On the Generalisation of Kepler's 3rd Law for the Vacuum Field of the Point-Mass

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I derive herein a general form of Kepler's 3rd Law for the general solution to Einstein's vacuum field. I also obtain stable orbits for photons in all the configurations of the point-mass. Contrary to the accepted theory, Kepler's 3rd Law is modified by General Relativity and leads to a finite angular velocity as the proper radius of the orbit goes down to zero, without the formation of a black hole. Finally, I generalise the expression for the potential function of the general solution for the point-mass in the weak field.

1 Introduction

In previous papers [1, 2] I derived the general solution for Einstein's vacuum field and showed that black holes do not exist in Einstein's universe. In those papers I also obtained expressions for Kepler's 3rd Law for the simple (i. e. non-rotating) point-mass and the simple point-charge. In this paper I obtain expressions for Kepler's 3rd Law for the rotating point-mass and the rotating point-charge. Owing to the rotation of the source of the field, Kepler's 3rd Law for the polar orbit is not the same as that for the equatorial orbit, so that stable photon orbits are also different in the polar and equatorial orbits, showing that in the rotating configurations spacetime is no longer isotropic.

The expressions I obtain readily reduce to those I have previously derived for the non-rotating configurations.

2 Definitions

I have already shown [3] that the most general static metric for the point-mass is,

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$D = |r - r_0|,$$

$$A, B, C > 0,$$

where r_0 is an arbitrary real number. With respect to this metric I identify the coordinate radius, the r-parameter, the radius of curvature, and the proper radius thus:

1. *The coordinate radius is* $D = |r - r_0|$.
1. *The* r*-parameter is the variable* r.
2. *The radius of curvature is* $R = \sqrt{C(D)}$.
3. *The proper radius is* $R_p = \int \sqrt{B(D)}\, dD$.

3 The equatorial orbit

The general Kerr-Newman form in Boyer-Lindquist coordinates is,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ - \frac{\sin^2\theta}{\rho^2}\left[\left(R^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2.$$

This can be written as,

$$ds^2 = \left(\frac{\Delta - a^2\sin^2\theta}{\xi}\right)dt^2 - \frac{\xi}{\Delta}dR^2 - \\ - \xi d\theta^2 + \left[\frac{a^2\Delta\sin^4\theta - \left(R^2 + a^2\right)^2\sin^2\theta}{\xi}\right]d\varphi^2 - \\ - \left[\frac{2a\Delta\sin^2\theta - 2a\left(R^2 + a^2\right)\sin^2\theta}{\xi}\right]dt d\varphi, \tag{1}$$

where I have previously shown [2, 3] in the case of the rotating point-charge,

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re,\ r \in \Re,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2},$$

$$a^2 + q^2 < m^2, \quad n \in \Re^+, \quad \xi = \rho^2 = R^2 + a^2\cos^2\theta,$$

$$a = \frac{L}{m}, \quad \Delta = R^2 - \alpha R + a^2 + q^2,$$

$$0 < |r - r_0| < \infty,$$

where L is the angular momentum, and n and r_0 are arbitrary.

I have also shown previously that Kepler's 3rd Law for the simple point-mass is,

$$\omega^2 = \frac{\alpha}{2R^3}, \tag{2}$$

where

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = R_0 = \alpha = 2m \ \forall\ r_0,$$

is a scalar invariant; and for the simple point-charge is,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}, \tag{3}$$

where, $\forall\, r_0$,

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = R_0 = \beta = m + \sqrt{m^2 - q^2}, \; q^2 < m^2 \,,$$

is a scalar invariant.

In the case of the equatorial orbit, $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, so (1) becomes,

$$\begin{aligned} ds^2 = & \left(\frac{\Delta - a^2}{\xi}\right) dt^2 - \frac{\xi}{\Delta} dR^2 + \\ & + \left[\frac{a^2\Delta - \left(R^2 + a^2\right)^2}{\xi}\right] d\varphi^2 - \\ & - \left[\frac{2a\Delta - 2a\left(R^2 + a^2\right)}{\xi}\right] dt d\varphi \,. \end{aligned} \tag{4}$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}} \,,$$

$$\beta = m + \sqrt{m^2 - q^2} \,, \; q^2 < m^2 \,,$$

$$\xi = R^2 \,, \; \Delta = R^2 - \alpha R + a^2 + q^2 \,,$$

$$0 < |r - r_0| < \infty \,.$$

Consider the associated Lagrangian, where the dot indicates $\partial/\partial\tau$,

$$\begin{aligned} L = & \frac{1}{2}\left[\frac{\Delta - a^2}{\xi}\dot{t}^2 - \frac{\xi}{\Delta}\dot{R}^2\right] + \\ & + \frac{1}{2}\left[\frac{a^2\Delta - \left(R^2 + a^2\right)^2}{\xi}\right]\dot{\varphi}^2 - \\ & - \frac{1}{2}\left[\frac{2a\Delta - 2a\left(R^2 + a^2\right)}{\xi}\right]\dot{t}\,\dot{\varphi} \,. \end{aligned} \tag{5}$$

