A Unified Field Theory of Gravity, Electromagnetism, and the Yang-Mills Gauge Field

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In this work, we attempt at constructing a comprehensive four-dimensional unified field theory of gravity, electromagnetism, and the non-Abelian Yang-Mills gauge field in which the gravitational, electromagnetic, and material spin fields are unified as intrinsic geometric objects of the space-time manifold $\mathbb{S}_4$ via the connection, with the generalized non-Abelian Yang-Mills gauge field appearing in particular as a sub-field of the geometrized electromagnetic interaction.

1 Introduction

In our previous work [1], we developed a semi-classical conformal theory of quantum gravity and electromagnetism in which both gravity and electromagnetism were successfully unified and linked to each other through an "external" quantum space-time deformation on the fundamental Planck scale. Herein we wish to further explore the geometrization of the electromagnetic field in [1] which was achieved by linking the electromagnetic field strength to the torsion tensor built by means of a conformal mapping in the evolution (configuration) space. In so doing, we shall in general disregard the conformal mapping used in [1] and consider an arbitrary, very general torsion field expressible as a linear combination of the electromagnetic and material spin fields.

Herein we shall find that the completely geometrized Yang-Mills field of standard model elementary particle physics, which roughly corresponds to the electromagnetic, weak, and strong nuclear interactions, has a more general form than that given in the so-called rigid, local isospace.

We shall not simply describe our theory in terms of a Lagrangian functional due to our unease with the Lagrangian approach (despite its versatility) as a truly fundamental physical approach towards unification. While the meaning of a particular energy functional (to be extremized) is clear in Newtonian physics, in present-day space-time physics the choice of a Lagrangian functional often appears to be non-unique (as it may be concocted arbitrarily) and hence devoid of straightforward, intuitive physical meaning. We shall instead, as in our previous works [1–3], build the edifice of our unified field theory by carefully determining the explicit form of the connection.

2 The determination of the explicit form of the connection for the unification of the gravitational, electromagnetic, and material spin fields

We shall work in an affine-metric space-time manifold $\mathbb{S}_4$ (with coordinates x^μ) endowed with both curvature and torsion. As usual, if we denote the symmetric, non-singular, fundamental metric tensor of $\mathbb{S}_4$ by g, then $g_{\mu\lambda}g^{\nu\lambda}=\delta^\nu_\mu$, where δ is the Kronecker delta. The world-line s is then given by the quadratic differential form $ds^2=g_{\mu\nu}\,dx^\mu dx^v$. (The Einstein summation convention employed throughout this work.)

As in [1], for reasons that will be clear later, we define the electromagnetic field tensor F via the torsion tensor of space-time (the anti-symmetric part of the connection Γ) as follows:

$$F_{\mu\nu}=2\,\frac{mc^2}{e}\,\Gamma^\lambda_{[\mu\nu]}u_\lambda\,,$$

where m is the mass (of the electron), c is the speed of light in vacuum, and e is the electric charge, and where $u^\mu=\frac{dx^\mu}{ds}$ are the components of the tangent world-velocity vector whose magnitude is unity. Solving for the torsion tensor, we may write, under very general conditions,

$$\Gamma^\lambda_{[\mu\nu]}=\frac{e}{2mc^2}\,F_{\mu\nu}u^\lambda+S^\lambda{}_{\mu\nu}\,,$$

where the components of the third-rank material spin (chirality) tensor ${}^3S$ are herein given via the second-rank anti-symmetric tensor ${}^2S$ as follows:

$$S^\lambda{}_{\mu\nu}=S^\lambda{}_\mu u_\nu-S^\lambda{}_\nu u_\mu\,.$$

As can be seen, it is necessary that we specify the following orthogonality condition:

$$S_{\mu\nu}u^\nu=0\,,$$

such that

$$S^\lambda{}_{\mu\nu}u_\lambda=0\,.$$

We note that ${}^3S$ may be taken as the intrinsic angular momentum tensor for microscopic physical objects which may be seen as the points in the space-time continuum itself. This way, ${}^3S$ may be regarded as a microspin tensor describing the internal rotation of the space-time points themselves [2]. Alternatively, ${}^3S$ may be taken as being "purely material" (entirely non-electromagnetic).

