thus the Synge-Weber equation provides a means for the calculation of the relative oscillations of test-particles, derived from the presence of the space curvature (gravitational fields). Weber proposed a gravitational wave detector consisting of two particles connected by a spring that imitates a non-gravitational interaction between them. In his analysis he made the substantial supposition that the under action of gravitational waves the model will behave like a harmonic oscillator where the forcing power is in the Riemann-Christoffel curvature tensor. Weber made calculations and theoretical propositions for the behaviour of this model. This model is known as the *quadrupole mass-detector* [17].

The Weber calculations served as theoretical grounds for creating a whole industry, the main task of which has been the building of resonance type detectors, known as the Weber detectors (the Weber pigs). It is supposed that the body of a Weber detector, having cylindrical form, should be deformed under the action of a gravitational wave. This deformation should lead to a piezoelectric effect. Thus, oscillations of atoms in the cylindrical pig, resulting from a gravitational wave, could be registered. To amplify the effect in measurements, the level of noise was lowered by cooling the cylinder pigs down to temperature close to 0 K.

But the fact that gravitational waves have not yet been discovered does not imply that the waves do not exist in Nature. The corner-stone of this problem is that the Weber theory of detection is linked to a search for waves of only a specific kind — weak transverse waves of the space deformation (*weak deformation transverse waves*). However, besides the Weber theory, there is the *theory of strong gravitational waves*, which is independent of the Weber theory. Studies of the theory of strong gravitational waves reached its peak in the 1950's.

Generally speaking, all theoretical studies of gravitational waves can be split into three main groups:

1. The first group consists of studies whose task is to give an invariant definition for gravitational waves. These are studies made by Pirani [18, 19], Lichnerowicz [20, 21], Bel [22, 23, 24], Debever [25, 26, 27], Hély [28], Trautman [29], Bondi [19, 30], and others.
2. The second group joins studies around a search for such solutions to the Einstein equations for gravitational fields, which, proceeding from physical considerations, could describe gravitational radiations. These are studies made by Bondi [31], Einstein and Rosen [32, 33], Peres [34], Takeno [35, 36], Petrov [37], Kompaneetz [38], Robinson and Trautman [39], and others.
3. The task of works related to the third group is to study gravitational inertial waves, covariant with respect of transformations of spatial coordinates and also invariant with respect of transformations of time [40, 41].

*As it is well-known, the space curvature is linked to the gravitational field by the Einstein equations.

The studies are based on the theory of physically observable quantities — Zelmanov's theory of chronometric invariants [42, 43].

Most criteria for gravitational waves are linked to the structure of the Riemann-Christoffel curvature tensor, hence one assumes space curvature the source of such waves.

Besides these three main considerations, the theory of gravitational waves is directly linked to the algebraical classification of spaces given by Petrov [37] (*Petrov classification*), according to which three kinds for spaces (gravitational fields) exist. They are dependent on the structure of the Riemann-Christoffel curvature tensor:

1. Fields of gravitation of the 1st kind are derived from island distributions of masses. An instance of such a field is the that of a spherical distribution of matter (a spherical mass island) described by the Schwarzschild metric [44]. Spaces containing such fields approach a flat space at an infinite distance from the gravitating island.

2–3. Spaces containing gravitational fields of the 2nd and 3rd kinds cannot asymptotically approach a flat space even, if they are empty. Such spaces can be curved themselves, independently of the presence of gravitating matter. Such fields satisfy most of the invariant definitions given to gravitational waves [40, 45, 46, 47].

It should be noted that the well-known solution that gives weak plane gravitational waves [14, 15] is related to fields of the sub-kind N of the 2nd kind by Petrov's classification (see p. 38). Hence the theory of weak plane gravitational waves is a particular case of the theory of strong gravitational waves. But, bseides this well-studied particular case, the theory of strong gravitational waves contains many other approaches to the problem and give other methods for the detection of gravitational waves, different to the Weber detectors in principle (see [48], for instance).

We need to look at the gravitational wave problem from another viewpoint, by studying other cases of the theory of strong gravitational waves not considered before. Exploring such new approaches to the theory of gravitational waves is the main task of this research.

At the present time there are many solutions of the gravitational wave problem, but none of them are satisfactory. The principal objective of this research is to extract that which is common to every one of the theoretical approaches.

We will see further that this analysis shows, according to most definitions given for gravitational waves, that a gravitational field is assumed a wave field if the space where it is located has the specific curvature described by numerous particular cases of the Riemann-Christoffel curvature tensor.

Note that we mean the Riemannian (four-dimensional) curvature, whose formula contains accelerations, rotations, and deformations of the observer's reference space. Analysis of most wave solutions to the gravitational field equations (Einstein's equations) shows that such gravitational waves have a *deformation nature* — they are waves of the space deformations. The true nature of gravitational waves can be found by employing the mathematical methods of chronometric invariants (the theory of physically observable quantities in the General Theory of Relativity), which show that the space deformation (non-stationarity of the spatial observable metric) consists of two factors:

1. Changes of the observer's scale of distance with time (deformations of the 1st kind);
2. Possible vortical properties of the acting gravitational inertial force field (deformations of the 2nd kind).

Waves of the space deformations (of the 1st or 2nd kind) underlie the detection attempts of the experimental physicists.

Because such gravitational waves are expected to be weak, one usually uses the metric for weak plane gravitational waves of the 1st kind (which are derived from changes of the distance scale with time).

The basis for all the experiments is the Synge-Weber equation (the world-lines deviation equation), which sets up a relation between relative oscillations of test-particles and the Riemann-Christoffel curvature tensor. Unfortunately Joseph Weber himself gave only a rough analysis of his equation, aiming to describe the behaviour of a quadrupole mass-detector in the field of weak plane gravitational waves. In his analysis he assumed (without substantial reasons) that space deformation waves of the 1st kind must produce a resonance effect in a quadrupole mass-detector.

However, it would be more logical way, making no assumptions or propositions, to solve the Synge-Weber equation aiming exactly. Weber did not do this, limiting himself instead to only rough bounds on possible solutions.

In this research we obtain exact solutions to the Synge-Weber equation in the fields of weak plane gravitational waves. As a result we conclude that the expected relative oscillations of test-particles, which originate in the space deformation waves of the 1st kind, *are not of the resonance kind* as Weber alleged from his analysis, but are instead *parametric oscillations*.

This deviation between our conclusion and Weber's false conclusion is very important, because oscillations of a parametric kind appear only if test-particles are moving*, whilst in Weber's statement of the experiment the particles are at rest in the observer's laboratory reference frame. All activities in search of gravitational waves using the Weber pigs are concentrated around attempts to isolate the bulk pigs from external affects — experimental physicists place them in mines in the depths of mountains and cool them to 2 K,

*In other words, if their velocities are different from zero. Parametric oscillations merely add their effect to the relative motion of the moving particles. Parametric oscillations cannot be excited in a system of particles which are at rest with respect to each other and the observer.

so particles of matter in the pigs can be assumed at rest with respect to one another and to the observer. At present dozens of Weber pigs are used in such experiments all around the world. Experimental physicists spend billions and billions of dollars yearly on their experiments with the Weber pigs.

Parametric oscillations do not appear in resting particles, so the space deformation waves of the 1st kind can not excite parametric oscillations in the Weber pigs. Therefore the *gravitational waves expected by scientists cannot be registered by solid-body detectors of the resonance kind* (*the Weber pigs*).

Even so, everything said so far does not mean rejection of the experimental search for gravitational waves. We merely need to look at the problem from another viewpoint. We need to remember the fact that our world is not a three-dimensional space, but a four-dimensional space-time. For this reason we need to turn our attention to the fact that relative deviations of particles in the field of gravitational waves have both spatial components and a time component. Therefore it would be reasonable to propose an experiment by which, having a detector under the influence of gravitational waves, we could register both relative displacements of particles in the detector and also corrections to time flow in the detector due to the waves (the second task is much easier from the technical viewpoint).

Here are two aspects for consideration. First, in solving the Synge-Weber equations we must take its time component into account; we must not neglect the time component. Second, we should turn our attention to possible experimental effects derived from gravitational waves of the 2nd (deformation) kind, which appear if the acting gravitational inertial force field is vortical, as it will be shown further that in this case there is a field of the space rotation (stationary or non-stationary)*. Such experiments, aiming to register gravitational waves of the 2nd kind are progressive because they are much simpler and cheaper than the search for waves of the 1st kind.

2 Theoretical bases for the possibility of registering gravitational waves

Gravitational waves were already predicted by Einstein [37], but what space objects could be sources of the waves is not a trivial problem. Some link the possibility of gravitational radiations to clusters of black holes. Others await powerful gravitational radiations from super-dense compact stars of radii close to their gravitational radii† $r \sim r_g$. Although the "black hole solution", being under substantial criticism from the purely mathematical viewpoint [53, 54, 55], makes objects like black holes very doubtful, the existence of super-dense neutron stars is outside of doubt between astronomers. Gravitational waves at frequencies of 10^2–10^4 Hz should also be radiated in super-nova explosions by explosion of their super-dense remains [56].

The search for gravitational waves, beginning with Weber's observations of 1968–1971, is realized by using gravitational antennae, the most promising of which are:

1. Solid-body detectors (the Weber cylinder pigs);
2. Antennae built on free masses.

A solid-body detector of the Weber kind is a massive cylindrical pig of 1–3 metres in length, made with high precision. This experiment supposes that gravitational waves are waves of the space deformation. For this reason the waves cause a piezoelectic effect in the pig, one consequence of which is mechanical oscillations at low frequencies that can be registered in the experiment. It is supposed that such oscillations have a resonance nature. An immediate problem is that such resonance in massive pigs can be caused by very different external processes, not only waves of the space deformation. To remove other effects, experimental physicists locate the pigs in deep tunnels in mountains and cool the pigs down to temperature close to 0 K.

An antenna of the second kind consists of two masses, separated by $\Delta l \sim 10^3$–10^4 metres, and a laser range-finder which should register small changes of Δl. Both masses are freely suspended. This experiment supposes that waves of the space deformation should change the distance between the free masses, and should be registered by the laser range-finder. It is possible to use two satellites located in the same orbit near the Earth, having a range-finder in each of the satellites. Such satellites, being in free fall along the orbit, should be an ideal system for measurements, if it were not for effects due to the terrestrial globe. In practice it would be very difficult to divorce the effect derived from waves of the space deformation (supposed gravitational waves) and many other factors derived from the inhomogeneity of the Earth's gravitational field (purely geophysical factors).

The mathematical model for such an antenna consists of two free test-particles moving on neighbouring geodesic lines located infinitely close to one another. The mathematical model for a solid-body detector (a Weber pig) consists of two test-masses connected by a spring that gives a model for elastic interactions inside a real cylindrical pig, in which changes reveal the presence of a wave of the space deformation.

From the theoretical perspective, we see that the possibility of registering waves of the space deformation (supposed gravitational waves) is based on the supposition that particles which encounter such a wave should be set into relative oscillations, the origin of which is the space curvature. The

*There are well-known Hafele-Keating experiments concerned with displacing standard clocks around the terrestrial globe, where rotation of the Earth space sensibly changes the measured time flow [49, 50, 51, 52].

†According to today's mainstream concepts, the gravitational radius r_g of an object is that minimal distance from its centre to its surface, starting from which this object is in a special state — *collapse*. One means that any object going into collapse becomes a "black hole". From the purely mathematical viewpoint, under collapse, the potential w of the gravitational field of the object merely reaches its upper ultimate numerical value $\mathrm{w} = c^2$.

strong solution for this problem had been given by Synge for free particles [17]. He considered a two-parameter family of geodesic lines $x^\alpha = x^\alpha(s,v)$, where s is a parameter along the geodesics, v is a parameter along the direction orthogonal to the geodesics (it is taken in the plane normal to the geodesics). Along each geodesic line $v = \text{const}$.

He introduced two vectors

$$U^\alpha = \frac{\partial x^\alpha}{\partial s}\,, \qquad V^\alpha = \frac{\partial x^\alpha}{\partial v}\,, \tag{2.1}$$

where $\alpha = 0,1,2,3$ denotes four-dimensional (space-time) indexes. The vectors satisfy the condition

$$\frac{\mathrm{D}U^\alpha}{\partial v} = \frac{\mathrm{D}V^\alpha}{\partial s}\,, \tag{2.2}$$

(where D is the absolute derivative operator) that can be easy verified by checking the calculation. The parameter v is different for neighbouring geodesics; the difference is dv. Therefore, studying relative displacements of two geodesics $\Gamma(v)$ and $\Gamma(v+dv)$, we shall study the vector of their infinitesimal relative displacement

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\,dv = V^\alpha dv\,. \tag{2.3}$$

The deviation of the geodesic line $\Gamma(v+dv)$ from the geodesic line $\Gamma(v)$ can be found by solving the equation [17]

$$\begin{gathered}\frac{\mathrm{D}^2 V^\alpha}{ds^2} = \frac{\mathrm{D}}{ds}\frac{\mathrm{D}V^\alpha}{ds} = \frac{\mathrm{D}}{ds}\frac{\mathrm{D}U^\alpha}{dv} = \\ = \frac{\mathrm{D}}{dv}\frac{\mathrm{D}U^\alpha}{ds} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma,\end{gathered} \tag{2.4}$$

where $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ is the Riemann-Christoffel curvature tensor. This equality has been obtained using the relation [17]

$$\frac{\mathrm{D}^2 V^\alpha}{ds dv} - \frac{\mathrm{D}^2 V^\alpha}{dv ds} = R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma. \tag{2.5}$$

For two neighbour geodesic lines, the following relation is obviously true

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0\,, \tag{2.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ are Christoffel's symbols of the 2nd kind. Then (2.4) takes the form

$$\frac{\mathrm{D}^2 V^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma = 0\,, \tag{2.7}$$

or equivalently,

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = 0\,. \tag{2.8}$$

It can be shown [17] that,

$$\frac{\partial}{\partial s}(U_\alpha V^\alpha) = U_\alpha \frac{\mathrm{D}V^\alpha}{ds} = U_\alpha \frac{\mathrm{D}U^\alpha}{dv} = \frac{1}{2}\frac{\partial}{\partial v}(U_\alpha U^\alpha)\,. \tag{2.9}$$

The quantity $U_\alpha U^\alpha = g_{\alpha\beta}U^\alpha U^\beta$ takes the numerical value +1 for non-isotropic geodesics (substantial particles) or 0 for isotropic geodesics (massless light-like particles). Therefore

$$U_\alpha V^\alpha = \text{const}\,. \tag{2.10}$$

In the particular case where the vectors U_α and V^α are orthogonal to each to other at a point, where $U_\alpha V^\alpha$ is true, the orthogonality remains true everywhere along the $\Gamma(v)$.

Thus relative accelerations of free test-particles are caused by the presence the space curvature ($R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta} \neq 0$), and linear velocities of the particles are determined by the geodesic equations (2.6).

Relative accelerations of test-particles, connected by a force Φ^α of non-gravitational nature, are determined by the Synge-Weber equation [16]. The Synge-Weber equation is the generalization of equation (2.8) for that case where the particles, each having the rest-mass m_0, are moved along non-geodesic world-lines, determined by the equation

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = \frac{\Phi^\alpha}{m_0 c^2}\,. \tag{2.11}$$

In this case the world-lines deviation equation takes the form

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv\,, \tag{2.12}$$

which describes relative accelerations of the interacting masses. In this case

$$\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha, \tag{2.13}$$

so the angle between the vectors U^α and η^α does not remain constant for the interacting particles.

Equations (2.8) and (2.12) describe relative accelerations of free particles and interacting particles, respectively. Then, to obtain formulae for the velocity U^α it is necessarily to solve the geodesic equations for free particles (2.6) and the world-line equations for interacting particles (2.11). We consider the equations (2.8) and (2.12) as a mathematical base, with which we aim to calculate gravitational wave detectors: (1) antennae built on free particles, and (2) solid-body detectors of the resonance kind (the Weber detectors).

3 Invariant criteria for gravitational waves and their link to Petrov's classification

From the discussion in the previous paragraphs, one concludes that a physical factor enforcing relative displacements of test-particles (both free particles and interacting particles) is the space curvature — a gravitational field wherein the particles are located.

Here the next question arises. How well justified is the statement of the gravitational wave problem?

Generally speaking, in the General Theory of Relativity, there is a problem in describing gravitational waves in a mathematically correct way. This is a purely mathematical problem, not solved until now, because of numerous difficulties. In particular, the General Theory of Relativity does not contain a satisfactory general covariant definition for the energy of gravitational fields. This difficulty gives no possibility of describing gravitational waves as traveling energy of gravitational fields.

The next difficulty is that when one attempts to solve the gravitational wave problem using the classical theory of differential equations, he sees that the gravitational field equations (the Einstein equations) are a system of 10 non-linear equations of the 2nd order written with partial derivatives. No universal boundary conditions exist for such equations.

The gravitational field equations (the Einstein equations) are

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = -\kappa T_{\alpha\beta} + \lambda\, g_{\alpha\beta}\,, \tag{3.1}$$

where $R_{\alpha\beta} = R^{\sigma\cdots}_{\cdot\alpha\sigma\beta}$ is Ricci's tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $\kappa = \frac{8\pi G}{c^2}$ is Einstein's constant for gravitational fields, G is Gauss' constant of gravitation, λ is the cosmological constant (λ-term).

When studying gravitational waves, one assumes $\lambda = 0$. Sometimes one uses a particular case of the Einstein equations (3.1)

$$R_{\alpha\beta} = \kappa\, g_{\alpha\beta}\,, \tag{3.2}$$

in which case the space, where the gravitational field is located, is called an *Einstein space*. If $\kappa = 0$, we have an *empty space* (without gravitating matter). But even in empty spaces ($\kappa = 0$) gravitational fields can exist, if the spaces are of the 2nd and 3rd kinds by Petrov's classification.

In accordance with the classical theory of differential equations, those gravitational fields that describe gravitational waves are determined by solutions of the Einstein equations with initial conditions located in a characteristic surface. A wave is a Hadamard break in the initial characteristic surface; such a surface is known as the *wave front*. The wave front is determined as the characteristic isotropic surface $S\{\Phi(x^\alpha) = 0\}$ for the Einstein equations. Here the scalar function Φ satisfies the eikonal equation [20, 21]

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta = 0\,, \tag{3.3}$$

where ∇_α denotes covariant differentiation with respect to Riemannian coherence with the metric $g_{\alpha\beta}$. The trajectories along which gravitational waves travel (gravitational rays) are bicharacteristics of the field equations, having the form

$$\frac{dx^\alpha}{d\tau} = g^{\alpha\sigma}\nabla_\sigma\Phi\,, \tag{3.4}$$

where τ is a parameter along lines of the geodesic family.

But the general solution of the Einstein equations with initial conditions in the hypersurface is unknown. For this reason the next problem arises: it is necessary to formulate an effective criterion which could determine solutions to the Einstein equations with initial conditions in the characteristic hypersurface.

There is another difficulty: there is no general covariant d'Alembertian which, being in its clear form, could be included into the Einstein equations.

Therefore, solving the gravitational wave problem reduces to the problem of formulating an invariant criterion which could determine this family of the field equations as wave equations.