Then,

$$\begin{aligned} & \frac{\partial L}{\partial R} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{R}}\right) = 0 \Rightarrow \frac{\xi\Delta' - \xi'\left(\Delta - a^2\right)}{2\xi^2}\dot{t}^2 + \\ & + \frac{\xi\left[a^2\Delta' - 4R\left(R^2 + a^2\right)\right]}{2\xi^2}\dot{\varphi}^2 - \\ & - \frac{\xi'\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]}{2\xi^2}\dot{\varphi}^2 - \\ & - \frac{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2 + a^2\right)\right]}{2\xi^2}\dot{t}\dot{\varphi} + \\ & + \frac{\Delta\xi' - \xi\Delta'}{2\Delta^2}\dot{R}^2 + \frac{\xi}{\Delta}\ddot{R} = 0 \,. \end{aligned} \tag{6}$$

Taking $R = \text{const.}$ reduces (6) to,

$$\begin{aligned} & \Big\{\xi\left[a^2\Delta' - 4R\left(R^2 + a^2\right)\right] - \\ & - \xi'\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]\Big\}\omega^2 - \\ & - \left\{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2 + a^2\right)\right]\right\}\omega + \\ & + \xi\Delta' - \xi'\left(\Delta - a^2\right) = 0 \,, \end{aligned} \tag{7}$$

where $\omega = \frac{\dot{\varphi}}{\dot{t}}$. The solutions for ω are,

$$\omega = \frac{a\alpha R - 2aq^2 \pm R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4} \,.$$

In order for this to reduce to the non-rotating configurations, the plus sign must be taken so,

$$\omega = \frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4} \,, \tag{8}$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}} \,,$$

$$\beta = m + \sqrt{m^2 - q^2} \,, \; q^2 < m^2 \,,$$

$$\alpha = 2m \,,$$

$$0 < |r - r_0| < \infty \,.$$

Equation (8) is Kepler's 3rd Law for the equatorial plane of the rotating point-charge. I remark that the radius of curvature in the equatorial orbit is precisely that for the simple point-charge. The expression for Kepler's 3rd Law for the equatorial plane of the rotating point-mass is obtained from (8) by setting $q = 0$,

$$\omega = \frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4} \,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}} \,,$$

$$\alpha = 2m \,,$$

$$0 < |r - r_0| < \infty \,,$$

in which case the radius of curvature in the equatorial orbit is precisely that for the simple point-mass.

Taking the near-field limit on (8) gives,

$$\lim_{r \to r_0^{\pm}} \omega = \frac{a\alpha\beta - 2aq^2 + \beta^2\sqrt{2\alpha\beta - 4q^2}}{a^2\alpha\beta - 2a^2q^2 - 2\beta^4} \,, \tag{9}$$

which is a scalar invariant.

When $a = 0$ and $q \neq 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4} \,,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. If $a = q = 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} \,,$$

$$\beta = \alpha = 2m \,,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

When $a = 0$ and $q \neq 0$, (9) reduces in the near-field limit, to

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha}{2\beta^3} - \frac{q^2}{\beta^4},$$

$$\beta = m + \sqrt{m^2 - q^2},$$

the scalar invariant of Kepler's 3rd Law for the simple point-charge; and when $a = q = 0$, (9) reduces to the near-field limit,

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{1}{2\alpha^2},$$

$$\alpha = 2m,$$

the scalar invariant for Kepler's 3rd Law for the simple point-mass, as originally obtained by Karl Schwarzschild [4] for his particular solution.