The covariant derivative of an arbitrary tensor field T is given via the asymmetric connection Γ by

$$\nabla_\lambda T^{\mu\nu\dots}_{\rho\sigma\dots}=\partial_\lambda T^{\mu\nu\dots}_{\rho\sigma\dots}+\Gamma^\mu_{\alpha\lambda}T^{\alpha\nu\dots}_{\rho\sigma\dots}+\Gamma^\nu_{\alpha\lambda}T^{\mu\alpha\dots}_{\rho\sigma\dots}+\dots-$$
$$-\;\Gamma^\alpha_{\rho\lambda}T^{\mu\nu\dots}_{\alpha\sigma\dots}-\Gamma^\alpha_{\sigma\lambda}T^{\mu\nu\dots}_{\rho\alpha\dots}-\dots\,,$$

where $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$. Then, as usual, the metricity condition $\nabla_\lambda g_{\mu\nu} = 0$, or, equivalently, $\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}$ (where $\Gamma_{\mu\nu\lambda} = g_{\mu\rho}\Gamma^{\rho}_{\nu\lambda}$), gives us the relation

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \Gamma^{\lambda}_{[\mu\nu]} - \\ - g^{\lambda\rho}\left(g_{\mu\sigma}\Gamma^{\sigma}_{[\rho\nu]} + g_{\nu\sigma}\Gamma^{\sigma}_{[\rho\mu]}\right).$$

Hence we obtain, for the connection of our unified field theory, the following explicit form:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \\ + \frac{e}{2mc^2}\left(F_{\mu\nu}u^\lambda - F^{\lambda}{}_{\mu}u_\nu - F^{\lambda}{}_{\nu}u_\mu\right) + \\ + S^{\lambda}{}_{\mu\nu} - g^{\lambda\rho}\left(S_{\mu\rho\nu} + S_{\nu\rho\mu}\right),$$

where

$$\Delta^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right)$$

are the components of the usual symmetric Levi-Civita connection, and where

$$K^{\lambda}_{\mu\nu} = \frac{e}{2mc^2}\left(F_{\mu\nu}u^\lambda - F^{\lambda}{}_{\mu}u_\nu - F^{\lambda}{}_{\nu}u_\mu\right) + S^{\lambda}{}_{\mu\nu} - \\ - g^{\lambda\rho}\left(S_{\mu\rho\nu} + S_{\nu\rho\mu}\right)$$

are the components of the contorsion tensor in our unified field theory.

The above expression for the connection can actually be written alternatively in a somewhat simpler form as follows:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \\ + \frac{e}{2mc^2}\left(F_{\mu\nu}u^\lambda - F^{\lambda}{}_{\mu}u_\nu - F^{\lambda}{}_{\nu}u_\mu\right) + 2S^{\lambda}{}_{\mu}u_\nu\,.$$

At this point, we see that the geometric structure of our space-time continuum is also determined by the electromagnetic field tensor as well as the material spin tensor, in addition to the gravitational (metrical) field.

As a consequence, we obtain the following relations (where the round brackets on indices, in contrast to the square ones, indicate symmetrization):

$$\Gamma^{\lambda}_{(\mu\nu)} = \Delta^{\lambda}_{\mu\nu} - \frac{e}{2mc^2}\left(F^{\lambda}{}_{\mu}u_\nu + F^{\lambda}{}_{\nu}u_\mu\right) + S^{\lambda}{}_{\mu}u_\nu + S^{\lambda}{}_{\nu}u_\mu,$$

$$\varphi_\mu = K^{\lambda}_{\mu\lambda} = 2\Gamma^{\lambda}_{[\mu\lambda]} = \frac{e}{mc^2}F_{\mu\lambda}u^\lambda.$$