Following this approach, analogous the classical theory of differential equations, we encounter an essential problem. Are functions $g_{\alpha\beta}(x^\sigma)$ smooth when we set up the Cauchy problem for the Einstein equation? A gravitational wave is interpreted as Hadamard break for the curvature tensor field in the initial characteristic hypersurface. The curvature tensor field permits a Hadamard break only if the functions $g_{\alpha\beta}(x^\sigma)$ permit breaks in their first derivatives. In accordance with Hadamard himself [20], the second derivatives of $g_{\alpha\beta}$ can have a break in a surface $S\{\Phi(x^\alpha) = 0\}$

$$[\partial_{\rho\sigma}\, g_{\alpha\beta}] = a_{\alpha\beta}\, l_\rho\, l_\sigma\,, \qquad (l_\alpha \equiv \partial_\alpha\Phi) \tag{3.5}$$

only if a Hadamard break in the curvature tensor field $[R_{\alpha\beta\gamma\delta}]$ satisfies the equations [21]

$$l_\lambda[R_{\mu\alpha\beta\nu}] + l_\alpha[R_{\mu\beta\lambda\nu}] + l_\beta[R_{\mu\lambda\alpha\nu}] = 0\,. \tag{3.6}$$

Proceeding from such an interpretation of the characteristic hypersurface for the Einstein equations, and also supposing that a break $[R_{\alpha\beta\gamma\delta}]$ in the curvature tensor $R_{\alpha\beta\gamma\delta}$ located in the front of a gravitational wave is proportional to the tensor itself, Lichnerowitz [20, 21] formulated this criterion for gravitational waves:

Lichnerowitz' criterion The space curvature $R_{\alpha\beta\gamma\delta} \neq 0$ determines the state of "full gravitational radiations", only if there is a vector $l^\alpha = 0$ satisfying the equations

$$\begin{aligned} &l^\mu R_{\mu\alpha\beta\nu} = 0\,, \\ &l_\lambda R_{\mu\alpha\beta\nu} + l_\alpha R_{\mu\beta\lambda\nu} + l_\beta R_{\mu\lambda\alpha\nu} = 0\,, \end{aligned} \tag{3.7}$$

and thus the vector l^α is isotropic ($l_\alpha l^\alpha = 0$). If $R_{\alpha\beta} = 0$ (the space is free of masses, so it is empty), the equations (3.7) determine the state of "clear gravitational radiations".

There is also Zelmanov's invariant criterion for gravitational waves [40]*, it is linked to the Lichnerowitz criterion.

*This criterion is named for Abraham Zelmanov, although it had been published by Zakharov [40]. This happened because Zelmanov gave many of his unpublished results, his unpublished criterion included, to Zakharov, who completed his dissertation under Zelmanov's leadership at that time.

Zelmanov proceeded from the general covariant generalization given for the d'Alembert wave operator

$$\Box^{\sigma}_{\sigma} \equiv g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\,. \tag{3.8}$$

Zelmanov's criterion The space determines the state of gravitational radiations, only if the curvature tensor: (a) is not a covariant constant quantity ($\nabla_{\sigma}R_{\mu\alpha\beta\gamma}=0$); (b) satisfies the general covariant condition

$$\Box^{\sigma}_{\sigma}R_{\mu\alpha\beta\nu} = 0. \tag{3.9}$$

Thus, as it was shown in [40], any empty space that satisfies the Zelmanov criterion also satisfies the Lichnerowitz criterion. On the other hand, any empty space that satisfies the Lichnerowitz criterion (excluding that trivial case where $\nabla_{\sigma}R_{\mu\alpha\beta\gamma}=0$) also satisfies the Zelmanov criterion.

There are also other criteria for gravitational waves, introduced by Bel, Pirani, Debever, Maldybaeva and others [58]. Each of the criteria has its own advantages and drawbacks, therefore none of the criteria can be considered as the final solution of this problem. Consequently, it would be a good idea to consider those characteristics of gravitational wave fields which are common to most of the criteria. Such an integrating factor is Petrov's classification — the algebraic classification of Einstein spaces given by Petrov [37], in the frame of which those gravitational fields that satisfy the condition (3.2) are classified by their relation to the algebraic structure of the Riemann-Christoffel curvature tensor.

As is well known, the components of the Riemann-Christoffel tensor satisfy the identities

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}\,, \quad R_{\alpha[\beta\gamma\delta]} = 0\,. \tag{3.10}$$

Because of (3.10), the curvature tensor is related to tensors of a special family, known as *bitensors*. They satisfy two conditions:

1. Their covariant and contravariant valencies are even;
2. Both covariant and contravariant indices of the tensors are split into pairs and inside each pair the tensor $R_{\alpha\beta\gamma\delta}$ is antisymmetric.

A set of tensor fields located in an n-dimensional Riemannian space is known as a *bivector set*, and its representation at a point is known as a *local bivector set*. Every antisymmetric pair of indices $\alpha\beta$ is denoted by a common index a, and the number of the common indices is $N=\frac{n(n-1)}{2}$. It is evident that if $n=4$ we have $N=6$. Hence a bitensor $R_{\alpha\beta\gamma\delta}\rightarrow R_{ab}$, located in a four-dimensional space, maps itself into a six-dimensional bivector space. It can be metrised by introducing the specific metric tensor

$$g_{ab} \rightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\,. \tag{3.11}$$

The tensor g_{ab} ($a,b=1,2,\dots N$) is symmetric and non-degenerate. The metric g_{ab}, given for the sign-alternating $g_{\alpha\beta}$, can be sign-alternating, having a signature dependent on the signature of the $g_{\alpha\beta}$. So, for Minkowski's signature $(+---)$, the signature of the g_{ab} is $(+++---)$.

Mapping the curvature tensor $R_{\alpha\beta\gamma\delta}$ onto the metric bivector space R_N, we obtain the symmetric tensor R_{ab} ($a,b=1,2,\dots N$) which can be associated with a lambda-matrix

$$\left(R_{ab} - \Lambda\, g_{ab}\right). \tag{3.12}$$

Solving the classic problem of linear algebra (reducing the lambda-matrix to its canonical form along a real distance), we can find a classificaton for V_n under a given n. Here the *specific kind* of an Einstein space we are considering is set up by a *characteristic* of the lambda-matrix. This kind remains unchanged in that area where this characteristic remains unchanged.

Bases of elementary divisors of the lambda-matrix for any V_n have an ordinary geometric meaning as *stationary curvatures*. Naturally, the Riemannian curvature V_n in a two-dimensional direction is determined by an ordinary (single-sheet) bivector $V^{\alpha\beta}=V^{\alpha}_{(1)}V^{\beta}_{(2)}$, of the form

$$K = \frac{R_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}{g_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}\,. \tag{3.13}$$

If $V^{\alpha\beta}$ is not ordinary, the invariant K is known as the *bivector curvature in the given vector's direction*. Mapping K onto the bivector space, we obtain

$$K = \frac{R_{ab}V^{a}V^{b}}{g_{ab}V^{a}V^{b}}\,, \qquad a,b=1,2,\dots N. \tag{3.14}$$

Ultimate numerical values of the K are known as *stationary curvatures* taken at a given point, and the vectors V^a corresponding to the ultimate values are known as *stationary not simple bivectors*. In this case

$$V^{\alpha\beta} = V^{\alpha}_{(1)}V^{\beta}_{(2)}\,, \tag{3.15}$$

so the stationary curvature coincides with the Riemannian curvature V_n in the given two-dimension direction.

The problem of finding the ultimate values of K is the same as finding those vectors V^a where the K takes the ultimate values, that is, the same as finding *undoubtedly stationary directions*. The necessary and sufficient condition of stationary state of the V^a is

$$\frac{\partial}{\partial V^a}\,K = 0\,. \tag{3.16}$$

The problem of finding the stationary curvatures for Einstein spaces had been solved by Petrov [40]. If the space signature is sign-alternating, generally speaking, the stationary curvatures are complex as well as the stationary bivectors relating to them in the V_n.

For four-dimensional Einstein spaces with Minkowski signature, we have the following theorem [40]:

THEOREM *Given an ortho-frame $g_{\alpha\beta} = \{+1, -1, -1, -1\}$, there is a symmetric paired matrix (R_{ab})*

$$R_{ab} = \left(\begin{array}{c|c} M & N \\ \hline N & -M \end{array}\right), \tag{3.17}$$

where M and N are two symmetric square matrices of the 3rd order, whose components satisfy the relationships

$$m_{11}+m_{22}+m_{33} = -\kappa\,, \qquad n_{11}+n_{22}+n_{33} = 0\,. \tag{3.18}$$

After transformations, the lambda-matrix $(R_{ab} - \Lambda\, g_{ab})$ where $g_{\alpha\beta} = \{+1, +1, +1, -1, -1, -1\}$ takes the form

$$\begin{gathered}(R_{ab} - \Lambda\, g_{ab}) = \\ = \left(\begin{array}{c|c} M + iN + \Lambda\varepsilon & 0 \\ \hline 0 & M - iN + \Lambda\varepsilon \end{array}\right) \equiv \\ \equiv \begin{pmatrix} Q(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{pmatrix},\end{gathered} \tag{3.19}$$

where $Q(\Lambda)$ and $\bar{Q}(\Lambda)$ are three-dimensional matrices, the elements of which are complex conjugates, ε is the three-dimensional unit matrix. The matrix $Q(\Lambda)$ can have only one of the following types of characteristics:

(1) [111]; (2) [21]; (3) [3]. It is evident that the initial lambda-matrix can have only one characteristic drawn from:

(1) $[111, \overline{111}]$; (2) $[21, \overline{21}]$; (3) $[3, 3]$.

The bar in the second half of a characteristic implies that elementary divisors in both matrices are complex conjugates. There is no bar in the third kind because the elementary divisors there are always real.

Taking a lambda-matrix of each of the three possible kinds, Petrov deduced the canonical form of the matrix (R_{ab}) in a non-holonomic ortho-frame [40]

<u>The 1st Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \\ N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},\end{gathered} \tag{3.20}$$

where $\sum_{s=1}^{3} \alpha_s = -\kappa$, $\sum_{s=1}^{3} \beta_s = 0$ (so in this case there are 4 independent parameters, determining the space structure by an invariant form),

<u>The 2nd Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2{+}1 & 0 \\ 0 & 0 & \alpha_2{-}1 \end{pmatrix}, \\ N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{pmatrix},\end{gathered} \tag{3.21}$$

where $\alpha_1 + 2\alpha_2 = -\kappa$, $\beta_1 + 2\beta_2 = 0$ (so in this case there are 2 independent parameters determining the space structure by an invariant form),

<u>The 3rd Kind</u>

$$\begin{gathered}(R_{ab}) = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \\ M = \begin{pmatrix} -\frac{\kappa}{3} & 1 & 0 \\ 1 & -\frac{\kappa}{3} & 0 \\ 0 & 0 & -\frac{\kappa}{3} \end{pmatrix}, \\ N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\end{gathered} \tag{3.22}$$

so no independent parameters determining the space structure by an invariant form exist in this case.

Thus Petrov had solved the problem of reducing a lambda-matrix to its canonical form along a real path in a space of the sign-alternating metric. Although this solution is obtained only at given point, the classification obtained is invariant because the results are applicable to any point in the space.

Real curvatures take the form

$$\Lambda_s = \alpha_s + i\beta_s\,, \tag{3.23}$$

in gravitational fields (spaces) of the 3rd kind, where the quantities Λ_s are real: $\Lambda_1 = \Lambda_2 = \Lambda_3 = -\frac{\kappa}{3}$.

Values of some stationary curvatures in gravitational fields (spaces) of the 1st and 2nd kinds can be coincident. If they coincide, we have sub-kinds of the fields (spaces). The 1st kind has 3 sub-kinds: I ($\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$); D ($\Lambda_2 = \Lambda_3$); O ($\Lambda_1 = \Lambda_2 = \Lambda_3$). If the space is empty ($\kappa = 0$) the kind O means the flat space. The 2nd kind has 2 sub-kinds: II ($\Lambda_1 \neq \Lambda_2$); N ($\Lambda_1 = \Lambda_2$). Kinds I and II are called basic kinds.

In empty spaces (empty gravitational fields) the stationary curvatures become the unit value $\Lambda = 0$, so the spaces (fields) are called *degenerate*.

Studying the algebraic structure of the curvature tensor for known solutions to the Einstein equations, it was shown that the most of the solutions are of the 1st kind by Petrov's classification. The curvature decreases with distance from a gravitating mass. In the extreme case where the distance becomes infinite the space approaches the Minkowski flat space. The well-known Schwarzschid solution, describing a spherically symmetric gravitational field derived from a spherically symmetric island of mass located in an empty space, is classified as the sub-kind D of the 1st kind [44].

Invariant criteria for gravitational waves are linked to the algebraic structure of the curvature tensor, which should be associated with a given criterion from the aforementioned types. The most well-known solutions, which are interpreted as gravitational waves, are attributed to the sub-kind N (of the 1st kind). Other solutions are attributed to the 2nd kind and the 3rd kind. It should be noted that spaces of the 2nd and 3rd kinds cannot be flat anywhere, because components of the curvature tensor matrix $\|R_{ab}\|$ contain $+1$ and -1. This makes asymptotical approach to a curvature of zero impossible, i.e. excludes asymptotical approach to Minkowski space. Therefore, because of the structure of such fields, gravitational fields in a space of the 2nd kind (the sub-kind N) or the 3rd kind, are gravitational waves of the curvature traveling everywhere in the space. Pirani [18] holds that gravitational waves are solutions to gravitational fields in spaces of the 2nd kind (the sub-kind N) or the 3rd kind by Petrov's classification. The following solutions are classified as sub-kind N: Peres' solution [34] where he describes flat gravitational waves

$$ds^2 = (dx^0)^2 - 2\alpha(dx^0 + dx^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2; \tag{3.24}$$

Takeno's solution [35]

$$ds^2 = (\gamma + \rho)(dx^0)^2 - 2\rho dx^0 dx^3 - \alpha(dx^1)^2 - 2\delta dx^1 dx^2 - \beta(dx^2)^2 + (\rho - \gamma)(dx^3)^2, \tag{3.25}$$

where $\alpha = \alpha(x^1 - x^0)$, and γ, ρ, β, δ are functions of $(x^3 = x^0)$; Petrov's solution [37], studied also by Bondi, Pirani and Robertson in another coordinate system [19]

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \alpha(dx^2)^2 + 2\beta dx^2 dx^3 + \gamma(dx^3)^2, \tag{3.26}$$

where α, β, γ are functions of $(x^1 + x^0)$.

A detailed study of relations between the invariant criteria for gravitational waves and Petrov's classification had been undertaken by Zakharov [40]. He proved:

THEOREM *In order that a given space satisfies the state of "pure gravitational radiations" (in the Lichnerowicz sense), it is a necessary and sufficient condition that the space should be of the sub-kind N by Petrov's algebraical classification, characterized by equality to zero of the values of the curvature tensor matrix $\|R_{ab}\|$ in the bivector space.*

THEOREM *An Einstein space that satisfies Zelmanov's criterion can only be an empty space ($\varkappa = 0$) of the sub-kind N. And conversely, any empty space V_4 of the sub-kind N (excluding the sole symmetric space* of this kind), that is described by the metric*

$$ds^2 = 2dx^0 dx^1 - \mathrm{sh}^2 x^0 (dx^2)^2 - \sin^2 dx^0 (dx^3)^2, \tag{3.27}$$

satisfies the Zelmanov criterion.

With these theorems we obtain the general relation between the Zelmanov criterion for gravitational wave fields located in empty spaces and the Lichnerowicz criterion for "pure gravitational radiations":

> An empty V_4, satisfying the Zelmanov criterion for gravitational wave fields, also satisfies the Lichnerowicz criterion for "pure gravitational radiations". Conversely, any empty V_n, satisfying the Lichnerowicz criterion (excluding the sole trivial V_n described by the metric 3.27), satisfies the Zelmanov criterion. The relation between the criteria in the general case is still an open problem.

In [40] it was shown that all known solutions to the Einstein equations in vacuum, which satisfy the Zelmanov and Lichnerowicz criteria, can be obtained as particular cases of the more generalized metric whose space permits a covariant constant vector field l^α

$$\nabla_\sigma l^\alpha = 0\,. \tag{3.28}$$

It is evident that condition (3.10) leads automatically to the first condition (3.7), hence this empty V_4 is classified as sub-kind N by Petrov's classification and, also, there the vector l^α, playing a part of the gravitational field wave vector, is isotropic $l_\alpha l^\alpha = 0$ and unique. According to Eisenhart's theorem [60], the space V_4 containing the unique isotropic covariant constant vector l^α (the absolute parallel vector field l^α, in other words), has the metric

$$ds^2 = \varepsilon(dx^0)^2 + 2dx^0 dx^1 + 2\varphi dx^0 dx^2 + 2\psi dx^0 dx^3 + \alpha(dx^2)^2 + 2\gamma dx^2 dx^3 + \beta(dx^3)^2, \tag{3.29}$$

where ε, φ, ψ, α, β, γ are functions of x^0, x^2, x^3, and $l^\alpha = \delta^\alpha_1$. The metric (3.29), satisfying equations (3.2), is the exact solution to the Einstein equations for vacuum, and satisfies the Zelmanov and Lichnerowicz gravitational wave criteria. This solution generalizes well-known solutions deduced by Takeno, Peres, Bondi, Petrov and others, that satisfy the aforementioned criteria [40].

The metric (3.29), taken under some additional conditions [30], satisfies the Einstein equations in their general form (3.1) in the case where $\lambda = 0$ and the energy-momentum tensor $T_{\alpha\beta}$ describes an isotropic electromagnetic field where Maxwell's tensor $F_{\mu\nu}$ satisfies the conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.30}$$

$F^{*\mu\nu} = \frac{1}{2}\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the pseudotensor dual of the Maxwell tensor, $\eta^{\mu\nu\rho\sigma}$ is the discriminant tensor. Direct substitution shows that this metric satisfies the following requirements: the Riner-Wheeler condition [61]

$$R = 0\,, \qquad R_{\alpha\rho}R^{\rho\beta} = \frac{1}{4}\,\delta^\beta_\alpha\,(R_{\rho\sigma}R^{\rho\sigma}) = 0\,, \tag{3.31}$$

*A space is called *symmetric*, if its curvature tensor is a covariant constant, i.e. if it satisfies the condition $\nabla_\sigma R_{\alpha\beta\gamma\delta} = 0$.

and also the Nordtvedt-Pagels condition [62]

$$\eta_{\mu\varepsilon\gamma\sigma}\left(R^{\delta\gamma,\sigma}R^{\varepsilon\tau}-R^{\delta\varepsilon,\sigma}R^{\gamma\tau}\right), \tag{3.32}$$

where $R^{\delta\gamma,\sigma}=g^{\sigma\mu}\nabla_{\mu}R^{\delta\gamma}$, $\delta^{\alpha}_{\beta}=g^{\alpha}_{\beta}$.

From the physical viewpoint we have an interest in isotropic electromagnetic fields because an observer who accompanies it should be moving at the velocity of light [18, 21]. Hence, isotropic electromagnetic fields can be interpreted as fields of electromagnetic radiation without sources. On the other hand, according to Eisenhart theorem [60], a space V_4 with the metric (3.29) permits an absolute parallel vector field $l^{\alpha}=\delta^{\alpha}_{1}$. Taking this fact and also the Einstein equations into account, we conclude that the vector l^{α} considered in this case satisfies the Lichnerowicz criterion for "full gravitational radiations".

Thus the metric (3.29), satisfying the conditions

$$\begin{gathered} R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-\kappa T_{\alpha\beta}\,,\\ T_{\alpha\beta}=\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\alpha\beta}-F_{\alpha\sigma}F_{\beta\cdot}^{\cdot\sigma}\,,\\ F_{\alpha\beta}F^{\alpha\beta}=0\,,\qquad F_{\alpha\beta}F^{*\alpha\beta}=0 \end{gathered} \tag{3.33}$$

and under the additional condition [30]

$$R_{2323}=R_{0232}=R_{0323}=0\,, \tag{3.34}$$

is the exact solution to the Einstein equations which describes co-existence of both gravitational waves and electromagnetic waves. This solution does not satisfy the Zelmanov criterion in the general case, but the solution satisfies it in some particular cases where $T_{\alpha\beta}\neq 0$, and also under $R_{\alpha\beta}=0$.