4 Photons in equatorial orbit

Setting $\theta = \frac{\pi}{2}$ in (1) and setting (1) equal to zero gives,

$$\begin{aligned} &\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]\omega^2 - \\ &- \left[2a\Delta - 2a\left(R^2 + a^2\right)\right]\omega + \left(\Delta - a^2\right) = 0, \end{aligned} \tag{10}$$

from which it follows,

$$\omega = \frac{\dot{\varphi}}{\dot{t}} = \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2 R - a^2R^2 - R^4}. \tag{11}$$

Equating (8) to (11) gives,

$$\begin{aligned} &\frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4} = \\ &= \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2R - a^2R^2 - R^4}, \end{aligned} \tag{12}$$

$$\alpha = 2m,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2}, \; q^2 < m^2,$$

$$r_0 \in \Re, \; n \in \Re^+,$$

for the radius of curvature $R_{ph-e} = R = \sqrt{C_n(r_{ph-e})}$ of the equatorial orbit of a photon for the rotating point-charge. When $a = 0$ equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4},$$

recovering the stable radius of curvature for the photon orbit about the simple point-charge [2]. When $a = q = 0$, equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha}{2} = 3m, \tag{13}$$

which recovers the stable radius of curvature for the photon around the simple point-mass [1].

When $n = 1$ and $r_0 = \alpha$, equation (13) gives,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = r_{ph-e} = 3m,$$

This radius is taken incorrectly by the orthodox relativists as a measurable proper radius in the gravitational field of the simple point-mass. The actual proper radius associated with (13) is,

$$R_p = \frac{\alpha\sqrt{3}}{2} + \alpha\ln\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right),$$

which is a scalar invariant for the photon orbit about the point-mass.

The expression for the radius of curvature of the stable photon equatorial orbit for the rotating point-mass is obtained from (12) by setting $q = 0$, thus

$$\frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4} = \frac{a\alpha R - R^2\sqrt{R^2 - \alpha R + a^2}}{\alpha a^2 R + a^2R^2 + R^4},$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}},$$

$$\alpha = 2m,$$

$$r_0 \in \Re, \; n \in \Re^+.$$

5 The polar orbit

According to (1), if $R = \sqrt{C_n(r)}$ is a function of t,

$$R = R(t, \theta) = = \sqrt{C_n(r(t))} = \left(|r(t) - r_0|^n + \beta^n\right)^{\frac{1}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta},$$

so if $\dot{r} = 0$, $\dot{R} = 0$.

In the polar orbit there is no loss of generality in taking $\varphi = \text{const.}$, $\dot{\varphi} = 0$. Then (1) becomes,

$$ds^2 = \frac{\Delta - a^2\sin^2\theta}{\xi}dt^2 - \frac{\xi}{\Delta}dR^2 - \xi d\theta^2, \tag{14}$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re, \; r \in \Re,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2},$$

$$a^2 + q^2 < m^2, \quad n \in \Re^+, \quad \xi = \rho^2 = R^2 + a^2\cos^2\theta,$$

$$a = \frac{L}{m}, \quad \Delta = R^2 - \alpha R + a^2 + q^2,$$

$$0 < |r - r_0| < \infty.$$

Consider the associated Lagrangian,

$$L = \frac{1}{2}\left[\frac{\Delta - a^2\sin^2\theta}{\xi}\dot{t}^2 - \frac{\xi}{\Delta}\dot{R}^2 - \xi\dot{\theta}^2\right].$$

Then,

$$\frac{\partial L}{\partial R} - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{R}}\right) = \frac{1}{2}\left[\frac{\xi\Delta' - \xi'(\Delta - a^2 \sin^2\theta)}{\xi^2}\dot{t}^2\right] + \\ - \frac{1}{2}\left[\frac{(\Delta\xi' - \xi\Delta')}{\Delta^2}\dot{R}^2 + \xi'\dot{\theta}^2\right] + \frac{\xi}{\Delta}\ddot{R} = 0\,. \tag{15}$$

If $\dot{R} = 0$, then (15) yields,

$$\omega^2 = \frac{\dot{\theta}^2}{\dot{t}^2} = \frac{\xi\Delta' - \xi'(\Delta - a^2\sin^2\theta)}{\xi'\xi^2} = \\ = \frac{\alpha R^2 - \alpha a^2\cos^2\theta - 2q^2 R}{2R\left(R^2 + a^2\cos^2\theta\right)^2} = \\ = \frac{\alpha C_n - \alpha a^2 \cos^2\theta - 2q^2\sqrt{C_n}}{2\sqrt{C_n}\left(C_n + a^2\cos^2\theta\right)^2}\,, \tag{16}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,\ a^2 + q^2 < m^2\,,$$

$$n \in \Re^+\ r_0 \in \Re\,,$$

$$0 < |r - r_0| < \infty\,.$$

Equation (16) is Kepler's 3rd Law for the polar orbit of the rotating point-charge. I remark that the angular velocity depends upon azimuth.