We also have

$$\gamma_\mu = \Gamma^{\lambda}_{\mu\lambda} = \Delta^{\lambda}_{\mu\lambda} + \frac{e}{mc^2}F_{\mu\lambda}u^\lambda,$$

in addition to the usual relation

$$\Gamma^{\lambda}_{\lambda\mu} = \Delta^{\lambda}_{\lambda\mu} = \partial_\mu\left(\ln\sqrt{\det(g)}\right).$$

At this point, we may note that the spin vector φ is always orthogonal to the world-velocity vector as

$$\varphi_\mu u^\mu = 0\,.$$

In terms of the four-potential A, if we take the electromagnetic field tensor to be a pure curl as follows:

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu = \bar{\nabla}_\nu A_\mu - \bar{\nabla}_\mu A_\nu\,,$$

where $\bar{\nabla}$ represents the covariant derivative with respect to the symmetric Levi-Civita connection alone, then we have the following general identities:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \bar{\nabla}_\lambda F_{\mu\nu} + \bar{\nabla}_\mu F_{\nu\lambda} + \bar{\nabla}_\nu F_{\lambda\mu} = 0,$$

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = \\ = -2\left(\Gamma^{\rho}_{[\mu\nu]}F_{\lambda\rho} + \Gamma^{\rho}_{[\nu\lambda]}F_{\mu\rho} + +\Gamma^{\rho}_{[\lambda\mu]}F_{\nu\rho}\right).$$

The electromagnetic current density vector is then given by

$$J^\mu = -\frac{c}{4\pi}\nabla_\nu F^{\mu\nu}.$$

Its fully covariant divergence is then given by

$$\nabla_\mu J^\mu = -\frac{c}{4\pi}\nabla_\mu\left(\Gamma^{\mu}_{[\rho\sigma]}F^{\rho\sigma}\right).$$

If we further take $J^\mu = \rho_{em}u^\mu$, where ρ_{em} represents the electromagnetic charge density (taking into account the possibility of a magnetic charge), we see immediately that our electromagnetic current is conserved if and only if $\bar{\nabla}_\mu J^\mu = 0$, as follows

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^{\mu}_{\lambda\mu}J^\lambda = \\ = \bar{\nabla}_\mu J^\mu + \frac{e}{mc^2}F_{\lambda\mu}J^\lambda u^\lambda = \bar{\nabla}_\mu J^\mu.$$

In other words, for the electromagnetic current density to be conserved in our theory, the following conditions must be satisfied (for an arbitrary scalar field Φ):

$$J^\mu = -\frac{c}{4\pi}\Gamma^{\mu}_{[\rho\sigma]}F^{\rho\sigma},$$

$$\Gamma^{\lambda}_{[\mu\nu]} = \delta^{\lambda}_{\mu}\partial_\nu\Phi - \delta^{\lambda}_{\nu}\partial_\mu\Phi\,.$$

These relations are reminiscent of those in [1]. Note that we have made use of the relation $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\,\Phi = 2\Gamma^{\lambda}_{[\mu\nu]}\nabla_\lambda\Phi$.

Now, corresponding to our desired conservation law for electromagnetic currents, we can alternatively express the connection as

$$\Gamma^{\lambda}_{\mu\nu} = \Delta^{\lambda}_{\mu\nu} + 2\left(g^{\lambda\rho}g_{\mu\nu}\partial_\rho\Phi - \delta^{\lambda}_{\nu}\partial_\mu\Phi\right).$$

Contracting the above relation, we obtain the simple relation $\Gamma^{\lambda}_{\mu\lambda} = \Delta^{\lambda}_{\mu\lambda} - 6\partial_\mu\Phi$. On the other hand, we also have the relation $\Gamma^{\lambda}_{\mu\lambda} = \Delta^{\lambda}_{\mu\lambda} + \frac{e}{mc^2}F_{\mu\lambda}u^\lambda$. Hence we see that Φ

is a constant of motion as

$$\partial_\mu \Phi = -\frac{e}{6mc^2} F_{\mu\nu} u^\nu ,$$

$$\frac{d\Phi}{ds} = 0 .$$

These two conditions uniquely determine the conservation of electromagnetic currents in our theory.