Wave properties of recursion curvature spaces were studied in [63]. A recursion curvature space is a Riemannian space having a curvature which satisfies the relationship

$$\nabla_{\sigma}R_{\alpha\beta\gamma\delta}=l_{\sigma}R_{\alpha\beta\gamma\delta}\,. \tag{3.35}$$

Because of Bianchi's identity, such spaces satisfy

$$l_{\sigma}R_{\alpha\beta\gamma\delta}+l_{\alpha}R_{\beta\sigma\gamma\delta}+l_{\beta}R_{\sigma\alpha\gamma\delta}=0\,. \tag{3.36}$$

Total classification for recursion curvature spaces had been given by Walker [64]. His results [64] were applied to the basic space-time of the General Theory of Relativity, see [65] for the results. For the class of prime recursion spaces*, we are particularly interested in the two metrics

$$ds^2=\psi(x^0,x^2)(dx^0)^2+2dx^0dx^1-(dx^2)^2-(dx^3)^2, \tag{3.37}$$

$$\begin{aligned} ds^2&=2dx^0dx^1+\psi(x^1,x^2)(dx^1)^2-\\ &\quad-(dx^2)^2-(dx^3)^2,\quad \psi>0\,. \end{aligned} \tag{3.38}$$

*A recursion curvature space is known as prime or simple, if it contains $n-2$ parallel vector fields, which could be isotropic or non-isotropic. Here n is the dimension of the space.

For the metric (3.37) there is only one component of the Ricci tensor that is not zero, $R_{00}=-\dfrac{1}{2}\dfrac{\partial^2\psi}{\partial x^{2\,2}}$, in the metric (3.38) only $R_{11}=-\dfrac{1}{2}\dfrac{\partial^2\psi}{\partial x^{2\,2}}$ is not zero. Einstein spaces with such metrics can only be empty ($\kappa=0$) and flat ($R_{\alpha\beta\gamma\delta}=0$). This can be proven by checking that both metrics satisfy conditions (3.31) and (3.32), which describe isotropic electromagnetic fields.

Both metrics are interesting from the physical viewpoint: in these cases the origin of the space curvature is an isotropic electromagnetic field. Moreover, if we remove this field from the space, the space becomes flat. Besides these there are few metrics which are exact solutions to the Einstein-Maxwell equations, related to the class of isotropic electromagnetic fields. Neither of the said metrics satisfy the Zelmanov and Lichnerowicz criteria.

Minkowski's signature permits only two metrics for non-simple recursion curvature spaces. They are the metric

$$\begin{aligned} ds^2&=\psi(x^0,x^2,x^3)(dx^0)^2+2dx^0dx^1+\\ &\quad+K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2,\\ K_{22}&<0\,,\qquad K_{22}K_{33}-K_{23}^2<0\,, \end{aligned} \tag{3.39}$$

wherein $\psi=\chi_1(x_0)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+ +\chi_2(x^0)x^2+\chi_3(x^0)x^3$, and the metric

$$\begin{aligned} ds^2&=2dx^0dx^1+\psi(x^1,x^2,x^3)(dx^1)^2+\\ &\quad+K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2, \end{aligned} \tag{3.40}$$

wherein $\psi=\chi_1(x_1)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+ +\chi_2(x^1)x^2+\chi_3(x^1)x^3$. Here a_{ij}, K_{ij} ($i,j=2,3$) are constants.

Both metrics satisfy the conditions $R_{\alpha\beta}=\kappa g_{\alpha\beta}$ only if $\kappa=0$, reducing to the single relationship

$$K_{33}a_{22}+K_{22}a_{33}-2K_{23}a_{23}=0\,. \tag{3.41}$$

In this case both metrics are of the sub-kind N by Petrov's classification. It is interesting to note that the metric (3.40) is stationary and, at the same time, describes "pure gravitational radiation" by Lichnerowicz. Such a solution was also obtained in [65].

In the general case ($R_{\alpha\beta}\neq\kappa g_{\alpha\beta}$) the metrics (3.39) and (3.40) satisfy conditions (3.32) and (3.33), so the metrics are solutions to the Einstein-Maxwell equations that describe co-existing gravitational waves and electromagnetic waves without sources. In this general case both metrics satisfy the Zelmanov and Lichnerowicz invariant criteria. The solution (3.40) is stationary.

All that has been detailed above applies to gravitational waves as waves of the space curvature, which exist in any reference frame.

Additionally it would be interesting to study another approach to the gravitational radiation problem, where the main issue is gravitational inertial waves, connected to the given reference frame of an observer. This new approach is linked directly to the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), introduced by Zelmanov in 1944 [42, 43]. In order to understand the true results given by gravitational wave experiments it is necessary to master this mathematical apparatus, which is described concisely in the in the next section.

4 Basics of the theory of physical observable quantities

In brief, the essence of the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), developed by Zelmanov in 1940's [42, 43] is that, if an observer accompanies his reference body, his observable quantities are projections of four-dimensional quantities on his time line and the spatial section — *chronometrically invariant quantities*, made by the projecting operators $b^{\alpha}=\frac{dx^{\alpha}}{ds}$ and $h_{\alpha\beta}=-g_{\alpha\beta}+b_{\alpha}b_{\beta}$ which fully define his real reference space (here b^{α} is his velocity with respect to his real references). The chr.inv.-projections of a world-vector Q^{α} are $b_{\alpha}Q^{\alpha}=\frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_{\alpha}Q^{\alpha}=Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^{\alpha}b^{\beta}Q_{\alpha\beta}=\frac{Q_{00}}{g_{00}}$, $h^{i\alpha}b^{\beta}Q_{\alpha\beta}=\frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_{\alpha}h^k_{\beta}Q^{\alpha\beta}=Q^{ik}$. Physically observable properties of the space are derived from the fact that the chr. inv.-differential operators $\frac{{}^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}v_i\frac{{}^*\partial}{\partial t}$ are non-commutative, so that $\frac{{}^*\partial^2}{\partial x^i\partial t}-\frac{{}^*\partial^2}{\partial t\,\partial x^i}=\frac{1}{c^2}F_i\frac{{}^*\partial}{\partial t}$ and $\frac{{}^*\partial^2}{\partial x^i\partial x^k}-\frac{{}^*\partial^2}{\partial x^k\partial x^i}=\frac{2}{c^2}A_{ik}\frac{{}^*\partial}{\partial t}$, and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i=\frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right),\qquad \sqrt{g_{00}}=1-\frac{\mathrm{w}}{c^2} \tag{4.1}$$

$$A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\left(F_i v_k-F_k v_i\right), \tag{4.2}$$

$$D_{ik}=\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,,\quad D^{ik}=-\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,,\quad D^k_k=\frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{4.3}$$

where w is the gravitational potential, $v_i=-c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $h_{ik}=-g_{ik}+\frac{1}{c^2}v_iv_k$ is the metric chr.inv.-tensor, and $h=\det\|h_{ik}\|$, $h\,g_{00}=-g$, $g=\det\|g_{\alpha\beta}\|$. Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk}=h^{im}\Delta_{jk,m}$, which are built just like Christoffel's regular symbols $\Gamma^{\alpha}_{\mu\nu}=g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, but using h_{ik} instead of $g_{\alpha\beta}$.

In this way, any equations obtained using general covariant methods we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections with its real physically observable properties. From this we arrive at equations containing only quantities measurable in practice. Expressing $ds^2=g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}$ through the observable time interval

$$d\tau=\frac{1}{c}\,b_{\alpha}dx^{\alpha}=\left(1-\frac{\mathrm{w}}{c^2}\right)dt-\frac{1}{c^2}\,v_i dx^i \tag{4.4}$$

and also the observable spatial interval $d\sigma^2=h_{\alpha\beta}\,dx^{\alpha}dx^{\beta}=h_{ik}\,dx^idx^k$ (note that $b^i=0$ for an observer who accompanies his reference body). We arrive at the formula

$$ds^2=c^2d\tau^2-d\sigma^2. \tag{4.5}$$

From an"external" viewpoint, an observer's three-dimensional space is the *spatial section* $x^0=ct=\text{const}$. At any point of the space-time a local spatial section (a local space) can be placed orthogonal to the *time line*. If there exists a space-time enveloping curve for such local spaces, then it is a spatial section everywhere orthogonal to the time lines. Such a space is called *holonomic*. If no enveloping curve exists for such local spaces, so there only exist spatial sections locally orthogonal to the time lines, such a space is called *non-holonomic*. A spatial section, placed in a holonomic space, is everywhere orthogonal to the time lines, i. e. $g_{0i}=0$ is true there. In the presence of $g_{0i}=0$ we have $v_i=0$, hence $A_{ik}=0$. This implies that non-holonomity of the space and its three-dimensional rotation are the same. In a non-holonomic space $g_{0i}\neq 0$ and $A_{ik}\neq 0$. Hence $A_{ik}=0$ is the necessary and sufficient condition of holonomity of the space. So A_{ik} is the *tensor of the space non-holonomity*.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are linked to one another by two identities

$$\frac{{}^*\partial A_{ik}}{\partial t}+\frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i}-\frac{{}^*\partial F_i}{\partial x^k}\right)=0\,, \tag{4.6}$$

$$\begin{gathered}\frac{{}^*\partial A_{km}}{\partial x^i}+\frac{{}^*\partial A_{mi}}{\partial x^k}+\frac{{}^*\partial A_{ik}}{\partial x^m}+\\ +\frac{1}{2}\left(F_iA_{km}+F_kA_{mi}+F_mA_{ik}\right)=0\,,\end{gathered} \tag{4.7}$$

which are known as *Zelmanov's identities*.

Components of the usual Christoffel symbols

$$\Gamma^{\alpha}_{\mu\nu}=\frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}+\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right). \tag{4.8}$$

are linked to the chr.inv.-Christoffel symbols

$$\Delta^i_{jk}=\frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k}+\frac{{}^*\partial h_{km}}{\partial x^j}-\frac{{}^*\partial h_{jk}}{\partial x^m}\right), \tag{4.9}$$

and other chr.inv.-chractersitics of the accompanying reference space of the given observer by the relations

$$D^i_k + A^{\cdot i}_{k\cdot} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k} - \frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right), \qquad (4.10)$$

$$F^k = -\frac{c^2\,\Gamma^k_{00}}{g_{00}}\,, \qquad g^{i\alpha}g^{k\beta}\,\Gamma^m_{\alpha\beta} = h^{iq}h^{ks}\Delta^m_{qs}. \qquad (4.11)$$

Here is the four-dimensional generalization of the chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov, the 1960's [57]): $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta a_{\mu\nu}$, $D_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha\right)$, $d_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha\right)$.

Zelmanov also deduced formulae for chr.inv.-projections of the Riemann-Christoffel tensor [42]. He followed the same procedure by which the Riemann-Christoffel tensor was built, proceeding from the non-commutativity of the second derivatives of an arbitrary vector taken in the given space. Taking the second chr.inv.-derivatives of an arbitrary vector

$$^*\nabla_i{}^*\nabla_k\, Q_l - {}^*\nabla_k{}^*\nabla_i\, Q_l = \frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t} + H^{\cdot\cdot\cdot j}_{lki\cdot}Q_j\,, \qquad (4.12)$$

he obtained the chr.inv.-tensor

$$H^{\cdot\cdot\cdot j}_{lki\cdot} = \frac{^*\partial\Delta^j_{il}}{\partial x^k} - \frac{^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\,, \qquad (4.13)$$

which is like Schouten's tensor from the theory of non-holonomic manifolds [59]. The tensor $H^{\cdot\cdot\cdot j}_{lki}$ differs algebraically from the Riemann-Christoffel tensor because of the presence of rotation of the space A_{ik} in the formula (4). Nevertheless its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \qquad (4.14)$$

which possesses all the algebraic properties of the Riemann-Christoffel tensor in this three-dimensional space. Therefore Zelmanov called C_{iklj} the *chr.inv.-curvature tensor*, which actually is the tensor of the observable curvature of the observer's spatial section. This tensor, describing the observable curvature of the three-dimensional space of an observer, possesses all the properties of the Riemann-Christoffel curvature tensor in the three-dimensional space and, at the same time, the property of chronometric invariance. Its contraction

$$C_{kj} = C^{\cdot\cdot\cdot i}_{kij\cdot} = h^{im}C_{kimj}\,, \qquad C = C^j_j = h^{lj}C_{lj} \qquad (4.15)$$

gives the chr.inv.-scalar C whose sense is the *observable three-dimensional curvature* of this space.

Substituting the necessary components of the Riemann-Christoffel tensor into the formulae for its chr.inv.-projections $X^{ik} = -c^2\frac{R^{\cdot i\cdot k}_{0\cdot 0\cdot}}{g_{00}}$, $Y^{ijk} = -c\frac{R^{\cdot ijk}_{0\cdot\cdot\cdot}}{\sqrt{g_{00}}}$, $Z^{ijkl} = c^2R^{ijkl}$, and by lowering indices Zelmanov obtained the formulae

$$X_{ij} = \frac{^*\partial D_{ij}}{\partial t} - \left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D_{jl} + A_{jl}\right) + \frac{1}{2}\left(^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}\,F_iF_j\,, \qquad (4.16)$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}A_{ij}F_k\,, \qquad (4.17)$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2C_{iklj}\,, \qquad (4.18)$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like the Riemann-Christoffel tensor. Contraction of the spatial observable projection Z_{iklj} step-by-step gives

$$Z_{il} = D_{ik}D^k_l - D_{il}D + A_{ik}A^{\cdot k}_{l\cdot} + 2A_{ik}A^{k\cdot}_{\cdot l} - c^2C_{il}\,, \qquad (4.19)$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C\,. \qquad (4.20)$$

Besides these considerations, taken in an observer's accompanying reference frame, Zelmanov considered a *locally geodesic reference frame* that can be introduced at any point of the pseudo-Riemannian space. Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2\tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots\,, \qquad (4.21)$$

i. e. its components at a point, located in the vicinities, are different to those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can betaken as constant, while the first derivatives of the metric (the Christoffel symbols) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be defined. At the same time, at any point of this locally geodesic reference frame, a tangential flat Euclidean space can be defined so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space converge to values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basis vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). So the lengths of the basis vectors may be sometimes very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$. Then $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^2_{(3)}\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$. On the other hand, the same quantity is $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$. As a result we have $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)}\cos(x^\alpha; x^\beta)$. So we obtain

$$g_{00} = e_{(0)}^2\,, \qquad g_{0i} = e_{(0)}e_{(i)}\cos(x^0; x^i)\,, \tag{4.22}$$

$$g_{ik} = e_{(i)}e_{(k)}\cos(x^i; x^k)\,. \tag{4.23}$$

The gravitational potential is $\mathrm{w} = c^2(1-\sqrt{g_{00}})$. So, the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$, having the length $e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$ is smaller than unity the greater is the gravitational potential w.

The space rotation linear velocity v_i and, according to it, the chr.inv.-metric tensor h_{ik} are

$$v_i = -c\,e_{(i)}\cos(x^0; x^i)\,, \tag{4.24}$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]\,. \tag{4.25}$$

This representation enablkes us to see the geometric sense of physical quantities measurable in experiments, because we represent them through pure geometric characteristics of the observer's space — the angles between coordinate axes etc.

This completes the basics of Zelmanov's mathematical apparatus of chronometric invariants (physically observable quantities) that will be employed below with the aim of studying the gravitational wave problem.

5 Gravitational inertial waves and their link to the chronometrically invariant representation of Petrov's classification

Of all the experimental statements on the General Theory of Relativity, including the search for gravitational wave experiments, the most important case is that where the observer is at rest with respect to his laboratory reference frame and all physical standards located in it. Quantities measured by the observer in an *accompanying reference frame* are *chronometrically invariant quantities* (see the previous paragraph for the details). Keeping this fact in mind, Zelmanov formulated his *chronometrically invariant criterion for gravitational waves*. This criterion is invariant only for transformations of coordinates of that reference system which is at rest with respect to the laboratory references (the body of reference). Such an approach, in contrast to the invariant approach, permits us to interpret the results of measurement in terms of physically observable quantities, providing thereby a means of comparing results given by the theory of gravitational waves to results obtained from real physical experiments.

In order to solve the problem of interpretation of experimental data on gravitational waves it is appropriate to consider a more general case — fields of gravitational inertial waves. Such fields are more general because they are applicable to both gravitational fields and the inertial field of the observer's reference frame. The mathematical method that we propose to apply to this problem joins both fields into a common field. The method itself does not differ for each field: to set an invariant difference between gravitational fields and the observer's inertial field would be possible only by introducing an additional invariant criterion.

Gravitational waves are determined independently of both spatial coordinate frames and space-time reference frames. In contrast to gravitational waves, gravitational inertial waves are determined only in the reference frame of an observer, who observes them. They are determined with precision to within so-called "inner" transformations of coordinates

$$\left.\begin{aligned} &\text{(a)}\quad \tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3) \\ &\text{(b)}\quad \tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\} \tag{5.1}$$

which does not change the space-time reference frame itself.

Invariance with respect to (5.1) splits into invariance with respect to (5.1a), so-called *chronometric invariance*, and also invariance with respect to (5.1b), so-called *spatial invariance*. Therefore a definition given for gravitational inertial waves should be:

(1) chronometrically invariant;

(2) spatially covariant.

We then have a basis by which we introduce the chronometrically invariant spatially covariant d'Alembert operator [40]*

$$^*\Box = h^{ik}\,{}^*\nabla_i{}^*\nabla_k - \frac{1}{a^2}\frac{{}^*\partial^2}{\partial t^2}\,, \tag{5.2}$$

where $h^{ik} = -g^{ik}$ is the chr.inv.-metric tensor (the physically observable metric tensor) in its contravariant (upper-index) form, ${}^*\nabla_i$ is the symbol for the chr.inv.-derivative (the chr.inv.-analogue to the covariant derivative symbol ∇_σ), a is the linear velocity at which attraction of gravity spreads, $\frac{{}^*\partial}{\partial t}$ is the symbol for the chr.inv.-derivative with respect to time.

A chronometrically invariant criterion for gravitational inertial waves, formulated according to Zelmanov's idea, is:

Zelmanov's chr.inv.-criterion Chr.inv.-quantities f, characterising the observer's reference space, such as the gravitational inertial force vector F_i, the space non-holonomity (self-rotation) tensor A_{ik}, the space deformation rate tensor D_{ik}, the spatial curvature tensor C_{iklj}, and also scalar quantities, built on them, and also the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} must satisfy equations of the form

$$^*\Box f = A, \tag{5.3}$$

*This approach to the gravitational inertial wave problem was developed by Zelmanov, although it had first been published by Zakharov because the latter prepared his dissertation under Zelmanov's leadership: see footnote on page 35.

where A is an arbitrary function of four-dimensional world-coordinates, which has no more than first order derivatives of the f.

The Zelmanov chr.inv.-criterion (5.3) was applied in analyzing well-known solutions to the Einstein equations in emptiness [40]. This criterion is true for the metrics (3.25) in that case where the gravitational inertial force vector F^i is the wave function. But, at the same time, most of the invariant criteria for gravitational waves are related to some conditions and limitations imposed on the curvature tensor. Therefore it would be most interesting to study relations between gravitational wave criteria and gravitational inertial wave criteria in that case where the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} are the wave functions.