Let $a = 0$, $q \neq 0$, then (16) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,\ \beta = m + \sqrt{m^2 - q^2},$$

$$q^2 < m^2,\ r_0 \in \Re,\ n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. Setting $a = q = 0$ reduces (16) to,

$$\omega^2 = \frac{\alpha}{2R^3}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$n \in \Re^+,\ r(0) \in \Re\,,$$

$$0 < |r - r_0| < \infty,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

Taking the near-field limit on (16),

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha\beta^2 - \alpha a^2\cos^2\theta - 2q^2\beta}{2\beta\left(\beta^2 + a^2\cos^2\theta\right)^2}\,, \tag{17}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-charge.

When $q = 0$, $a \neq 0$, equation (16) reduces to,

$$\omega^2 = \frac{\alpha R^2 - \alpha a^2\cos^2\theta}{2R\left(R^2 + a^2\cos^2\theta\right)^2} = \\ = \frac{\alpha C_n - \alpha a^2\cos^2\theta}{2\sqrt{C_n}\left(C_n + a^2\cos^2\theta\right)^2}\,, \tag{18}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,,\ a^2 < m^2\,,$$

$$n \in \Re^+\ r_0 \in \Re\,,$$

$$0 < |r - r_0| < \infty\,.$$

This is Kepler's 3rd Law for the polar orbit of the rotating point-mass.

Taking the near-field limit on (18),

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha\beta^2 - \alpha a^2\cos^2\theta}{2\beta\left(\beta^2 + a^2\cos^2\theta\right)^2}\,, \tag{19}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-mass.

Thus, ω varies with azimuth as does $R = \sqrt{C_n(r)}$. At the poles of the rotating point-charge,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - a^2 - q^2}\,, \tag{20}$$

$$\omega^2 = \frac{\alpha R^2 - \alpha a^2 - 2q^2 R}{2R\left(R^2 + a^2\right)^2}\,,$$

and at the equator,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,, \tag{21}$$

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,.$$

It is noted that at the momentary equator in a polar orbit, the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-charge.

In the case of the rotating point-mass, at the poles,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - a^2}\,, \tag{22}$$

$$\omega^2 = \frac{\alpha R^2 - \alpha a^2}{2R\left(R^2 + a^2\right)^2}\,,$$

and at the equator,

$$R^2 = C_n(r) = \left(|r - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = 2m = \alpha\,, \tag{23}$$

$$\omega^2 = \frac{\alpha}{2R^3}\,.$$

At the momentary equator in a polar orbit the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-mass.

6 Photons in the polar orbit

Setting (14) equal to zero, with $\dot{R}=0$, gives

$$\omega^2=\frac{\Delta-a^2\sin^2\theta}{\xi^2}=\frac{R^2-\alpha R+a^2\cos^2\theta+q^2}{\left(R^2+a^2\cos^2\theta\right)^2}\,. \tag{24}$$

Denote the stable photon radius of curvature for a photon in polar orbit by $R_{ph-p}=\sqrt{C_n(r_{ph-p})}$. Then equating (24) to (16) gives,

$$\begin{gathered}2R^3_{ph-p}-3\alpha R^2_{ph-p}+\\+\left(2a^2\cos^2\theta+4q^2\right)R_{ph-p}+\alpha a^2\cos^2\theta=0\,,\\R^2_{ph-p}=C_n(r_{ph-p})=\left(|r_{ph}-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,, \qquad (25)\\\beta=m+\sqrt{m^2-a^2\cos^2\theta-q^2},\ a^2+q^2<m^2\,,\\r_0\in\Re,\quad n\in\Re^+\,.\end{gathered}$$

Equation (25) gives the stable photon radius of curvature in the polar orbit. The orbit has a variable radius of curvature with azimuth.