Furthermore, not allowing for external forces, the geodesic equation of motion in $\mathbb{S}_4$, namely,

$$\frac{Du^\mu}{Ds} = u^\nu \nabla_\nu u^\mu = 0 ,$$

must hold in $\mathbb{S}_4$ in order for the gravitational, electromagnetic, and material spin fields to be genuine intrinsic geometric objects that uniquely and completely build the structure of the space-time continuum.

Recalling the relation $\Gamma^\lambda_{(\mu\nu)} = \Delta^\lambda_{\mu\nu} - \frac{e}{2mc^2}\left(F^\lambda{}_\mu u_\nu + F^\lambda{}_\nu u_\mu\right) + S^\lambda{}_\mu u_\nu + S^\lambda{}_\nu u_\mu$, we obtain the equation of motion

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = \frac{e}{mc^2} F^\mu{}_\nu u^\nu ,$$

which is none other than the equation of motion for a charged particle moving in a gravitational field. This simply means that our relation $F_{\mu\nu} = 2\frac{mc^2}{e}\Gamma^\lambda_{[\mu\nu]} u_\lambda$ does indeed indicate a valid geometrization of the electromagnetic field.

In the case of conserved electromagnetic currents, we have

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = -6\, g^{\mu\nu} \partial_\nu \Phi.$$

3 The field equations of the unified field theory

The (intrinsic) curvature tensor R of $\mathbb{S}_4$ is of course given by the usual relation

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\, V_\lambda = R^\rho{}_{\lambda\mu\nu} V_\rho - 2\Gamma^\rho_{[\mu\nu]} \nabla_\rho V_\lambda ,$$

where V is an arbitrary vector field. For an arbitrary tensor field T, we have the more general relation

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\, T^{\alpha\beta\ldots}_{\rho\sigma\ldots} = R^\lambda{}_{\rho\mu\nu} T^{\alpha\beta\ldots}_{\lambda\sigma\ldots} + R^\lambda{}_{\sigma\mu\nu} T^{\alpha\beta\ldots}_{\rho\lambda\ldots} + \ldots - R^\alpha{}_{\lambda\mu\nu} T^{\lambda\beta\ldots}_{\rho\sigma\ldots} - R^\beta{}_{\lambda\mu\nu} T^{\alpha\lambda\ldots}_{\rho\sigma\ldots} - \ldots - 2\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda T^{\alpha\beta\ldots}_{\rho\sigma\ldots} .$$

Of course,

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} .$$

If we define the following contractions:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} ,$$

$$R = R^\mu{}_\mu ,$$

then, as usual,

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma}\right) + \frac{1}{6}\left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}\right) R ,$$

where C is the Weyl tensor. Note that the generalized Ricci tensor (given by its components $R_{\mu\nu}$) is generally asymmetric.

Let us denote the usual Riemann-Christoffel curvature tensor by $\bar{R}$, i.e.,

$$\bar{R}^\rho{}_{\lambda\mu\nu} = \partial_\mu \Delta^\rho_{\lambda\nu} - \partial_\nu \Delta^\rho_{\lambda\mu} + \Delta^\sigma_{\lambda\nu}\Delta^\rho_{\sigma\mu} - \Delta^\sigma_{\lambda\mu}\Delta^\rho_{\sigma\nu}.$$

The symmetric Ricci tensor and the Ricci scalar are then given respectively by $\bar{R}_{\mu\nu} = \bar{R}^\lambda{}_{\mu\lambda\nu}$ and $\bar{R} = \bar{R}^\mu{}_\mu$.