What is the relation between the Zelmanov invariant criterion (3.9) and his chr.inv.-criterion (5.3)? This problem was solved by Zakharov [40, 58]. His method was to express equation (3.9) in chr.inv.-form. In chr.inv..-form (in the terms of physically observable quantities) equation (3.9) takes the form

$$^*\Box X^{ij}=A^{ij}_{(1)}\,,\quad ^*\Box Y^{ijk}=A^{ijk}_{(2)}\,,\quad ^*\Box Z^{iklj}=A^{iklj}_{(3)}\,,\qquad (5.4)$$

where $A^{ij}_{(1)}$, $A^{ijk}_{(2)}$, $A^{iklj}_{(3)}$ are chronometrically invariant and spatially invariant tensors, which have no more than first order derivatives of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}. Thus those gravitational fields that satisfy the Zelmanov invariant criterion also satisfy the Zelmanov chr.inv.-criterion (5.3), where the Riemann-Christoffel curvature tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj} play the part of wave functions.

The necessary condition for gravitational inertial waves is the fact that the chr.inv.-d'Alembert operator (5.2) is non-trivial, mathematically expressed as follows:

1. Chr.inv.-quantities f are non-stationary, i. e. $\frac{^*\partial f}{\partial t}\neq 0$;
2. The quantities f are inhomogeneous, i. e. $^*\nabla_i f_k\neq 0$.

The wave functions X_{ij} (4.16), Y_{ijk} (4.17) and Z_{iklj} (4.18) satisfy these requirements only if the mechanical chr.inv.-characteristics of the observer's reference space (the chr.inv.-quantities F_i, A_{ik}, D_{ik}) and the geometric chr.inv.-characteristic of the space (the chr.inv.-quantity C_{iklj}) also satisfy these requirements. Zelmanov himself in [42] formulated conditions of inhomogeneity inside a finite region located in the observer's space

$$\left.\begin{array}{ll} ^*\nabla_i F_k\neq 0\,, & ^*\nabla_j A_{ik}\neq 0\,, \\ ^*\nabla_j D_{ik}\neq 0\,, & ^*\nabla_j C_{ik}\neq 0\,. \end{array}\right\}\qquad (5.5)$$

It is evident that under these conditions the wave functions X^{ij}, Y^{ijk}, Z^{iklj} shall be inhomogeneous.

The origin of non-stationary states of the gravitational inertial force vector F_i (4.1) is the non-stationarity of the gravitational potential w or the linear velocity of the space rotation v_i, consisting the force. Identities (4.6) and (4.7), linking quantities F_i and A_{ik}, lead us to conclude that the source of non-stationary states of v_i is the vortical nature of the vector F_i, i. e. $^*\nabla_k F_i - {}^*\nabla_i F_k\neq 0$. The origin of non-stationary states of the space deformation rate D_{ik} (4.3) and the space observable curvature C_{iklj} (4.14) is non-stationarity of the physical observable metric tensor h_{ik}, see [42],

$$h_{ik}=-g_{ik}+\frac{g_{0i}g_{0k}}{g_{00}}=-g_{ik}+\frac{1}{c^2}\,v_i v_k\,.\qquad (5.6)$$

Thus, the origin of non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} is the non-stationarity of components of the fundamental metric tensor $g_{\alpha\beta}$, namely:

(1) $g_{00}=\left(1-\frac{\mathrm{w}}{c^2}\right)^2$;

(2) $g_{0i}=-\frac{1}{c}\,v_i\left(1-\frac{\mathrm{w}}{c^2}\right)$;

(3) $g_{ik}=-h_{ik}+\frac{1}{c^2}\,v_i v_k$.

We consider each of the cases here,mindful of the need to find theoretical grounds for gravitational wave experiments:

1. Non-stationary states of g_{00} manifest as a result of time changes of the gravitational potential w. In experiments this non-stationarity is derived from very different geophysical sources, which, in a particular case, are due to changes in solar activity;
2. Non-stationary states of mixed components g_{0i} are derived from the non-stationarity of the space rotation linear velocity v_i and the gravitational potential w. The quantities g_{00} and g_{0i} are included in the formula for an interval of observable time $d\tau=\sqrt{g_{00}}\,dt+\frac{g_{0i}}{\sqrt{g_{00}}}\,dx^i$ [42, 43]. Thus under non-stationary states of g_{00} and g_{0i} in the observer's laboratory (his reference frame) a standard clock located there should have some corrections (which change with time) with respect to a standard clock located in an region where the quantities g_{00} and g_{0i} are stationary.
3. Non-stationary states of g_{ik} are usually considered as deformations of the three-dimensional space. But the theory of physically observable quantities introduces substantial corrections to this thesis. The approach of Classical Mechanic looks at the spatial deformations as $\frac{1}{2}\frac{\partial g_{ik}}{\partial t}$, but the theory of physically observable quantities, taking properties of the observer into account, gives rise to a corrected formula for the spatial deformations which is $D_{ik}=\frac{1}{2\sqrt{g_{00}}}\frac{\partial}{\partial t}\left(-g_{ik}+\frac{1}{c^2}\,v_i v_k\right)$.*

*The presence of the minus sign here is a consequence of the fact that we use the signature $(+---)$, where plus is related to the time coordinate while minus is attributed to spatial coordinates. The minus sign has been chosen for the g_{ik} in the h_{ik} formula, because in this case the observable spatial interval $d\sigma=h_{ik}\,dx^i dx^k$ is positive, which is an important fact in the theory of physically observable quantities [42, 43].

The formulae coincide in that particular case where $g_{00}=1$ (w = 0) and $g_{0i}=0$ ($v_i=0$). If $F_i=0$, according (to 4.6) the space rotation is stationary. If $v_i=0$, $A_{ik}=0$. Thus the necessary and sufficient condition to make w and v_i simultaneously zero is $F_i=0$ and $A_{ik}=0$ [42, 43]. In this case the observer's reference frame falls freely and is free of rotations. Such reference frames are known as *synchronous* [15], because there all clocks can be synchronized. Moreover, in this case time can be integrated: in calculations of the time interval $d\tau=dt$ between any two events, the integral of $d\tau$ is independent of the way we take this integral between the events (the path of integration). If $F_i\neq 0$ but $A_{ik}=0$, it is impossible to synchronize all the clocks simultaneously, but the synchronization itself can be realized because of the proportionality $d\tau=\sqrt{g_{00}}\,dt$ there. If $A_{ik}\neq 0$, the synchronization is impossible in principle, because the integral of $d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{\sqrt{g_{00}}}\,dx^i$ depends on the path of integration [42, 43].

Synchronous reference frames, because of their simplicity and associated simple calculations, are of broad utility in the General Theory of Relativity. In particular, they are used in relativistic cosmology and the gravitational wave problem. For instance, the well-known metric of weak plane gravitational waves takes the form [14, 15]

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + + 2b\,dx^2dx^3 - (1+a)(dx^3)^2, \tag{5.7}$$

where $a=a(ct\pm x^1)$, $b=b(ct\pm x^1)$. So in this metric there is no gravitational potential (w = 0) as soon as there is no space rotation ($v_i=0$). The condition w = 0 prohibits the ultimate transit to Newton's theory of gravity. For this reason we arrive at an important conclusion:

> Weak plane gravitational waves are derived from sources other than gravitational fields of masses*.

An analogous situation arises in relativistic cosmology, where, until now, the main part is played by the theory of a homogeneous isotropic universe. Foundations of this theory are built on the metric of a homogeneous isotropic space [42]

$$ds^2 = c^2dt^2 - - R^2\frac{(dx^1)^2+(dx^2)^2+(dx^3)^2}{\left[1+\frac{k}{4}\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right]\right]^2}, \tag{5.8}$$

$$R=R(t)\,, \qquad k=0,\pm 1\,.$$

When one substitutes this metric into the Einstein equations taken with a specific value of the cosmological constant ($\lambda=0$, $\lambda<0$, $\lambda>0$), he obtains a spectra of solutions, which are known as *Friedmann's cosmological models* [42].

Taking our previous conclusion on the origin of weak plane gravitational waves into account, we come to a new and important conclusion:

> No gravitational fields derived from masses exist in any Friedmann universe. Moreover, any Friedmann universe is free of space rotations.

Currently there is no indubitable observational data supporting the absolute rotation of the Universe. This problem has been under considerable discussion between astronomers and physicists over last decade, and remains open. Rotations of bulk space bodies like planets, stars, and galaxies are beyond any doubt, but these rotations do not imply the absolute rotation of the whole Universe, including the absolute rotation of its gravitational field if one will describe it by the Friedmann models.

Looking back at the question of whether or not gravitational inertial waves exist, or whether or not non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} exist, we conclude that non-stationary states of the quantities are derived from:

1. A vortical nature of the field of the acting gravitational inertial force F_i;
2. Non-stationary states of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$.

In the first case, the effect of gravitational inertial waves manifests as non-stationary corrections to the observer's time flow.

In the second case, the observer's time flow remains unchanged, but gravitational waves are waves of only the space deformation. Such pure deformation waves will deform a detector itself, so one simply waits for a gravitational wave to cause a resonance effect in a solid-body detector of the Weber kind [16]. Whether this conclusion is true or false will be considered in §7 and §8. Here we consider only the general theory of gravitational inertial waves and its relation to the invariant theory of gravitational waves.

As we showed above, those gravitational fields that satisfy the Zelmanov invariant criterion (3.9) also satisfy the Zelmanov chr.inv.-criterion (5.3), where the wave functions f are the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj}. As it was shown in the previous paragraph, "empty gravitational fields" (we mean gravitational fields permeating empty spaces, where no mass islands of matter exist) that satisfy the Zelmanov invariant criterion (3.9) are related to the 2nd kind (the sub-kind N) by Petrov's classification. Therefore it is appropriate to specify the algebraical kinds of the Riemann-Christoffel tensor in terms of physically observable quantities (chronometric invariants).

The whole problem of representing Petrov's classification in chronometrically invariant form has been solved in [66]. This solution, obtained Petrov in general covariant form [37],

*See §7 and §8 below for detailed calculations for the effect due to weak plane gravitational waves in solid-body detectors of the Weber kind (the Weber pigs) and also in antennae built on free masses.

was obtained for an ortho-frame, taken at an arbitrary fixed point of the space.

Chr.inv.-components of the Riemann-Christoffel curvature tensor have the properties

$$X_{ij} = X_{ji}\,, \quad X^k_k = -\kappa\,, \qquad Y_{[ijk]} = 0\,, \quad Y_{ijk} = -Y_{ikj}\,. \tag{5.9}$$

Equations (4.16), (4.17), (4.18) in an ortho-frame are

$$X_{ij} = -c^2 R_{0i0j}\,, \quad Y_{ijk} = -cR_{0ijk}\,, \quad Z_{iklj} = c^2 R_{iklj}\,. \tag{5.10}$$

When we write equations $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ in the orth-frame, we take the relationships (5.10) into account. Then, introducing three-dimensional matrices x and y such that

$$x \equiv \| x_{ik} \| = -\frac{1}{c^2}\, \| X_{ik} \|\,, \qquad y \equiv \| y_{ik} \| = -\frac{1}{2c}\, \| \varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn} \|\,, \tag{5.11}$$

where ε_{imn} is the three-dimensional discriminant tensor, we represent the six-dimensional matrix R_{ab} as follows

$$\| R_{ab} \| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|, \qquad a, b = 1, 2, \dots 6\,, \tag{5.12}$$

satisfying the relations

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.13}$$

Now, let us compose a lambda-matrix

$$\| R_{ab} - \Lambda g_{ab} \| = \left\| \begin{matrix} x + \Lambda\varepsilon & y \\ y & -x - \Lambda\varepsilon \end{matrix} \right\|, \tag{5.14}$$

where ε is the three-dimensional unit matrix. Then, after transformations, we reduce this lambda-matrix to the form

$$\left\| \begin{matrix} x + iy + \Lambda\varepsilon & 0 \\ 0 & -x - iy - \Lambda\varepsilon \end{matrix} \right\| = \left\| \begin{matrix} \bar{Q}(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{matrix} \right\|. \tag{5.15}$$

The initial lambda-matrix can have one of the following characteristics:

$$(1)\ [111, \overline{111}]; \quad (2)\ [21, \overline{21}]; \quad (3)\ [3, 3]. \tag{5.16}$$

Then, using Petrov's had obtained the canonical form of the matrix $\|R_{ab}\|$ in the non-holonomic ortho-frame for each of the three kinds of the curvature tensor [37], we express the matrix $\|R_{ab}\|$ through components of the chr.inv.-tensors X_{ij} and Y_{ijk} [66]. We obtain

<u>The 1st Kind</u>

$$\| R_{ab} \| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|,$$

$$x = \left\| \begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{matrix} \right\|, \qquad y = \left\| \begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{matrix} \right\|, \tag{5.17}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.18}$$

Using (5.11) we also express values of the stationary curvatures Λ_i $(i = 1, 2, 3)$ through the Riemann-Christoffel tensor's physically observable components

$$\Lambda_1 = -\frac{1}{c^2}\, X_{11} + \frac{i}{c}\, Y_{123}\,, \quad \Lambda_2 = -\frac{1}{c^2}\, X_{22} + \frac{i}{c}\, Y_{231}\,, \quad \Lambda_3 = -\frac{1}{c^2}\, X_{33} + \frac{i}{c}\, Y_{312}\,. \tag{5.19}$$

Thus, the components X_{ik} are included in the real parts of the stationary curvatures Λ_i, and components Y_{ijk} are included in the imaginary parts. In spaces of the sub-kind D $(\Lambda_2 = \Lambda_3)$ we have: $X_{22} = X_{33}$, $Y_{231} = Y_{312}$. In spaces of the sub-kind O $(\Lambda_1 = \Lambda_2 = \Lambda_3)$ we have: $X_{11} = X_{22} = X_{33} = -\frac{\kappa}{3}$, $Y_{123} = Y_{231} = Y_{312}$. Hence Einstein spaces of the sub-kind O have only real curvatures, while being empty they are flat.

For the 2nd kind we have

<u>The 2nd Kind</u>

$$\| R_{ab} \| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|,$$

$$x = \left\| \begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22}+1 & 0 \\ 0 & 0 & x_{33}-1 \end{matrix} \right\|, \qquad y = \left\| \begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 1 \\ 0 & 1 & y_{22} \end{matrix} \right\|, \tag{5.20}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad x_{22} - x_{33} = 2\,, \quad y_{11} + 2y_{22} = 0\,. \tag{5.21}$$

The stationary curvatures are

$$\Lambda_1 = -\frac{1}{c^2}\, X_{11} + \frac{i}{c}\, Y_{123}\,, \quad \Lambda_2 = -\frac{1}{c^2}\, X_{22} - 1 + \frac{i}{c}\, Y_{231}\,, \quad \Lambda_3 = -\frac{1}{c^2}\, X_{33} + 1 + \frac{i}{c}\, Y_{312}\,. \tag{5.22}$$

From this we conclude that values of the stationary curvatures Λ_2 and Λ_3 can never become zero, so Einstein spaces (gravitational fields) of the 2nd kind are curved in any case — they cannot approach Minkowski flat space.

In spaces of the sub-kind N ($\Lambda_1 = \Lambda_2$) in an ortho-frame the relations are true

$$X_{11} = X_{22} - c^2 = X_{33} + c^2, \qquad Y_{123} = Y_{231} = Y_{312} = 0\,, \tag{5.23}$$

so the stationary curvatures are real. In an empty space the matrices x and y become degenerate (its determinant becomes zero). For this reason spaces of the sub-kind N are *degenerate*, and, respectively, gravitational fields in spaces of the sub-kind N are known as *gravitational fields of the 2nd degenerate kind by Petrov's classification*. In emptiness ($\kappa = 0$) some elements of the matrices x and y take the numerical values $+1$ and -1 thereby making an ultimate transition to the Minkowski flat space impossible.

For the 3rd kind we have

The 3rd Kind

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|, \quad x = \left\| \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|, \quad y = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{matrix} \right\|. \tag{5.24}$$

Here the stationary curvatures are zero and both of the matrices x and y are degenerate. Einstein spaces of the 3rd kind can only be empty ($\kappa = 0$), but, at the same time, they can never be flat.

From the equations deduced for the canonical form of the matrix $\|R_{ab}\|$, we conclude: $Y_{ijk} = 0$ can be true only in gravitational fields of the 1st kind, which are derived from island masses of matter in emptiness or vacuum. Therefore we conclude that those gravitational fields where $Y_{ijk} = 0$ is true in the observer's accompanying reference frame can only be of the 1st kind, having stationary curvatures which are real.

Furthermore, in accordance with most of the criteria, the presence of gravitational waves is linked to spaces of the 2nd (N) kind and the 3rd kind, where the matrix y_{ik} has components equal to $+1$ or -1. Moreover, in fields of the 2nd (N) and 3rd kinds the values $+1$ or -1 are attributed also to components of the matrix x. This implies that:

> Those spaces which contain gravitational fields, satisfying the invariant criteria for gravitational waves, are curved independently of whether or not they are empty ($T_{\alpha\beta} = 0$) or filled with matter (in such spaces $T_{\alpha\beta} = g_{\alpha\beta}$). In any case, gravitational radiations are derived from interaction between two observable components X_{ij}, Y_{ijk} of the Rimeann-Christoffel curvature tensor.

The classification of gravitational fields built here applies only to Einstein spaces, because solving this problem for spaces of general kind, where $T_{\alpha\beta} \neq \kappa g_{\alpha\beta}$, would be very difficult, for mathematical reasons. Considering the details of these difficulties, we see that, having an arbitrary distribution of matter in a space, the matrix $\|R_{ab}\|$, taken in a non-holonomic ortho-frame, is not symmetrically doubled; on the contrary, the matrix takes the form

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y' & z \end{matrix} \right\|, \tag{5.25}$$

where the three-dimensional matrices x, y, z are built on the following elements, respectively*

$$x_{ik} = -\frac{1}{c^2}\,X_{ik}\,, \qquad z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}, \qquad y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}\,, \tag{5.26}$$

and y' implies transposition. It is evident that reduction of this matrix to its canonical form is a very difficult problem.

Nevertheless Petrov's classification permits us to conclude:

> The physically observable components X^{ij} and Y^{ijk} of the Riemann-Christoffel curvature tensor are different in their physical origin†. Metrics can exist where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$. Such spaces are of the 1st kind by Petrov's classification; they have real stationary curvatures. Such spaces do not satisfy the invariant criteria for gravitational waves. Thus no wave fields of gravity exist in spaces where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$.

And further:

> In solutions of the Einstein equations there are no metrics where $Y^{ijk} \neq 0$ but $X^{ij} = 0$ and $Z^{iklj} = 0$. Thus in wave fields of gravity $Y^{ijk} \neq 0$ and $X^{ij} \neq 0$ (and as well $Z^{iklj} \neq 0$: see the footnote) everywhere and always.

*In ortho-frames there is no difference between upper and lower indices (see [37]). For this reason we can write $z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z_{mnpq}$ and $y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{kmn}$ instead of $z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}$ and $y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}$ in formula (3.26). This note relates to all formulae written in an ortho-frame. We met a similar case in formula (5.11), where we can also write $y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\|\varepsilon_{imn}Y_{kmn}\|$ instead of $y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\left\|\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}\right\|$.

†We do not mention the third observable component Z^{iklj}, because in an ortho-frame the matrices x and z are connected by the equation $x = -z$.

We will show that in Einstein spaces filled with gravitational fields where the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj} play a part of the wave functions, the quantity X^{ij} is analogous to the electric component of an electromagnetic field, while Y^{ijk} is analogous to its magnetic component. All this will be discussed in §7.