When $a=0,\ q\neq 0$, equation (25) reduces to

$$R_{ph-p}=\sqrt{C_n(r_{ph-p})}=\frac{3\alpha+\sqrt{9\alpha^2-32q^2}}{4}\,, \tag{26}$$

$$C_n(r_{ph-p})=\left(\left|r_{ph-p}-r_0\right|^n+\beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta=m+\sqrt{m^2-q^2},\ q^2<m^2\,,$$

$$r_0\in\Re,\ n\in\Re^+\,,$$

which recovers the radius of curvature for the stable orbit of a photon about the simple point-charge. When $a=q=0$, (25) reduces to,

$$R_{ph-p}=\sqrt{C_n(r_{ph-p})}=\frac{3\alpha}{2}\,, \tag{27}$$

$$C_n(r_{ph-p})=\left(\left|r_{ph-p}-r_0\right|^n+\alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha=2m,\ r_0\in\Re,\ n\in\Re^+\,,$$

which recovers the curvature radius for the stable orbit of a photon about the simple point-mass. When $n=1$ and $r_0=\alpha$, equation (27) gives,

$$R_{ph-p}=\sqrt{C_n(r_{ph-p})}=r_{ph-p}=3m\,,$$

which is the stable radius of curvature for the photon about the simple point-mass, but which is misinterpreted by the orthodox relativists as a measurable proper radius.

To obtain the stable photon radius of curvature of the polar orbit for the rotating point-mass, set $q=0$ in (25),

$$\begin{gathered}2R^3_{ph-p}-3\alpha R^2_{ph-p}+2a^2\cos^2\theta R_{ph-p}+\\+\alpha a^2\cos^2\theta=0\,,\\R^2_{ph-p}=C_n(r_{ph-p})=\left(|r_{ph}-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,, \qquad (28)\\\beta=m+\sqrt{m^2-a^2\cos^2\theta},\ a^2<m^2\,,\\r_0\in\Re,\quad n\in\Re^+\,.\end{gathered}$$

7 Potential functions in the weak field

In the case of the rotating point-charge,

$$g_{00}=\frac{\Delta-a^2\sin^2\theta}{\rho^2}\,, \tag{29}$$

$$\Delta=C_n(r)-\alpha\sqrt{C_n(r)}+a^2+q^2\,,$$

$$\rho^2=C_n(r)+a^2\cos^2\theta\,.$$

The potential Φ for a general metric is given by,

$$g_{00}=(1-\Phi)^2=1-2\Phi+\Phi^2\,.$$

In the weak field,

$$g_{00}\approx 1-2\Phi\,.$$

Now (29) gives,

$$g_{00}=\frac{C_n(r)-\alpha\sqrt{C_n(r)}+a^2\cos^2\theta+q^2}{C_n(r)+a^2\cos^2\theta}==1-\frac{\alpha\sqrt{C_n(r)}-q^2}{C_n(r)+a^2\cos^2\theta}\,,$$

so the potential is,

$$\Phi=\frac{\alpha\sqrt{C_n(r)}-q^2}{2\left(C_n(r)+a^2\cos^2\theta\right)}\,, \tag{30}$$

$$C_n(r)=\left(\left|r-r_0\right|^n+\beta^n\right)^{\frac{2}{n}},\quad r_0\in\Re\,,$$

$$\beta=m+\sqrt{m^2-a^2\cos^2\theta-q^2}\,,$$

$$a^2+q^2<m^2\,,\quad n\in\Re^+\,,$$

$$0<|r-r_0|<\infty\,.$$

The potential therefore depends upon azimuth.

The potential for the rotating point-mass is obtained from (30) by setting $q = 0$,

$$\Phi = \frac{\alpha\sqrt{C_n(r)}}{2\left(C_n(r) + a^2\cos^2\theta\right)}\,, \tag{31}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,,$$

$$a^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

If $a = 0$ the potential for the simple point-charge is recovered from (30),

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}} - \frac{q^2}{2C_n(r)}\,, \tag{32}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,$$

$$q^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,,$$

and if $a = q = 0$ the potential for the simple point-mass is recovered,

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}}\,, \tag{33}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

According to (30), orbit in the equatorial gives equations (32) for the simple point-charge. According to (31), orbit in the equatorial gives equations (33) for the simple point-mass. For orbits in the polar, equations (32) and (33) are momentarily realised at the equator for a test particle orbiting the rotating point-charge and the rotating point-mass respectively. Thus, the effects of rotation of the source of the field do not manifest for a test particle in an equatorial orbit.

Taking the near-field limit on (30) gives,

$$\lim_{r \to r_0^{\pm}} \Phi = \frac{\alpha\beta - q^2}{2\left(\beta^2 + a^2\cos^2\theta\right)}\,, \tag{34}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,.$$

The potential approaches a finite limit with azimuth. The limiting values for the simpler configurations are easily obtained from (34) in the obvious way.

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