# The Algebraic Rainich Conditions

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In the literature, the algebraic Rainich conditions are obtained using special methods such as spinors, duality rotations, an eigenvalue problem for certain $4\times 4$ matrices or artificial tensors of 4th order. We give here an elementary procedure for deducing an identity satisfied by a determined class of second order tensors in arbitrary $\Re^4$, from which the Rainich expressions are immediately obtained.

## 1 Introduction

Rainich [1–5] proposed a unified field theory for the geometrization of the electromagnetic field, whose basic relations can be obtained from the Einstein-Maxwell field equations:

$$R_{ij} - \frac{1}{2} R\, g_{ij} = -8\pi \left( F_{ib} F_{j\cdot}^{\cdot b} - \frac{1}{4} F_{ab} F^{ab} g_{ij} \right), \qquad (1)$$

where $R_{ac} = R_{ca}$, $R = R^{b\cdot}_{\cdot b}$ and $F_{ac} = -F_{ca}$ are the Ricci tensor, scalar curvature and Faraday tensor [6], respectively.

If in (1) we contract $i$ with $j$ we find that:

$$R = 0 \qquad (2)$$

then (1) adopts the form:

$$R_{ij} = 2\pi F_{ab} F^{ab} g_{ij} - 8\pi F_{ib} F_{j\cdot}^{\cdot b} \qquad (3)$$

used by several authors [1, 2, 5, 7, 8] to obtain the identity:

$$R_{ic} R_{j\cdot}^{\cdot c} = \frac{1}{4} \left( R_{ab} R^{ab} \right) g_{ij}\,. \qquad (4)$$

If $F_{ar}$ is known, then (3) is an equation for $g_{ij}$ and our situation belongs to general relativity. The Rainich theory presents the inverse process: To search for a solution of (2) and (4) (plus certain differential restrictions), and after with (3) to construct the corresponding electromagnetic field; from this point of view $F_{ar}$ is a consequence of the spacetime geometry.

In the next Section we give an elementary proof of (4), without resorting to duality rotations [2], spinors [7], eigenvalue problems [8] or fourth order tensors [9, 10].

## 2 The algebraic Rainich conditions

The structure of (3) invites us to consider tensors with the form:

$$C_{ij} = A\, g_{ij} + B_{ik} F_{j\cdot}^{\cdot k} \qquad (5)$$

where $A$ is a scalar and $B_{ac}$, $F_{ij}$ are arbitrary antisymmetric tensors. Then from (5) it is easy to deduce the expression:

$$C_{ia} C^{a\cdot}_{\cdot j} - \frac{C}{2} C_{ij} - \frac{1}{4} \left( C_{ab} C^{ba} - \frac{C^2}{2} \right) g_{ij} = D_{ij} \qquad (6)$$

with $C = C^{r\cdot}_{\cdot r}$ and

$$D_{ij} = B_{ik} F^{ak} B_{am} F_{j\cdot}^{\cdot m} - \frac{1}{2} \left( B^{nm} F_{nm} \right) B_{ib} F_{j\cdot}^{\cdot b} + \frac{1}{8} \left[ \left( B^{nm} F_{nm} \right)^2 - 2 B_{bk} F^{a\cdot}_{\cdot k} B_{a\cdot}^{\cdot m} \right] g_{ij}\,. \qquad (7)$$

But in four dimensions we have the following identities between antisymmetric tensors and their duals [11–13]:

$$B^{m\cdot}_{\cdot c} F^{ic} - {}^*B^{ic}\, {}^*F^{m\cdot}_{\cdot c} = \frac{1}{2} \left( B_{cd} F^{cd} \right) g^{im}, \qquad B^{k\cdot}_{\cdot r}\, {}^*B^{ir} = \frac{1}{4} \left( B_{ab}\, {}^*B^{ab} \right) g^{ik}. \qquad (8)$$

With (7) and (8) it is simple to prove that $D_{ij} = 0$. Therefore (6) implies the identity:

$$C_{ia} C^{a\cdot}_{\cdot j} - \frac{C}{2} C_{ij} = \frac{1}{4} \left( C_{ab} C^{ba} - \frac{C^2}{2} \right) g_{ij}\,. \qquad (9)$$

If now we consider the particular case:

$$A = 2\pi F_{ab} F^{ab}, \qquad B_{ij} = -8\pi F_{ij}\,, \qquad (10)$$

then (5) reproduces (3) and $C = R = 0$, and thus (9) leads to (4), q.e.d.

Our procedure shows that the algebraic Rainich conditions can be deduced without special techniques.

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