6 Wave properties of Einstein's equations

In §2 we have showed that the gravitational field equations (the Einstein equations) do not contain a general covariant d'Alembert operator derived from the fundamental metric tensor $g_{\alpha\beta}$ (where $g_{\alpha\beta}$ is considered as a "four-dimensional gravitational potential"). Nevertheless this problem has been solved in linear approximation in the case where gravitational fields are occupy an empty space ($R_{\alpha\beta}=0$, "empty gravitational fields") [14, 15]. In this case a gravitational field is considered as a tiny addition to a flat space background described by the Minkowski metric. Thus

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}\,, \tag{6.1}$$

where $\delta_{\alpha\beta}$ are components of the fundamental metric tensor in a Galilean reference frame $\delta_{\alpha\beta}=\{+1,-1,-1,-1\}$, and $\gamma_{\alpha\beta}$ describes weak corrections for the gravitational fields. The contravariant fundamental metric tensor $g^{\alpha\beta}$ to within the first order approximation of the $\gamma_{\alpha\beta}$ is

$$g^{\alpha\beta} = \delta^{\alpha\beta} - \gamma^{\alpha\beta}, \tag{6.2}$$

so the determinant of the tensor $g_{\alpha\beta}$ is

$$g = -\left(1+\gamma\right), \qquad \gamma = \det\|\gamma_{\alpha\beta}\|\,. \tag{6.3}$$

The requirement that components of the "additional" metric $\gamma_{\alpha\beta}$ must be infinitesimal fixes a prime reference frame. If this requirement is true in a reference frame, it will also be true after transformations

$$\tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}, \tag{6.4}$$

where ξ^{α} are infinitesimal quantities $\xi^{\alpha} \ll 1$. Then we have

$$\tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{\partial \xi_{\alpha}}{\partial x^{\beta}} - \frac{\partial \xi_{\beta}}{\partial x^{\alpha}}\,. \tag{6.5}$$

Because of (6.1), we impose an additional requirement on the tensor $\gamma_{\alpha\beta}$; this requirement is [15]

$$\frac{\partial \psi^{\alpha}}{\partial x^{\beta}} = 0\,, \qquad \psi^{\alpha}_{\beta} = \gamma^{\alpha}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\gamma\,. \tag{6.6}$$

Taking (6.6) into account, the Ricci tensor takes the form

$$R_{\alpha\beta} = \frac{1}{2}\,\Box\gamma_{\alpha\beta}\,, \tag{6.7}$$

where

$$\Box \equiv g^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,, \qquad \Delta = \frac{\partial^2}{\partial x^{1\,2}} + \frac{\partial^2}{\partial x^{2\,2}} + \frac{\partial^2}{\partial x^{3\,2}}\,. \tag{6.8}$$

Here $\Box$ is the d'Alembert operator, Δ is the Laplace operator. The calibrating requirements (6.6) are true in any metric $\gamma_{\alpha\beta}$ only if the quantities ξ^{α} are solutions of the equation

$$\Box\xi^{\alpha} = 0\,. \tag{6.9}$$

In [15] the requirement

$$\Box\gamma_{\alpha\beta} = 0 \tag{6.10}$$

was imposed on the quantities $\gamma_{\alpha\beta}$, which is interpreted as the *equation of weak gravitational waves in emptiness* — this formula (6.10) is a standard wave equation that describes a wave of the tensor field $\gamma_{\alpha\beta}$, traveling at the velocity c in emptiness.

One usually considers the equation (6.10) as the basis for the claim that the General Theory of Relativity predicts gravitational waves, which travel at the speed of light.

If we have a weak plane gravitational wave, so the field has changes along a single spatial direction (the x^1 axis, for instance), the formula (6.10) takes the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^{1\,2}}\right)\gamma_{\alpha\beta} = 0\,, \tag{6.11}$$

and solutions of it can be any function of $ct \pm x^1$. After numerous transformations of the function $\gamma_{\alpha\beta}$ [14, 15] it obtains that in the field of a weak plane gravitational wave only the following components are non-zero: $\gamma_{22}=-\gamma_{33}\equiv a$, $\gamma_{23}\equiv b$. Thus, those weak plane gravitational waves that satisfy the Einstein equations in emptiness are transverse.

Thus if some additional requirements are imposed upon the Einstein equations in emptiness, the equations describe weak plane waves of the space deformation, the space metric of which is [15]

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1+a)(dx^2)^2 + 2b\,dx^2dx^3 - (1-a)(dx^3)^2, \tag{6.12}$$

where a and b are functions of $ct \pm x^1$. The field of gravitation, described by the metric (6.12), is of the sub-kind N by Petrov's classification, so it satisfies most of invariant criteria for gravitational waves.

The metric (6.12) has been written in a synchronous reference frame, so its space deforms, falls freely, and, at the same time, has no rotations. Hence, *under the given assumptions, weak plane gravitational waves are waves of "pure" deformation of the space*. This conclusion is the main reason why experimental physicists, and Weber in particular [16], expect that gravitational waves will cause a "pure" deformation effect in detectors.

Calculations for the interaction between a Weber solid-body detector and a weak plane gravitational wave field will be given in §7. Here we continue our argument for the wave nature of the Einstein equations in *strong gravitational fields* in the case where matter is arbitrarily distributed in the space. This research will be given in the terms of physically observable quantities for the reason that we will consider situations derived from different factors, generating gravitational wave fields, not only the space deformation.

The Einstein equations in the case where matter is arbitrarily distributed are [42]

$$\frac{^*\partial D}{\partial t}+D_{jl}D^{jl}+A_{jl}A^{lj}+\left({}^*\nabla_j-\frac{1}{c^2}F_j\right)F^j= -\frac{\kappa}{2}\left(\rho c^2+U\right)+\lambda c^2, \qquad (6.13)$$

$${}^*\nabla_j\left(h^{ij}D-D^{ij}-A^{ij}\right)+\frac{2}{c^2}F_jA^{ij}=\kappa J^i, \qquad (6.14)$$

$$\frac{^*\partial D_{ik}}{\partial t}-\left(D_{ij}+A_{ij}\right)\left(D^j_k+A^{\cdot j}_{k\cdot}\right)+DD_{ik}+3A_{ij}A^{\cdot j}_{k\cdot}+\frac{1}{2}\left({}^*\nabla_iF_k+{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k-c^2C_{ik}=\frac{\kappa}{2}\left(\rho c^2h_{ik}+2U_{ik}-Uh_{ik}\right)+\lambda c^2h_{ik}. \qquad (6.15)$$

Here ${}^*\nabla_j$ denotes the chr.inv.-derivative, while the quantities $\rho=\frac{T_{00}}{g_{00}}$, $J^i=\frac{cT^i_0}{\sqrt{g_{00}}}$, $U^{ik}=c^2T^{ik}$ (from which we have $U=h^{ik}U_{ik}$) are the chr.inv.-components of the energy-momentum tensor $T_{\alpha\beta}$ of matter: the physically observable density ρ, the physically observable impulse density vector J^i, and the physically observable stress-tensor U^{ik}.

Zelmanov had deduced [42] that the chr.inv.-spatial curvature tensor C_{iklj} is linked to a chr.inv.-tensor H_{iklj}, which is like Schouten's tensor [67], by the equation

$$H_{lkij}=C_{lkij}+\frac{1}{c^2}\left(2A_{kj}D_{jl}+A_{ij}A_{kl}+A_{jk}D_{il}+A_{kl}D_{ij}+A_{li}D_{jk}\right) \qquad (6.16)$$

and contracted tensors $H_{lk}=H_{lik\cdot}^{\cdots i}$ and $C_{lk}=C_{lik\cdot}^{\cdots i}$ are related as follows

$$H_{lk}=C_{lk}+\frac{1}{c^2}\left(A_{kj}D^j_l+A_{lj}D^j_k+A_{kl}D\right). \qquad (6.17)$$

Taking the definition $D_{ik}=\frac{1}{2}\frac{^*\partial h_{ik}}{\partial t}$ into account, and C_{lk} from (6.17), we reduce (6.15) to the form

$$\frac{1}{2}\frac{^*\partial^2h_{ik}}{\partial t^2}-D_{ij}D^j_k+D\left(D_{ik}-A_{ik}\right)+2A_{ij}A^{\cdot j}_{k\cdot}+\frac{1}{2}\left({}^*\nabla_iF_k-{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k-c^2H_{ik}=\kappa U_{ik}+\lambda c^2h_{ik}. \qquad (6.18)$$

The quantity H_{ik}, by definition, is

$$H_{ik}=H_{ijk\cdot}^{\cdots j}=\frac{^*\partial\Delta^j_{ij}}{\partial x^k}-\frac{^*\partial\Delta^j_{ik}}{\partial x^j}+\Delta^m_{ij}\Delta^j_{km}-\Delta^m_{ik}\Delta^j_{jm}, \qquad (6.19)$$

where $\Delta^m_{jm}=\frac{^*\partial\ln\sqrt{h}}{\partial x^j}$.

Taking into account (6.17), (6.19), and also Zelmanov's identities (4.6), (4.7) that link F_i and A_{ik}, we reduce (6.18) to the form

$$\begin{aligned}{}^*\Box h_{ik}&=2\frac{^*\partial^2\ln\sqrt{h}}{\partial x^i\partial x^k}-\frac{2}{c^2}\left({}^*\nabla_iF_k+\frac{^*\partial A_{ik}}{\partial t}\right)-\\&-\frac{4}{c^2}\left(A_{ij}A^{\cdot j}_{k\cdot}-D_{ij}D^j_k\right)-\frac{2D}{c^2}\left(D_{ik}+A_{ik}\right)+\\&+2\left(h^{pq}\Delta^m_{pq}\Delta_{ik,m}+\Delta^m_{ij}\Delta^j_{km}\right)-\\&-h^{pm}\frac{^*\partial}{\partial x^p}\left(\frac{^*\partial h_{im}}{\partial x^k}+\frac{^*\partial h_{km}}{\partial x^i}\right)+\\&+\kappa\left(\rho h_{ik}+\frac{2}{c^2}U_{ik}-\frac{U}{c^2}h_{ik}\right)+2\lambda h_{ik}\,,\end{aligned} \qquad (6.20)$$

where ${}^*\Box$ is the chr.inv.-d'Alembert operator, applied here to the chr.inv.-metric tensor h_{ik} (the observable metric tensor of the observer's three-dimensional space)*.

If we equate the right part of (6.20) in zero, the whole equation becomes a wave equation with respect to h_{ik}, namely

$${}^*\Box h_{ik}=\frac{1}{c^2}\frac{^*\partial^2h_{ik}}{\partial t^2}-h^{jm}\frac{^*\partial^2h_{ik}}{\partial x^j\partial x^m}\,. \qquad (6.21)$$

In this case the spatial components of the Einstein equations describe gravitational inertial waves of the spatial metric h_{ik}, which travel at the velocity $u=c\left(1-\frac{\mathrm{w}}{c^2}\right)$ which depends on the value of the gravitational potential w. This coincides with the results recently obtained by Rabounski [48]. If $\mathrm{w}=0$, the waves travel at the velocity of light. The greater is w the smaller is u. The wave's velocity u becomes zero in the extreme case where $\mathrm{w}=c^2$ which occurs under collapse, hence under collapse gravitational waves stop — they become *standing gravitational waves*.

It is evident from the mathematical viewpoint, that reducing the right side of (6.20) to zero is a very difficult task, because the whole equation is a system of 6 nonlinear equations of the 2nd order, in which numerous variables are linked by relationships (6.13) and (6.14). Systems such as this cannot be solved analytically in general, but we can obtain solutions for various specific metrics.

Because experimental physicists. in their search for gravitational waves, propound experimental statements for detecting weak wave fields of gravitation, we are going to study a linearized form of the equation (6.20).

*Components of the chr.inv.-metric tensor h_{ik} satisfy the requirements ${}^*\nabla_jh_{ik}={}^*\nabla_jh^k_i={}^*\nabla_jh^{ik}=0$. For this reason we can apply the chr.inv.-d'Alembert operator ${}^*\Box=\frac{1}{c^2}\frac{^*\partial^2}{\partial t^2}-h^{ik\,*}\nabla^*_i\nabla_k$ to it.

For (6.20) in emptiness, the linear approximation is*

$$ {}^*\Box h_{ik} = 2\,\frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} - \frac{2}{c^2}\left({}^*\nabla_i F_k + \frac{{}^*\partial A_{ik}}{\partial t}\right). \qquad (6.22) $$

As a matter of fact, equation (6.22) describes weak plane gravitational inertial waves without sources, if the wave field satisfies the obvious chr.inv.-condition

$$ \frac{{}^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} = \frac{1}{c^2}\left({}^*\nabla_i F_k + \frac{{}^*\partial A_{ik}}{\partial t}\right). \qquad (6.23) $$

In other words, the field of the observable metric tensor h_{ik} is a wave field if there are some relations between the inhomogeneity of the gravitational inertial force field, the non-stationary rotation of the space, and the volume transformations of the space element, taken in the field†. The condition (6.23) is true for the well-known metric of weak plane gravitational waves (6.12), because in the metric (6.12) we have $F_i = 0$, $A_{ik} = 0$, $\sqrt{h} = \sqrt{1-a^2-b^2} \approx 1$. Thus:

> Weak plane gravitational waves in emptiness are also weak plane gravitational inertial waves of the spatial observable metric h_{ik}.

As shown in [41], the metric (6.12) satisfies the Zelmanov chr.inv.-criterion for gravitational waves, where the wave functions are the Riemann-Christoffel tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj}. Hence weak plane gravitational inertial waves (waves of the space curvature) can exist in emptiness, because of the Einstein equations. We have shown above that such wave gravitational fields can also exist in spaces of the sub-kind N by Petrov's classification (such spaces are curved themselves, and matter contributes only an additional component to the initial curvature). Hence such fields satisfy most of the known invariant criteria for gravitational waves.

As we showed above, on page 46, that fields of gravitational radiations cannot exist in spaces of the 1st kind by Petrov's classification. In spaces of the 1st kind $Y^{ijk} = 0$. Therefore it would be logical to express the Einstein equations in the physically observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor, aiming to find relations between the ch.inv.-quantities X^{ij}, Y^{ijk}, Z^{iklj} and the physically observable components of the energy-momentum tensor $T_{\alpha\beta}$ of distributed matter (ρ, J^i, U_{ik}, see page 48).

In chr.inv.-components the Einstein equations become

$$ \left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = \kappa\left(\rho c^2 + U\right) - 2\lambda c^2, \\ & Y_{\cdot\cdot m}^{im\cdot} = \kappa J^i, \\ & X_{ik} - X h_{ik} + Z_{\cdot imk}^{m\cdot\cdot\cdot} = \\ & = \frac{\kappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,, \end{aligned}\right\} \qquad (6.24) $$

if matter is distributed arbitrarily. Here $X = h^{ik} X_{ik}$ is the trace (spur) of the tensor X_{ik}.

From here we see that the physical observable components of the Riemann-Christoffel tensor have different physical origins:

1. Quantities X^{ij} (and as well Z^{iklj}) are linked to the mass density ρ and the stress-tensor U_{ik};
2. Quantities Y^{ijk} are linked to the impulse density J^i of matter.

As we showed above, on page 46, in all the widely known metrics which satisfy both the invariant criteria and the chr.inv.-criterion for gravitational waves, we have $Y^{ijk} \neq 0$, although X^{ij} (and as well Z^{iklj}) can be zero. This fact leads us to a very important conclusion:

> Gravitational waves and gravitational inertial waves are mainly waves of the field of the Y^{ijk} physically observable component of the Riemann-Christoffel curvature tensor‡.

But this conclusion does not mean that only waves of the field Y^{ijk} can be discovered. As we will see in §7, relative accelerations of test-particles are derived from wave fields of all three observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel tensor. Our conclusion means:

> If in a space, filled with a gravitational field, $Y^{ijk} = 0$ is true, the structure of the space itself prohibits the gravitational field from being a wave.

Contracting (6.26) and taking (6.24) into account, we obtain

$$ X = \frac{\kappa}{2}\left(U - \rho c^2\right) - 2\lambda c^2. \qquad (6.25) $$

In an empty space where there are no λ-fields, the trace of X^{ij} and the contracted quantity $Z_{mk\cdot\cdot}^{\cdot\cdot mk}$ are zero, as well as the contracted quantity $Y_{\cdot\cdot m}^{im\cdot}$. Thus the chr.inv.-Einstein equations (6.24) in emptiness take the form§

$$ \left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0\,, \qquad X = 0\,, \\ & Y_{\cdot\cdot m}^{im\cdot} = 0\,, \\ & X_{ik} + Z_{\cdot imk}^{m\cdot\cdot\cdot} = 0\,, \end{aligned}\right\} \qquad (6.26) $$

so, while the quantities X^{ik} and Z^{iklj} are connected to one another, the quantity Y^{ijk} (which, being non-zero, $Y^{ijk} \neq 0$, permits gravitational fields to be a wave) is the independent observable component of the Riemann-Christoffel tensor.

*In obtaining this formula, in the initial equation (6.20), we neglect products of the chr.inv.-quantities and of their derivatives.

†The integral of $\sqrt{h}\,dx^1 dx^2 dx^3$ is the volume of an element of the space. Here the differentials dx^i themselves and an interval, where values of the x^i change where we take the integral, do not depend on x^0 [42].

‡Quadrupole mass-detectors, in particular, solid-body detectors (the Weber pigs) can only register waves of the X^{ij} component, not waves of Y^{ijk} if its particles are at rest in the initial moment of time (see §7 and §8 for details). Thus, the Weber experimental statement is false at its base.

§As a matter of fact, equality to zero of inflected forms of a tensor does not imply that the tensor quantity itself is zero. Thus, equalities $X = 0$, $Y_{\cdot\cdot m}^{im\cdot} = 0$, $Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0$ do not imply that the quantities X^{ik}, Y^{ijk}, Z^{iklj} themselves are zero. Therefore the chr.inv.-Einstein equations in emptiness (6.24) permit gravitational waves if, of course, $Y^{ijk} \neq 0$.

7 Expressing Synge-Weber equation (the world-lines deviation equation) in the terms of physical observable quantities, and its exact solutions

In the previous paragraphs we focused our attention on general criteria, which differentiate gravitational wave fields from other gravitational fields in the General Theory of Relativity. As a result, we have found the main properties of gravitational wave fields.

We are now going to introduce a substantial criticism of the contemporary theoretical foundations of current attempts to detect gravitational waves by solid-body detectors of the resonance kind (the Weber pigs) and quadrupole mass detectors in general.

As we showed in the previous paragraphs, only gravitational fields located in spaces where the Riemann-Christoffel curvature tensor has a specific structure, permit the presence of gravitational waves. Therefore it would be reasonable to design experiments by which a physical detector could register wave changes of the four-dimensional (space-time) curvature* — the waves of the Riemann-Christoffel curvature tensor field.

Such a physical detector could be a system of two test-particles: their relative world-trajectories will necessarily undergo changes through the action of a wave of the space curvature. These systems are described by the world-lines deviation equation — the Synge equation of geodesic deviation (2.8) if these are two free particles, and the Synge-Weber equation (2.12) if the particles are connected by a force of non-gravitational nature.

We propose gravitational wave detectors of two possible kinds. The system of two free particles is known as a *detector built on free masses*. In practice such a detector consists of two freely suspended massive bodies, separated by a suitable distance. The system of two particles connected by a spring is known as a *quadrupole mass-detector*— this is a detector of the resonance kind, a typical instance of which is the Weber cylindrical pig.

To understand how a graviational wave would affect the different types of detectors we need to make specific calculations for their behaviour in gravitational wave fields. But before making the calculations, it is required to describe the behaviour of two test-particles in regular gravitational fields (of non-wave nature) in the terms of physically observable quantities (chronometric invariants). This analysis will show how different kinds of gravitational inertial waves cause relative deviation (both spatial and time displacements) of two test-particles.

We will solve this problem first for a system two free particles as described by the Synge equation (2.8) where the right side is zero. The problem for spring-connected particles, described by the Synge-Weber equation (2.12), will be solved in the same way except that there will be a non-gravitational force acting, so that the right side of the equation will be non-zero.

Relative accelerations of free test-particles $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ are derived from components of the Riemann-Christoffel world-tensor, contracted with components of the particles' four-dimensional velocity vector U^β and their relative deviation vector η^γ, namely — from the quantity $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma$. To determine what effect is introduced by each observable component of the Riemann-Christoffel tensor into the spatial and time relative displacements, described by the relative displacement world-vector η^α, we consider the geodesic deviation equation (2.8), keeping the term $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ without expressing it in terms of the Christoffel symbols and their derivatives.

As well as any general covariant equation, the geodesic deviation equation (2.8) can be projected onto the observer's time line and spatial section (his three-dimensional space) as given in [42, 43] or on page 40 herein. Denoting

$$M^\alpha \equiv \frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0\,, \tag{7.1}$$

let us find equations which are its projection on the time line

$$\frac{M_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha}}{\sqrt{g_{00}}}M^\alpha = \sqrt{g_{00}}\,M^0 - \frac{1}{c}v_iM^i = 0\,, \tag{7.2}$$

and its projection on the spatial section

$$M^i = 0\,. \tag{7.3}$$

To find the equations in expanded form we need first to find the chr.inv.-projections of them, consisting of the quantities η^α and U^α. Projections of the η^α onto the time line and spatial section are, respectively

$$\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}\,, \qquad n^i \equiv \eta^i, \tag{7.4}$$

other components of the η^α are expressed through its physically observable components φ and n_i as follows

$$\eta^0 = \frac{\varphi + \frac{1}{c}v_kn^k}{\sqrt{g_{00}}}\,, \qquad \eta_i = -\frac{\varphi}{c}v_i - n_i\,. \tag{7.5}$$

The time and spatial components of the particles' world-velocity vector U^α are derived from the chr.inv.-definitions given by the theory of chronometric invariants for the space-time interval ds and the observable chr.inv.-velocity vector v^i

$$ds = cd\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,, \quad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \quad \mathrm{v}^2 = h_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{7.6}$$

*It is important to note that the expected gravitational waves are waves of the *space-time* curvature, not merely of the spatial curvature of the three-dimensional space. Consequently, waves of the four-dimensional curvature must produce changes not only in the distance between test-particles in a detector, but also in the time flow for the particles.

so the required quantities U^0 and U^i are

$$U^0 = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dt}{d\tau}\,, \qquad U^i = \frac{\mathrm{v}^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{7.7}$$

A formula for the time function $dt/d\tau$ is obtained from*

$$g_{\alpha\beta}\,U^\alpha U^\beta = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1\,, \tag{7.8}$$

which can be reduced to the quadratic equation

$$\begin{gathered}\left(\frac{dt}{d\tau}\right)^2 - \frac{2\,v_i\mathrm{v}^i}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)}\frac{dt}{d\tau} + \\ + \frac{1}{\left(1-\frac{\mathrm{w}}{c^2}\right)^2}\left(\frac{1}{c^2}\,v_i v_k \mathrm{v}^i\mathrm{v}^k - 1\right) = 0\,,\end{gathered} \tag{7.9}$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i + 1}{1-\frac{\mathrm{w}}{c^2}}\,, \quad \left(\frac{dt}{d\tau}\right)_2 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i - 1}{1-\frac{\mathrm{w}}{c^2}}\,. \tag{7.10}$$

The first solution is related to a space where time flows from past into future (a regular observer's space), the second solution is related to a space where time flows from future into past with respect to a regular observer's time flow (the mirror Universe [70, 71]). Taking only the first root, U^0 takes the form

$$U^0 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i + 1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\left(1-\frac{\mathrm{w}}{c^2}\right)}\,. \tag{7.11}$$

Substituting formulae (7.5), (7.7), (7.11) into $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0$ (7.1), and expressing the components of the Riemann-Christoffel tensor $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ in terms of its physically observable components X^{ij}, Y^{ijk}, Z^{iklj}, we obtain a formula for the relative spatial oscillations of two free test-particles

$$\frac{\mathrm{D}^2\eta^i}{ds^2} = \frac{1}{c^2-\mathrm{v}^2}\left(Y_{mk\cdot}^{\cdot\cdot i}\,\mathrm{v}^k - X^i_m - \frac{1}{c^2}Z_{mk\cdot n}^{\cdot\cdot i\cdot}\,\mathrm{v}^k\mathrm{v}^n\right)\eta^m. \tag{7.12}$$

From this formula we see that:

> The relative spatial deviations of two free particles can be caused by all three observable components of the Riemann-Christoffel curvature tensor. Moreover, each of the components acts on the particles in a different way: (1) the field of X^{ik} acts the particles only if they are at rest with respect to the observer's space references, so the field of X^{ik} can move particles only if they are at rest at the initial moment of time; (2) the fields of Y^{ijk} and Z^{iklj} can displace the particles with respect of each to other only if they are in motion ($\mathrm{v}^i \neq 0$) — the effect of Z^{iklj} is perceptible if the particles move at speeds close to the velocity of light.

*That is the evident equality.

Thus, with measurement taken by any observer, the physically observable components of the Riemann-Christoffel curvature tensor are of 3 different kinds:

1. The component X^{ik} —of "electric kind", because it can displace even resting particles;
2. The component Y^{ijk} — of "magnetic kind", because it can displace only moving particles;
3. The component Z^{iklj} of "magnetic relativistic kind", because it causes an effect only in particles moving at relativistic speeds.

Besides the observable spatial component η^i of the relative deviation vector η^α there is also its observable time component φ, which indicates the difference between time flows measured by clocks located at each of the particles.

We then obtain the relative time deviation equation for two free test-particles

$$\begin{gathered}\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}\,v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = \\ = -\sqrt{g_{00}}R^{0\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma + \frac{1}{c}\,v_i R^{i\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma.\end{gathered} \tag{7.13}$$

Taking (7.10) into account and substituting the formulae for U^0, η^0, U^i, η^i into (7.11), then, expressing $R^{0\cdots}_{\cdot\beta\gamma\delta}$ in terms of physically observable quantities, we reduce formula (7.13) to its final form

$$\begin{gathered}\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}\,v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = \\ = \frac{1}{c^2-\mathrm{v}^2}\left[\frac{1}{c}X_{ik}\left(n^i - \frac{\varphi}{c}\mathrm{v}^k\right)\mathrm{v}^k + \frac{1}{c}Y_{imk}\mathrm{v}^i\mathrm{v}^k\eta^m\right].\end{gathered} \tag{7.14}$$

Looking at this formula we note one simple thing about the effect of gravitational waves on the system of two free particles:

> The time observable component of the relative deviation vector for two free particles undergoes oscillations due only to the X^{ik} and Y^{ijk} observable components of the Riemann-Christoffel curvature tensor, not its Z^{iklj} component. Moreover, the fields of both the components X^{ik} and Y^{ijk} act on the particles only if they are in motion with respect to the space references. If the particles are at rest with respect to each other and the observer ($\mathrm{v}^i = 0$), the fact that the space has a Riemannian curvature makes no difference to the time flow measured in the particles.

It should be added that if the particles are in motion with respect to the space references and the observer, the effect of X^{ik} is both linearly and quadratically dependent on the speed, whilst the effect of Y^{ijk} is only quadratically dependent on the speed.

Thus, there is no complete analogy between the physically observable components of the Riemann-Christoffel curvature tensor and Maxwell's electromagnetic field tensor.

The components X^{ik} can be interpreted "electric" only in *relative spatial displacements* of two particles. In relative time deviations between the particles (the difference between the time flow measured in the them both) both X^{ik} and Y^{ijk} act on them depending on the particles' velocity with respect to the space references and the observer, so in this case both X^{ik} and Y^{ijk} are of the "magnetic" kind. Therefore the terms "electric" and "magnetic" are only applicable relative to observable components of the Riemann-Chrstoffel curvature tensor. This terminology is strictly true in that case where the particles have only relative spatial deviations, while the time flow is the same on the both world lines.

A formula for the observable relative time deviation $\varphi = \frac{\eta_0}{\sqrt{g_{00}}}$ between two free particles can be obtained from the requirement that the scalar product $U_\alpha \eta^\alpha$ remains unchanged along geodesic trajectories, so $U_\alpha \eta^\alpha = \text{const}$ must be true along trajectories of free particles. For this reason, if the vectors U^α and η^α are orthogonal, they are orthogonal on the entire world-trajectory [17]. Formulating the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$ in terms of physically observable quantities, we introduce some corrections to the results obtained in [17].

In terms of physically observable quantities the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$, because it is actually the same as $U_0 \eta^0 + U_k \eta^k = \text{const}$, reduces to

$$\varphi - \frac{1}{c}\, n_i \mathrm{v}^i = \text{const} \times \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}\,. \tag{7.15}$$

From this we see that the vectors U^α and η^α are orthogonal only if $\mathrm{v}^2 = c^2$, i. e. U^α is isotropic: $g_{\alpha\beta} U^\alpha U^\beta = 0$. So if U^α and η^α are orthogonal, we have the deviation equation for *two isotropic geodesics* — world-lines of light-like particles moving at the velocity of light. We defer this case for the moment and consider only the case of *two neighbour non-isotropic geodesics*. In the particular case when two particles are moving on neighbouring geodesics, and are at rest with respect to the observer and his references (only the time flow is different in the particles), formula (7.15) leads to $\varphi = \text{const}$.

This formula verifies our previous conclusion that the particles have a time deviation only if they are in motion. The greater their velocity with respect to the space reference and the observer, the greater the deviation between the time flow on both world-lines. Thus measurement of time deviations between two particles in gravitational waves and gravitational inertial waves would be easier in experiments where the particles move at high speeds. In practice such an experimental statement could be realized using light-like particles (in particular, photons). A time deviation of two photons in gravitational wave fields can manifest as changes in the frequencies of two parallel light rays (laser beams, for instance), while a spatial deviation of the photons can manifest as changes in the phases of the light rays. Calculations of these effects will be presented in future article. Here we focus our attention on particles of non-zero rest-mass $m_0 \neq 0$ (so-called mass-bearing particles), which are at rest with respect to the space references and the observer or, alternatively, moving at sub-light speeds.

In equations (7.10) and (7.12), we kept the second absolute derivative $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ of the relative deviation vector η^α as a whole, because we were concerned only with the effects introduced by the Riemannian curvatureto the relative spatial acceleration $\frac{\mathrm{D}^2\eta^i}{ds^2}$ and relative time acceleration $\frac{\mathrm{D}^2\eta^0}{ds^2}$ of two free test-particles.

But if we wish to obtain solutions to the world-lines deviation equation, we need to express the quantity $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and also $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ in terms of the Christoffel symbols and their derivatives.

We are now going to obtain solutions to the deviation equation for geodesic lines (the Synge equation).

Taking the definition

$$\frac{\mathrm{D}\eta^\alpha}{ds} = \frac{d\eta^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\eta^\mu U^\nu \tag{7.16}$$

into account, we obtain

$$\begin{gathered}\frac{\mathrm{D}^2\eta^\alpha}{ds^2} = \frac{d^2\eta^\alpha}{ds^2} + \frac{d\Gamma^\alpha_{\mu\nu}}{ds}\eta^\mu U^\nu + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \\ + \Gamma^\alpha_{\mu\nu}\eta^\mu\frac{dU^\nu}{ds} + \Gamma^\alpha_{\rho\sigma}\Gamma^\rho_{\mu\nu}\eta^\mu U^\nu U^\sigma = 0\,.\end{gathered} \tag{7.17}$$

We write $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ as

$$R^{\alpha\cdots}_{\cdot\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\alpha_{\gamma\sigma} - \Gamma^\sigma_{\beta\gamma}\Gamma^\alpha_{\sigma\delta}\,, \tag{7.18}$$

express $\frac{dU^\alpha}{ds}$ via the geodesic equations

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\mu\nu}U^\mu U^\nu, \tag{7.19}$$

and use the definition

$$\frac{d\Gamma^\alpha_{\mu\nu}}{ds} = \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\sigma}U^\sigma. \tag{7.20}$$

Using the auxiliary formulae we obtain from (7.17) the Synge equation (the geodesic lines deviation equation) in its final form

$$\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma = 0\,. \tag{7.21}$$

This is a differential equation of the 2nd order with respect to the quantity η^α: the equation is a system of 4 differential equations with respect to the quantities η^0 and η^i $(i = 1, 2, 3)$. The variable coefficients $\Gamma^\alpha_{\mu\nu}$ and their derivatives must be taken for that gravitational field, whose waves act on two free test-particles in our experiment. The world-quantities U^ν ($\nu = 0, 1, 2, 3$) can be found as solutions to the geodesic equations

$$\frac{dU^\nu}{ds} + \Gamma^\nu_{\mu\rho} U^\mu U^\rho = 0 \tag{7.22}$$

only if the particles move with respect to the space references and the observer. If the particles are at rest with respect to the observer and his references, the components of their world-velocity vector U^ν are

$$U^0 = \frac{1}{\sqrt{g_{00}}}\,, \qquad U^i = 0\,, \tag{7.23}$$

and, according to (7.13–7.15), their relative time deviation is zero, $\varphi = 0$ (the time flow measured on both geodesic lines is the same).

Current detectors used in the search for gravitational wave radiations are of such a construction that the particles therein, which detect the waves, are almost at rest with respect of each other and the observer. Experimental physicists, following Joseph Weber and his methods, think that gravitational waves can cause the rest-particles to undergo a relative displacement. With the current theory of the gravitational wave experiment, the experimental physicists limit themselves to the expected amplitude and energy of waves arriving from a proposed source of a gravitational wave field.

However, to set up the gravitational wave experiment correctly, we need to eliminate all extraneous assumptions and traditions. We merely need to obtain exact solutions to the world-lines deviation equation, applied to detectors of that kind which this experiment uses.

Detectors described by the geodesic lines deviation equation (the Synge equation), which we consider in this section, are known as "antennae built on free masses". We shall consider such detectors first.

The detectors consist of two freely suspended masses which are at rest with respect of each other and the observer, and separated by an appreciable distance. These could be two mirrors, located in a near-to-Earth orbit, for instance. Each of the mirrors is fitted with a laser range-finder, so we can measure the distance between them with high precision.

In order to interpret the possible results of such an experiment, we need to solve the Synge equation (7.21), expressing its solutions in the terms of physically observable quantities (chronometric invariants). Following "tradition", we solve the Synge equation for particles which are at rest with respect to each other and the observer's space references. So we consider that case where the particles' observable velocities are zero ($\mathrm{v}^i = 0$).

At first, because we are going to obtain solutions to the Synge equation in chr.inv.-from, we need to know the physically observable characteristics of the observer's reference space through which we express the solutions. We find the chr.inv.-characteristics from the geodesic equations taken in the main chr.inv.-from [42]

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \tag{7.24}$$

$$\begin{aligned} \frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + {}\\ + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \end{aligned} \tag{7.25}$$

for each of the particles (because both particles are at rest with respect to one another, their geodesic equations are the same). Here m is the particle's relativistic mass, which, because in the case we are considering $\mathrm{v}^i = 0$, reduces to the rest-mass $m = m_0$. Then the geodesic equations take the very simple form

$$\frac{dm}{d\tau} = 0\,, \tag{7.26}$$

$$mF^i = 0\,, \tag{7.27}$$

so in this case the chr.inv.-vector of gravitational inertial force is $F^i = 0$: the particles are in free fall. In this case we can transform coordinates so that $g_{00} = 0$ and $\frac{\partial g_{0i}}{\partial t} = 0$ [42]. This implies that the Synge initial equation (7.19) can be solved correctly only for gravitational fields where the potential is weak $\mathrm{w} = 0$ (i. e. $g_{00} = 1$) and where the space rotation is stationary $\frac{\partial A_{ik}}{\partial t} = 0$. It should be noted that the metric of weak plane gravitational waves, the only metric used in the theory of gravitational wave experiments, satisfies these requirements.

Because $\varphi = \frac{1}{c}\,n_i\mathrm{v}^i$ (7.15), in the case we are considering the time observable component φ of the relative deviation vector η^α is zero $\varphi = 0$. For this reason we consider only the observable spatial component of the Synge equation (7.21).

In the accompanying reference frame (where the observer accompanies his references), according to the theory of chronometric invariants [42, 43], in the absence of gravitational fields $\mathrm{w} = 0$ and also gravitational inertial forces $F_i = 0$, we have: $\frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau}$, $U^0 = \frac{1}{\sqrt{g_{00}}} = 1$, $U^i = \frac{1}{c}\,\mathrm{v}^i$, $\eta^0 = -g_{0i}\eta^i$, $\Gamma^i_{00} = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^i = 0$, $\Gamma^i_{0k} = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\Big(D^i_k + A^{\cdot i}_{k\cdot} + \frac{1}{c^2}\,v_k F^k\Big) = \frac{1}{c}\big(D^i_k + A^{\cdot i}_{k\cdot}\big)$. Employing now the formulae for the Synge equation (7.21) under $\mathrm{v}^i = 0$, we obtain the *Synge equation in chr.inv.-form**

$$\frac{d^2\eta^i}{d\tau^2} + 2\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\frac{d\eta^k}{d\tau} = 0\,. \tag{7.28}$$

The quantity $\frac{d}{d\tau} = \frac{{}^*\partial}{\partial t} + \mathrm{v}^i\,\frac{{}^*\partial}{\partial x^i}$ [42, 43] here is

$$\frac{d}{d\tau} = \frac{\partial}{\partial t}\,, \tag{7.29}$$

*As we mentioned, if the particles are at rest $\mathrm{v}^i = 0$, the chr.inv.-time component of the Synge equation becomes zero.

so the chr.inv.Synge -equation (7.28) takes its final form

$$\frac{\partial^2 \eta^i}{\partial t^2} + 2\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{\partial \eta^k}{\partial t} = 0 \,. \qquad (7.30)$$

We find the exact solution to the Synge chr.inv.-equation (7.30) in the field of weak plane gravitational waves*. In the case we are considering ($\mathrm{v}^i = 0$) we have

$$F_i = 0 \,, \qquad A_{ik} = 0 \,,$$
$$D_{22} = -D_{33} = \frac{1}{2}\frac{\partial a}{\partial t} \,, \quad D_{23} = \frac{1}{2}\frac{\partial b}{\partial t} \,. \qquad (7.31)$$

Substituting the requirements into the initial equation (7.30) we obtain a system of three equations

$$\frac{\partial^2 \eta^1}{\partial t^2} = 0 \,, \qquad (7.32)$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + \frac{\partial a}{\partial t}\frac{\partial \eta^2}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^3}{\partial t} = 0 \,, \qquad (7.33)$$

$$\frac{\partial^2 \eta^3}{\partial t^2} - \frac{\partial a}{\partial t}\frac{\partial \eta^3}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^2}{\partial t} = 0 \,. \qquad (7.34)$$

The solution of (7.32) is

$$\eta^1 = \eta^1_{(0)} + \dot{\eta}^1_{(0)} t \,, \qquad (7.35)$$

where $\eta^1_{(0)}$ is the particle's initial deviation, $\dot{\eta}^1_{(0)}$ is its initial velocity.

This system can be easy solved in two particular cases of a linear polarized wave: (1) $b = 0$, and (2) $a = 0$.

In the first case ($b = 0$) we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1\, e^{-a}, \qquad \frac{\partial \eta^3}{\partial t} = C_2\, e^{+a}, \qquad (7.36)$$

where C_1 and C_2 are integration constants. Because values of a are weak, we can decompose e^{-a} into series. Then, assuming higher order terms infinitesimal, we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1 \left(1 - a\right), \qquad \frac{\partial \eta^3}{\partial t} = C_2 \left(1 + a\right). \qquad (7.37)$$

Assuming also that a falling gravitational wave is monochrome, bearing a constant amplitude A and a frequency ω,

$$a = A \sin\frac{\omega}{c}\left(ct \pm x^1\right), \qquad (7.38)$$

we integrate the system (7.37). As a result we obtain

$$\eta^2 = C_1\left[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + D_1 \,, \qquad (7.39)$$

$$\eta^3 = C_2\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + D_2 \,, \qquad (7.40)$$

where D_1 and D_2 are integration constants. Assuming $x^1 = 0$ at the initial moment of time $t = 0$, we easily express the integration constants C_1, C_2, D_1, D_2 through the initial conditions. Finally, we obtain solutions

$$\eta^2 = \dot{\eta}^2_{(0)}\left[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^2_{(0)} - \frac{A}{\omega}\,\dot{\eta}^2_{(0)} \,, \quad (7.41)$$

$$\eta^3 = \dot{\eta}^3_{(0)}\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^3_{(0)} - \frac{A}{\omega}\,\dot{\eta}^3_{(0)} \,, \quad (7.42)$$

where $\eta^2_{(0)}$, $\eta^3_{(0)}$ and $\dot{\eta}^2_{(0)}$, $\dot{\eta}^3_{(0)}$ are the initial numerical values of the relative deviation η and relative velocity $\dot{\eta}$ of the particles along the x^2 and x^3 axes, respectively.

We have now obtained the exact solutions to the Synge equation (the geodesic lines deviation equation). From the solutions we see,

> If at the initial moment of time $t = 0$, two free particles are at rest with respect to each other and the observer $\dot{\eta}^2_{(0)} = \dot{\eta}^3_{(0)} = 0$, weak plane gravitational waves of the deformation kind (waves of the Riemannian curvature) cannot force the particles to go into relative motion. If at the initial moment of time the particles are in motion, the waves augment the particles' initial motion, accelerating them.

Thus our purely mathematical analysis of detectors built on free masses leads to the final conclusion:

> Weak plane gravitational waves of the deformation kind (the Riemannian curvature's waves) cannot be detected by any antenna composed on free masses, if the masses are at rest with respect to each other and the observer.

8 Criticism of Weber's conclusions on the possibility of detecting gravitational waves by solid-body detectors of the resonance kind

Historically, the first gravitational wave detector was the quadrupole mass-detector built in 1964 by Prof. Joseph Weber with his students David Zipoy and Robert Forward at Maryland University [70]. It was an aluminium cylindrical pig weighing 1.5 tons, suspended by a steel "thread" in a vacuum camera. At the point of connection between the pig and the thread, the pig was covered by a piezoelectric quartz film linked to a highly sensitive voltmeter. Weber expected that *a falling gravitational wave should make relative displacements of the butt-ends of the cylindrical pig — extension or compression of the pig*. In other words, they expected that falling gravitational waves will deform the pig, necessarily causing a piezoelectric effect in it. Modified by Sinsky [71], the first detector gave a possibility of registering a 10^{-16}cm relative displacement of its butt-ends.

Later, Weber built a system of two pigs. That system worked through the principle of coincident frequencies of

*Where the metric (5.7) is $ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + 2bdx^2dx^3 - (1+a)(dx^3)^2$.

the signals registered in both pigs. The pigs had a relaxation time about 30 sec, were tuned for the frequency 10^4 rps, and were separated by 2 km. In 1967 Weber and his team registered coincident signals (to a precision within 0.2 sec) which appeared about once a month [1]. The registered relative displacements of the butt-ends in the pigs were $\sim 3\times10^{-10}$ cm. Weber supposed that the origin of the observed signals were gravitational wave radiations.

Weber subsequently even used 6 pigs, one of which was located at Argonne National Laboratory (Illinois), the other 5 pigs located in his laboratory at Maryland University. The distance between the laboratories wss about 1000 km. The detectors were tuned for 1660 Hz — the frequency of supposed gravitational radiations excited from collapsing supernovae. During several months of observations, numerous coincident signals were registered [72]. A second cycle of the observations gave the same positive result [73]. Weber interpreted the registered signals as proof that strong gravitational radiations exist in the Galaxy. A peculiarity of those experiments was that the pigs located both in Illinois and Maryland were isolated as much as possible from external electromagnetic and seismic influences.

After Weber's pioneering experiments, experimental physicists built many similar detectors, much more sensitive than those of Weber. However, in contrast to those of Weber, not one of them registered any signals.

Therefore, using the world-lines deviation theory developed here in the terms of physically observable quantities, we are going to:

(1) investigate what in principle can be registered by a solid-body detector (a Weber pig) and

(2) compare our conclusion with that explanation given by Weber himself for his observed signals.

From the theoretical viewpoint we can conceive of a solid-body cylindrical detector as consisting of two test-particles, connected by a spring [16]. It is supposed that the system falls freely. It is also supposed that at the initial moment of time, when we start our measurements, the particles are at rest with respect to us (the observers) and each other. This is the standard problem statement, not only of Weber [16] or ourselves, but also of any other theoretical physicist.

The behaviour of two neighbouring particles in their motion along their neighbouring world-lines is described by the *world-lines deviation equation*. If the particles are not free, but connected by a non-gravitational force Φ^α (a spring, for instance), the Synge-Weber equation (2.12) applies, namely $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = \frac{1}{m_0c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}dv$. This is an inhomogeneous differential equation of the 2nd order with respect to the relative deviation vector η^α of the particles. In order to solve the world-lines deviation equation we need to write $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and $\frac{\mathrm{D}\Phi^\alpha}{dv}dv$ in expanded form. Because both terms contain the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, it would be reasonable to express the components of the Riemann-Christoffel tensor $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of the $\Gamma^\alpha_{\mu\nu}$ and their derivatives: in collecting similar terms some of them will cancel out (the same situation arose when we made the same calculations for the geodesic lines deviation equation).

Using formulae (7.17) and (7.18), and the quantity $\frac{dU^\alpha}{ds}$ from the world-line equation of a particle moved by a non-gravitational force Φ^α (2.11), we obtain

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\rho\sigma}U^\rho U^\sigma + \frac{\Phi^\alpha}{m_0c^2}\,. \tag{8.1}$$

Expanding the formula for $\frac{\mathrm{D}\Phi^\alpha}{dv}dv$

$$\frac{\mathrm{D}\Phi^\alpha}{dv}dv = \frac{\partial\Phi^\alpha}{\partial v}dv + \Gamma^\alpha_{\mu\nu}\Phi^\mu\frac{\partial x^\nu}{\partial v}dv = \frac{\partial\Phi^\alpha}{\partial x^\sigma}\eta^\sigma + \Gamma^\alpha_{\mu\nu}\Phi^\mu\eta^\nu \tag{8.2}$$

and substituting this into the world-lines deviation equation in its initial form (2.12), takeing into account that (8.1) and (8.2), we obtain

$$\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma = \frac{1}{m_0c^2}\frac{\partial\Phi^\alpha}{\partial x^\sigma}\eta^\sigma. \tag{8.3}$$

This is the final form of the world-lines deviation equation for two test-particles connected by a spring. The quantities η^α and U^α are connected by (2.13): $\frac{\partial}{\partial s}(U_\alpha\eta^\alpha) = \frac{1}{m_0c^2}\Phi_\alpha\eta^\alpha$.

In a gravitational wave detector like Weber's, the cylindrical pig is isolated as much as possible from external influences of thermal, electromagnetic, seismic and another origins. To minimise external influences, experimental physicists place the detectors in mines located deep inside mountains or otherwise deep beneath the terrestrial surface, and cool the pigs to 2 K. Therefore particles of matter in the butt-ends and the pig in general, can be assumed at rest with respect to one another and to the observer.

Following Weber, experimental physicists expect that a falling gravitational wave will deform the pig, displacing its butt-ends with respect to each other. Relative displacements of the butt-ends of a pig are supposed to result in a piezo-electric effect which can be registered by a piezo-detector. In other words, experimental physicists expect that oscillations of the acting gravitational wave field give rise to a force in the world-lines deviation equation (the Synge-Weber equation), thereby displaceing the test-particles which were at rest at the initial moment of time. Oscillations of the acting gravitational wave field force the butt-ends of the pig to oscillate. As soon as the frequency of the pig's oscillations coincides with the falling wave's frequency, the amplitude of the pig's oscillations will increase significantly because of resonance, so the amplitude becomes measurable. Therefore the Weber detectors are said to be of the *resonance kind*.

Before ratifying the aforesaid conclusions it would be reasonable to study the world-lines deviation equation for two interacting test-particles that model a Weber pig, because this equation is the theoretical basis of all experimental attempts to register gravitational waves made by Weber and his followers during more than 30 years.

We will study this equation, proceeding from its form (8.3), because formula (8.3) gives a possibility of obtaining exact solutions to the relative deviation vector η^α; not the initial equation (2.12). Following analysis of the solutions we will come to a conclusion as to what effect a falling gravitational wave has on the detector*.

When we need to give a theoretical interpretation of experimental results, it is very important to analyse the results in the terms of physically observable quantities because such quantities can be registered in practice. For this reason we will study the behaviour of the Weber model (the system of two particles, connected by a non-gravitational force) in the terms of physically observable quantities (chronometric invariants) as we did in §7 when we solved a similar problem for the system of two free particles.

In detail, our task here is to consider commonly the world-lines equation (2.11) and the world-lines deviation equation (8.3), both written in chr.inv.-form. Note that the relationship (2.13), that is $\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha$, gives the exact solution for the quantity φ. The φ is the chr.inv.-time component of the relative deviation world-vector η^α with respect to which the world-lines deviation equation (8.3) is written. For this reason there is no need here to solve the chr.inv.-time projection of the world-lines deviation equation (8.3). We solve instead the relationship (2.13).

The world-lines equation (2.11) in chr.inv.-form is

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k = \frac{\sigma}{c}\,, \tag{8.4}$$

$$\frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + \Delta^i_{kn}\mathrm{v}^k\mathrm{v}^n = f^i, \tag{8.5}$$

where $\sigma \equiv \frac{\Phi_0}{\sqrt{g_{00}}}$ and $f^i \equiv \Phi^i$ are chr.inv.-components of the prevailing non-gravitational force Φ^α. In the case of the Weber model where the particles are at rest with respect to the observer ($\mathrm{v}^i = 0$), the chr.inv.-equations (8.4) and (8.5) become

$$\sigma = 0\,, \tag{8.6}$$

$$m_0 F^i = -f^i. \tag{8.7}$$

The condition $\sigma = 0$ comes from the fact that, when a particle is at rest its relativistic mass becomes the rest-mass $m = m_0$. Thus resting particles are under the action of only the spatial observable components f^i of the non-gravitational force Φ^α, so that the f^i are of the same value as the acting gravitational inertial force F^i, but acts in the opposite direction. Looking at definition (4.1), given by the theory of physical observable quantities for the gravitational inertial force F^i, we see that in this case the non-gravitational force f^i acts on a resting particle only if at least one of the following factor holds:

1. Inhomogeneity of the gravitational potential $\frac{\partial \mathrm{w}}{\partial x^i} \neq 0$;
2. Non-stationarity of the vector of the space rotation linear velocity $\frac{\partial v_i}{\partial t} \neq 0$.

If neither factor holds, $F^i = 0$ and hence $f^i = 0$, in which case both interacting particles, which are at rest with respect to each other and the observer, behaviour like free particles: their connecting force Φ^α does not manifest. Looking at the well-known metric (5.7) that describes weak plane gravitational waves, we see there that $F^i = 0$, $A_{ik} = 0$ and hence $v_i = 0$. Therefore:

> Weak plane gravitational waves described by the metric (5.7) **cannot be registered** by a solid-body detector of the resonance kind (a Weber detector).

Writing the relationship (2.13) in chr.inv.-form, we obtain

$$\frac{d}{d\tau}\left(\frac{\varphi - \frac{1}{c}\,n_i \mathrm{v}^i}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\right) = \frac{\sigma\varphi - f_i n^i}{mc}\,, \tag{8.8}$$

where again, $\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}$ and $n^i \equiv \eta^i$ are chr.inv.-components of the relative deviation world-vector η^α.

From this we see that the angle between the vectors U_α and η^α is a variable depending on many factors, including the velocity v^i of the particles. At speeds close to that of light c, the angle increases. At $\mathrm{v} = c$ formula (8.8) becomes senseless: the denominator on the left side becomes zero.

If both particles are at rest, formula (8.8) becomes

$$\frac{d\varphi}{d\tau} = -\frac{f_i n^i}{m_0 c} = \frac{F_i n^i}{c}\,, \tag{8.9}$$

so that in the case of interacting rest-particles, in contrast to free ones, there is the time observable component φ of the relative deviation vector η^α. This implies that there are not only relative spatial displacements of the particles, but also a deviation between measurements of time made by the clocks of both particles. The "time deviation" φ can be found by integrating (8.9). We obtain

$$\varphi = \frac{1}{c}\int F_i n^i + \mathrm{const}\,, \tag{8.10}$$

so the value of the "time deviation" φ increases with time. It Note that $\frac{\varphi}{\tau} = 0$ only if the vector F^i (and hence, in this case, also f^i) is orthogonal to the vector n^i, so that $F_i n^i = -\frac{1}{m_0} f_i n^i = 0$.

*It is evident that equation (8.3) can be solved also for other forcing fields, which can be of a non-wave origin.

The integral (8.10) is the solution to the chr.inv.-time component of the world-lines deviation equation (8.3). This solution for φ itself, being a chronometric invariant, is a physically observable quantity.

We are now going to obtain solutions to the remaining three chr.inv.-equations with respect to η^α:

$$\frac{1}{c^2}\frac{d^2\eta^i}{d\tau^2}+2\Gamma^i_{00}\frac{1}{c}\frac{d\eta^0}{d\tau}U^0+2\Gamma^i_{k0}\frac{1}{c}\frac{d\eta^k}{d\tau}U^0+ \\ +\frac{\partial\Gamma^i_{00}}{\partial x^0}U^0U^0\eta^0+\frac{\partial\Gamma^i_{00}}{\partial x^k}U^0U^0\eta^k=\frac{1}{m_0c^2}\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma, \tag{8.11}$$

— the chr.inv.-spatial components of the world-lines deviation equation (8.3), in the case of two rest-particles $ds=cd\tau$.

In the left side of (8.11) we substitute the formulae for the quantities Γ^i_{00}, Γ^i_{00}, U^0, η^0, given on page 53, and also derivatives of the quantities. Then we transform the right part of (8.11) as follows

$$\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma=\frac{\partial f^i}{\partial x^0}\eta^0+\frac{\partial f^i}{\partial x^k}\eta^k=\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial f^i}{\partial x^k}n^k, \tag{8.12}$$

where we use the definitions of the chr.inv.-derivative operators (see page 40): $\frac{^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}v_i\frac{^*\partial}{\partial t}$.

The initial equation (8.11) becomes

$$\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}-\frac{2}{c}\frac{d\varphi}{d\tau}F^i+\frac{2}{c^2}F_kn^kF^i- \\ -\frac{\varphi}{c}\frac{^*\partial F^i}{\partial t}-\frac{^*\partial F^i}{\partial x^k}n^k=\frac{1}{m_0}\left(\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial F^i}{\partial x^k}n^k\right). \tag{8.13}$$

Owing to the particular conditions (8.7) and (8.9), derived from the requirement that the particles are at rest ($\mathrm{v}^i=0$), formula (8.13) becomes much more simple

$$\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}=0\,, \tag{8.14}$$

which is the final form for the chr.inv.-spatial deviation equation for two rest-particles, connected by a non-gravitational force.

Equation (8.14) is like the chr.inv.-spatial deviation equation for two free rest-particles (7.30) — the chr.inv.-spatial part of the Synge general covariant equation. The difference is that (8.14) contains derivatives $\frac{d}{d\tau}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$, while (7.30) contains $\frac{\partial}{\partial t}$. This difference is derived from the fact that (7.30) is applicable to gravitational fields where $F_i=0$, the potential w is neglected and hence $\frac{\partial v_i}{\partial t}=0$, while (8.14) describes the relative deviation of two particles located in gravitational fields where $F_i\neq 0$.

The required condition $F_i\neq 0$ implies:

1. In this region the gravitational potential is $\mathrm{w}\neq 0$, hence, because the interval of physical observable time is $d\tau=\left(1-\frac{\mathrm{w}}{c^2}\right)dt-\frac{1}{c^2}v_idx^i$, the time flow differences at different points inside the region. In particular, if $v_i=0$, synchronization of clocks located at different points cannot be conserved. In the more general case where $v_i\neq 0$, clocks located at different points cannot be synchronized [42, 43];*
2. If the gravitational inertial force field F^i is vortical, the space rotation is non-stationary $\frac{\partial v_i}{\partial t}\neq 0$.

Let us get back to the chr.inv.-spatial equation for two particles connected by a non-gravitational force (8.14). There are quantities D_{ik} and A_{ik}, so relative accelerations of the particles can be derived from both the space deformations and rotation. In this problem statement, w implies that the gravitational potential of a distant source of gravitational radiations. So in a gravitational wave experiment we should specify the acting gravitational field as weak $\frac{\mathrm{w}}{c^2}\approx 0$, hence in the experiment the chr.inv.-gravitational inertial force vector F_i (4.1) becomes $F_i\approx\frac{\partial\mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}$. There are as well $\frac{d}{d\tau}=\frac{\partial}{\partial t}$.

We solve equation (8.14) in two cases, aiming to find what kind of gravitational field fluctuations were registered by Weber and his colleagues.

First case: $A_{ik}=0$, $D_{ik}\neq 0$.

In this case equation (8.14), with $\frac{\mathrm{w}}{c^2}\approx 0$, is the same as the chr.inv.-world-lines deviation equation for two free particles (7.30). As it was shown in §7, with solutions of equation (7.30) considered, a gravitational wave can affect the system of two free particles only if the particles are in motion at the initial moment of time. In that case a gravitational wave can only augment the initial motion of the particles. If they are at rest gravitational waves can have no effect on the particles.

*To realize the condition $\mathrm{w}\neq 0$ it is not necessary to have a wave gravitational field. In particular, $\mathrm{w}\neq 0$ is true even in stationary gravitational fields derived from island masses (like Schwarzschild's metric). Moreover, the phenomenon of different time flow in the Earth gravitational field is well-known from experimental tests of the General Theory of Relativity: a standard clock, located on the terrestrial surface, shows time which is $\sim 10^{-9}$sec different from time measured by the same standard clock, located in a balloon a few kilometers above the terrestrial surface (the difference increases with the duration of the experiment). But such corrections of time are not linked to the presence of gravitational waves.

There time corrections can also be registered, the origin of which are wave changes of the gravitational potential w. They can be interpreted as waves of the gravitational inertial force field F^i. In this case corrections to standard clocks, located at different points, should bear a relation to wave changes of w.

The presence of the space rotation $v_i\neq 0$ changes the time flow as well. Experiments, where a standard clock was moved by a jet plane around the world [49, 50, 51, 52], showed differential time flow with respect to the same standard clock located at rest at the air force base. Such difference of measured time, called the *desynchronization correction*, depends on the flight direction — with or opposite to Earth's rotation. Although such corrections are derived from the Earth rotation (the reference space rotation), in the "background" of such corrections there could also be registered additional tiny corrections derived from the rapid stationary rotation field of a massive space body, located far from the Earth.

When $A_{ik} = 0$ the chr.inv.-world-lines deviation equation (8.14), describing a Weber detector, coincides with equation (7.30), and we conclude:

> A Weber detector (a solid-body detector of the resonance kind) will have no response to a falling gravitational wave of the pure deformation kind, if the particles of which the detector is composed are at rest at the initial moment of measurements (the situation assumed in the Weber experiment).

Second case: $D_{ik} = 0$, $A_{ik} \neq 0$.

We assume that the space rotation has a constant angular velocity ω around the x^3 axis, while the linear velocity of this rotation is $v^i = \omega_{k\cdot}^{\cdot i} x^k \ll c$. For the background metric, following the classical approach [14, 15], we use the Minkowski line element, where the gravitational waves are superimposed as tiny corrections to it. Then the components of v^i are

$$v^1 = -\omega x^2, \quad v^2 = \omega x^1, \quad v^3 = 0\,, \tag{8.15}$$

and the space metric in its expanded form is

$$\begin{aligned} ds^2 = c^2 dt^2 + 2\omega\left(x^2 dx^1 - x^1 dx^2\right) dt - \\ - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \end{aligned} \tag{8.16}$$

This metric describes the four-dimensional space of a uniformly rotating reference frame, whose rotational linear velocity is negligible with respect to c.

Components of the tensor A_{ik} are

$$A_{2\cdot}^{\cdot 1} = \omega_{2\cdot}^{\cdot 1} = -\omega\,, \quad A_{1\cdot}^{\cdot 2} = \omega_{1\cdot}^{\cdot 2} = \omega\,, \quad A_{1\cdot}^{\cdot 3} = 0\,. \tag{8.17}$$

Substituting (8.17) into the chr.inv.-world-lines deviation equation (8.14) we obtain a system of deviation equations

$$\frac{\partial^2 \eta^1}{\partial t^2} - 2\omega \frac{\partial \eta^2}{\partial t} = 0\,, \tag{8.18}$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + 2\omega \frac{\partial \eta^1}{\partial t} = 0\,, \tag{8.19}$$

$$\frac{\partial^2 \eta^3}{\partial t^2} = 0\,, \tag{8.20}$$

which commonly describe behaviour of two neighboring rest-particles in a uniformly rotating reference frame.

Equation (8.20) can be integrated immediately

$$\eta^3 = \eta^3_{(0)} + \dot{\eta}^3_{(0)} t\,, \tag{8.21}$$

where $\eta^3_{(0)}$ and $\dot{\eta}^3_{(0)}$ are the initial values of the relative displacement and velocity of the particles along the x^3 axis.

In integrating equations (8.19) and (8.20), we introduce the notation $\frac{\partial \eta^1}{\partial t} \equiv x$ and $\frac{\partial \eta^2}{\partial t} \equiv y$. Then we have

$$\left.\begin{aligned} \dot{x} - 2\omega y = 0 \\ \dot{y} + 2\omega x = 0 \end{aligned}\right\}. \tag{8.22}$$

We differentiate the first equation with respect to t

$$\ddot{x} = 2\omega \dot{y} \tag{8.23}$$

and substitute $\dot{y} = \ddot{x}/2\omega$ into the second one. We obtain a harmonic oscillation equation

$$\ddot{x} + 4\omega^2 x = 0\,, \tag{8.24}$$

with respect to the relative velocity $x = \frac{\partial \eta^1}{\partial t}$ of the particles. The solution to (8.24) is

$$x = \frac{\partial \eta^1}{\partial t} = C_1 \cos 2\omega t + C_2 \sin 2\omega t\,, \tag{8.25}$$

where C_1 and C_2 are integration constants, which can be obtained from the initial conditions. Thus we obtain

$$\frac{\partial \eta^1}{\partial t} = \left(\frac{\partial \eta^1}{\partial t}\right)_{(0)} \cos 2\omega t + \frac{1}{2\omega}\left(\frac{\partial^2 \eta^1}{\partial t^2}\right)_{(0)} \sin 2\omega t\,, \tag{8.26}$$

where terms marked with zero are the initial values of the relative velocity and acceleration of the particles. Integrating (8.26), we obtain

$$\eta^1 = \frac{\dot{\eta}^1_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^1_{(0)}}{4\omega^2} \cos 2\omega t + B_1\,, \tag{8.27}$$

where B_1 is an integration constant. Obtaining B_1 from the initial conditions, we obtain the final formula for η^1

$$\eta^1 = \frac{\dot{\eta}^1_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^1_{(0)}}{4\omega^2} \cos 2\omega t + \eta^1_{(0)} + \frac{\ddot{\eta}^1_{(0)}}{4\omega^2}\,. \tag{8.28}$$

In the same fashion we obtain a formula for η^2

$$\eta^2 = \frac{\dot{\eta}^2_{(0)}}{2\omega} \sin 2\omega t - \frac{\ddot{\eta}^2_{(0)}}{4\omega^2} \cos 2\omega t + \eta^2_{(0)} + \frac{\ddot{\eta}^2_{(0)}}{4\omega^2}\,. \tag{8.29}$$

By the exact solutions (8.21), (8.28), (8.29), obtained for the world-lines deviation equation taken in chr.inv.-form (8.14), it follows that:

> Stationary rotations of the space cannot force two neighbouring particles to initiate relative motion, if they are at rest at the initial moment of time.

In common with the result obtained in §7, where we discussed gravitational wave detectors built on free masses, we arrive at a final conclusion for the possibilities of gravitational wave detectors:

> Behaviour of both a gravitational wave detector built on free masses and a sold-body detector (a Weber pig) are **similar**. The only difference is that a solid-body detector can register both the time observable component and spatial observable components of the relative deviation vector, while a free-mass detector can register only spatial observable deviations. Deformations and stationary rotation of the space do not affect detectors of either kind.

Thus neither deformations nor stationary rotation of the space can not induce relative motion in the butt-ends of a Weber detector, if they are at rest. However Weber and his teamregistered signals. The question therefore arises:

> What signals did Weber register, and why, during the past 30 years, have his signals remained undetected by other researchers using superior detectors of the Weber kind?

We assume that the signals registered by Weber and his team, were much more than noise, and beyond doubt. Therefore, according to our theoretical analysis of the behaviour of solid-body detectors in weak gravitational waves, we make the following suppositions:

1. Weber registered signals which were an effect made in the pig by a vortex of the gravitational inertial force field. In other words, the origin of the signals could be rapid non-stationary rotation of a distant object in the depths of space;
2. The particles of the aluminium cylindrical pig, used by Weber, had substantial thermal motions. In this case parametric oscillations could appear as an effect of a falling gravitational wave. But in order to get such a real effect, the "background thermal oscillations" should be substantial;
3. The signals were registered only by Weber and his team. Not one signal has been registered by other experimental physicist during the subsequent 30 years, using superior detectors of the Weber kind. Either Weber registered gravitational waves derived from a non-stationary rotating object in the Universe, which occurred as a unique and short-lived phenomenon, or his original detector had a substantial peculiarity that made it differ in principle from the detectors used by other scientists.

We consider Weber's theory, aiming to ascertain what he registered with his solid-body detector.

9 Criticism of Weber's theory of detecting gravitational waves

In his book in 1960 [16], Weber propounded his theoretical arguments for the detection of gravitational waves by means of a solid-body detector of the resonance kind. He built his theory on the world-lines deviation equation for two particles, connected by a non-gravitational force (a spring, for instance). This is equation (2.12), being a modification of the well-known deviation equation for two free particles deduced by Synge (2.8), is known as the Synge-Weber equation. We considered both equations in detail above.

There is no doubt that the Synge-Weber equation is valid. Our main claim here is that Weber himself, in his analysis of the equation in order to build the theory for detecting gravitational waves, introduced a substantial assumption:

Weber's assumption 1 A falling gravitational wave should produce relative displacements of the butt-ends of a cylindrical pig.

So he obtained the same principle that he introduced, precluding himself from any possibility of obtaining anything else.

This line of reasoning constitutes a vicious circle. It would be been more reasonable and honest to have solved the world-lines deviation equation. Then he would have obtained exact solutions to the equation as was done in the previous sections herein.

In detail Weber's assumption 1 leads to the fact that, having a system of two test-particles connected by a spring, the resulting distance vector between them should be [16]

$$\eta^\alpha = r^\alpha + \xi^\alpha, \qquad r^\alpha \gg \xi^\alpha, \tag{9.1}$$

where the initial distance vector r^α is the such that

$$\frac{\mathrm{D}r^\alpha}{ds} = 0\,. \tag{9.2}$$

He supposed as well that $\eta^\alpha \to r^\alpha$ in the ultimate case where the friction rises infinitely or the Riemann-Christoffel curvature tensor becomes zero $R^{\alpha\cdots}_{\cdot\beta\gamma\delta} = 0$ [16].

Taking the main supposition (9.1) into account, Weber transforms the Synge-Weber equation (2.12) into

$$\frac{\mathrm{D}^2\xi^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \left(r^\gamma + \xi^\gamma\right) = \frac{1}{m_0 c^2}\, f^\alpha, \tag{9.3}$$

where f^α is the difference between non-gravitational forces of the particles' interaction. Weber assumes f^α the sum of the elasticity force $f^\alpha_1 = -\mathcal{K}^\alpha_\sigma \xi^\sigma$ that restores the particles, and the oscillation relaxing force $f^\alpha_2 = -c\,\mathcal{D}^\alpha_\sigma \frac{\mathrm{D}\xi^\sigma}{ds}$, where $\mathcal{K}^\alpha_\sigma$ and $\mathcal{D}^\alpha_\sigma$ are the elasticity and friction coefficients, respectively. Then (9.3) takes the form

$$\frac{\mathrm{D}^2\xi^\alpha}{ds^2} + \frac{1}{m_0 c}\,\mathcal{D}^\alpha_\sigma \frac{\mathrm{D}\xi^\sigma}{ds} + \frac{1}{m_0 c^2}\,\mathcal{K}^\alpha_\sigma\, \xi^\sigma = -R^{\alpha\cdots}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \left(r^\gamma + \xi^\gamma\right). \tag{9.4}$$

Weber introduced additional substantial assumptions:

Weber's assumption 2 The whole detector is in the state of free falling;

Weber's assumption 3 The reference frame in his laboratory is such that the Christoffel symbols can be assumed zero.

Because of these assumptions, and the condition $|r| \gg |\xi|$, Weber writes equation (9.4) as follows

$$\frac{d^2\xi^\alpha}{dt^2} + \frac{1}{m_0}\,\mathcal{D}^\alpha_\sigma \frac{d\xi^\sigma}{dt} + \frac{1}{m_0 c^2}\,\mathcal{K}^\alpha_\sigma\, \xi^\sigma = -c^2 R^{\alpha\cdots}_{\cdot 0\sigma 0}\, r^\sigma. \tag{9.5}$$

Looking at the right side of Weber's equation (9.5) we see his fourth hidden assumption:

Weber's assumption 4 Particles located on two neighbouring world-lines in the Weber experimental statement (the butt-ends of his cylindrical pig) are at rest at the initial moment of time, so $U^i = 0$.

In §7, where we considered chr.inv.-equations of motion for two particles connected by a non-gravitational force (8.4) and (8.5), we came to the conclusion: a reference frame where interacting particles ($\Phi^\alpha \neq 0$) are at rest ($\mathrm{v}^i = 0$) cannot be in astate of free fall. Really, the free fall condition is $F^i = 0$. Equation $m_0 F^i = -f^i$ (8.7), which is the chr.inv.-form of spatial equations of motion of the interacting particles, implies that when $F^i = 0$, $f^i = 0$. Therefore:

> The Weber assumption 2 is **inapplicable** to his experimental statement.

Moreover, a reference frame where the Christoffel symbols are zero can be applicable only at a point, it is unapplicable to a finite region. At the same time, in the Weber experimental statement, the detector itself is a system of two particles located at the distance η from each other. In a Riemannian space the Riemann-Christoffel curvature tensor is different from zero, so the Riemannian coherence objects (the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$) cannot be reduced to zero by coordinate transformations. We can merely choose a reference system where, at a given point P, the coherence objects are zero $(\Gamma^\alpha_{\beta\gamma})_P = 0$. Such a reference frame is known as a *geodesic reference frame* [37]. Therefore:

> The Weber assumption 3 is **inapplicable** to his experimental statement.

Thus if we retain the rest-condition $U^i = 0$ and the free fall condition in the Weber equation (9.4), there must still be the non-gravitational force $\Phi^\alpha = 0$. So the Weber equation becomes the free particles deviation equation, which in chr.inv.-form is (7.30).

If we reject free fall in the Weber equation (9.4), but retain $U^i = 0$, it takes the same form as (8.14), which is not a free oscillation equation, in which case weak plane gravitational waves can act on the particles only if they are in motion at the initial moment of time.

Collecting these results we conclude that:

> The Weber equation (9.4) is **incorrect**, because the free fall condition in common with the rest-condition for two neighbouring particles, connected by a non-gravitational force, lead to the requirement that this force should be zero, thus contradicting the initial conditions of the Weber experimental statement.

It is evident that in aiming to determine the sort of effects a falling gravitational wave has on a free-mass detector or a Weber detector, it would be reasonable to consider a case where the particles are in motion $U^i \neq 0$. In this case, before solving the deviation equation (2.8) for two free particles or (2.12) for two interacting particles (depending on the type of detector used), we should solve the equations of motion for free particles (2.6) or forced particles (2.11), respectively.

It should be noted that the main structure of motion is determined by the left (geometrical) side of equations of motion, while the right side introduces only an additional effect into the motion.

In my previous articles [74, 75, 76] common exact solutions to the geodesic equations and the deviation equation had been obtained in the field of weak plane gravitational waves, described by the metric (6.12). The exact solutions had been obtained in general covariant form.

The solutions to the equations of motion for a free particle, equations (2.6), in a linear polarized harmonic wave $a = A \sin \frac{\omega}{c}(ct \pm x^1)$, $b = 0$ are as follows

$$U^0 + U^1 = \varepsilon = \text{const}\,, \tag{9.6}$$

$$U^1 = -\frac{1}{4\varepsilon}\Big[\big(U^2_{(0)}\big)^2 e^{2A\sin\frac{2\omega}{c}(ct\pm x^1)} + \big(U^3_{(0)}\big)^2 e^{-2A\sin\frac{\omega}{c}(ct\pm x^1)}\Big]_a + U^1_{(0)}\,, \tag{9.7}$$

$$U^2 = U^2_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.8}$$

$$U^3 = U^3_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.9}$$

where $U^1_{(0)}$, $U^2_{(0)}$, $U^3_{(0)}$ are the initial values of the particle's velocity along each of the spatial axes.

From the solutions two important conclusions follow:

1. A weak plane gravitational wave, falling in the x^1 direction, acts on free particles only if they have non-zero velocities in directions x^2 and x^3 orthogonal to the wave motion.
2. The presence of transverse oscillations in the plane (x^2, x^3) leads also to longitudinal oscillations in the direction x^1.

The solutions to the free-particles deviation equation (2.8) in the field of a weak plane gravitational wave are

$$\eta^1 = \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\Big(\eta^1_{(0)} + \dot{\eta}^1_{(0)} t\Big) \times \sin\frac{\omega}{c}(ct\pm x^1) + \frac{AL}{2\varepsilon\omega}\cos\frac{\omega}{c}(ct\pm x^1) + \left\{\dot{\eta}^1_{(0)} - \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\,\omega\eta^1_{(0)}\right\} t + \eta^1_{(0)} - \frac{AL}{2\varepsilon}\,, \tag{9.10}$$

$$\eta^2 = \dot{\eta}^2_{(0)}\Big[t + \frac{A}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] + \eta^2_{(0)} - \frac{A}{\omega}\dot{\eta}^2_{(0)} - \frac{AU^2_{(0)}}{\varepsilon}\left\{\dot{\eta}^1_{(0)}\Big[t - \frac{1}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] - \Big(\eta^1_{(0)} + \dot{\eta}^1_{(0)} t\Big)\cos\frac{\omega}{c}(ct\pm x^1) + \frac{\dot{\eta}^1_{(0)}}{\omega} + \eta^1_{(0)}\right\}, \tag{9.11}$$

$$\eta^3 = \dot{\eta}^3_{(0)}\left[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\right] + \eta^3_{(0)} + \frac{A}{\omega}\dot{\eta}^3_{(0)} - \\ + \frac{AU^3_{(0)}}{\varepsilon}\left\{\dot{\eta}^1_{(0)}\left[t - \frac{1}{\omega}\cos\frac{\omega}{c}(ct \pm x^1)\right] - \\ - \left(\eta^1_{(0)} + \dot{\eta}^1_{(0)}\,t\right)\cos\frac{\omega}{c}(ct \pm x^1) - \frac{\dot{\eta}^1_{(0)}}{\omega} - \eta^1_{(0)}\right\}, \tag{9.12}$$

where

$$L = U^2_{(0)}\dot{\eta}^2_{(0)} - U^3_{(0)}\dot{\eta}^3_{(0)} = \frac{\eta^1_{(0)}}{\varepsilon}\left[\left(U^3_{(0)}\right)^2 - \left(U^2_{(0)}\right)^2\right]. \tag{9.13}$$

The solutions η^1, η^2, η^3 are the relative deviations of two free particles in directions orthogonal to the direction of the wave's motion. The deviations are actually generalizations of the solutions (7.41) and (7.42), where the particles were at rest. The only difference is that here (9.10–9.12) there are additional parts, where the particles' initial velocities $U^2_{(0)}$ and $U^3_{(0)}$ are added.

Here we see that, besides regular harmonic oscillations, the term $t\cos\frac{\omega}{c}\left(ct \pm x^1\right)$ describes oscillations with an amplitude that increases without bound with time. Another substantial difference is that, in contrast to solutions (7.35), (7.42), (7.43) given for rest-particles, the solutions (9.10), (9.11), (9.12) contain longitudinal oscillations — they are described by solution (9.10). Both harmonic oscillations and unbounded-rising oscillations exist there only if, at the initial moment of time, the particles are in motion along x^2 and x^3 (orthogonal to the x^1 direction of the wave's motion).

So, we come to our final conclusions on both free-mass detectors and solid-body detectors of gravitational waves:

> The greater the velocities of particles (atoms and molecules) in a gravitational wave detector (built on either free masses or of the Weber kind), the more sensitive is the detector to a falling weak plane gravitational wave. In current experiments researchers cool the Weber pigs to super low temperatures, about 2 K, aiming to minimize the inherent oscillations of the particles of which they consist. This is a counter-productive procedure by which experimental physicists actually reduce the sensitivity of the Weber detectors to practically zero. We see the same vicious drawback in current experiments with free-mass detectors, where such a detector consists of two satellites located in the same orbit near the Earth. Because the observer (a laser range-finder) is located in one of the satellites, both satellites are at rest with respect to each other and the observer. All the current experiments cannot register gravitational waves in principle. In a valid experiment for discovering gravitational waves, the particles of which the detector consists must be in as rapid motion as possible. It would be better to design a detector using two laser beams directed parallel to each other, because of the light velocity of the moving particles (photons). The indicative quantities to be observed are the light frequency and phase.

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