Furthermore, we obtain the following decomposition:

$$R^\rho{}_{\lambda\mu\nu} = \bar{R}^\rho{}_{\lambda\mu\nu} + \bar{\nabla}_\mu K^\rho_{\lambda\nu} - \bar{\nabla}_\nu K^\rho_{\lambda\mu} + K^\sigma_{\lambda\nu} K^\rho_{\sigma\mu} - K^\sigma_{\lambda\mu} K^\rho_{\sigma\nu}.$$

Hence, recalling that $\varphi_\mu = K^\lambda_{\mu\lambda} = 2\Gamma^\lambda_{[\mu\lambda]}$, we obtain

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\lambda K^\lambda_{\mu\nu} - K^\lambda_{\mu\rho} K^\rho_{\lambda\nu} - \bar{\nabla}_\nu \varphi_\mu + 2K^\lambda_{\mu\nu}\varphi_\lambda ,$$

$$R = \bar{R} - 2\bar{\nabla}_\mu \varphi^\mu - \varphi_\mu \varphi^\mu - K_{\mu\nu\lambda} K^{\mu\lambda\nu}.$$

We then obtain the following generalized Bianchi identities:

$$R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\mu\nu\lambda} + R^\rho{}_{\nu\lambda\mu} = -2(\partial_\nu \Gamma^\rho_{[\lambda\mu]} + \partial_\lambda \Gamma^\rho_{[\mu\nu]} + \partial_\mu \Gamma^\rho_{[\nu\lambda]} + \Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{[\mu\nu]} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{[\nu\lambda]} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{[\lambda\mu]}) ,$$

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\lambda} + \nabla_\sigma R_{\mu\nu\lambda\rho} = 2\left(\Gamma^\alpha_{[\rho\sigma]} R_{\mu\nu\alpha\lambda} + \Gamma^\alpha_{[\sigma\lambda]} R_{\mu\nu\alpha\rho} + \Gamma^\alpha_{[\lambda\rho]} R_{\mu\nu\alpha\sigma}\right) ,$$

$$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 2g^{\mu\nu}\Gamma^\lambda_{[\rho\mu]} R^\rho{}_\lambda + \Gamma^\lambda_{[\rho\sigma]} R^{\rho\sigma\nu}{}_\lambda ,$$

in addition to the standard Bianchi identities

$$\bar{R}^\rho{}_{\lambda\mu\nu} + \bar{R}^\rho{}_{\mu\nu\lambda} + \bar{R}^\rho{}_{\nu\lambda\mu} = 0 ,$$

$$\bar{\nabla}_\lambda \bar{R}_{\mu\nu\rho\sigma} + \bar{\nabla}_\rho \bar{R}_{\mu\nu\sigma\lambda} + \bar{\nabla}_\sigma \bar{R}_{\mu\nu\lambda\rho} = 0 ,$$

$$\bar{\nabla}_\mu\left(\bar{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \bar{R}\right) = 0 .$$

(See [2–4] for instance.)

Furthermore, we can now obtain the following explicit expression for the curvature tensor R:

$$\begin{aligned}
R^\rho{}_{\lambda\mu\nu} &= \partial_\mu \Delta^\rho_{\lambda\nu} - \partial_\nu \Delta^\rho_{\lambda\mu} + \Delta^\sigma_{\lambda\nu}\Delta^\rho_{\sigma\mu} - \Delta^\sigma_{\lambda\mu}\Delta^\rho_{\sigma\nu} + \\
&+ \frac{e}{2mc^2}\Big\{\left(\partial_\mu F_{\lambda\nu} - \partial_\nu F_{\lambda\mu}\right) u^\rho + \left(\partial_\nu F^\rho{}_\mu - \partial_\mu F^\rho{}_\nu\right) u_\lambda + \\
&+ u_\mu \partial_\nu F^\rho{}_\lambda - u_\nu \partial_\mu F^\rho{}_\lambda + F_{\lambda\nu}\partial_\mu u^\rho - F_{\lambda\mu}\partial_\nu u^\rho + \\
&+ F^\rho{}_\mu \partial_\nu u_\lambda - F^\rho{}_\nu \partial_\mu u_\lambda + \left(\partial_\nu u_\mu - \partial_\mu u_\nu\right) F^\rho{}_\lambda + \\
&+ \left(F_{\lambda\nu} u^\sigma - F^\sigma{}_\lambda u_\nu - F^\sigma{}_\nu u_\lambda\right)\Delta^\rho_{\sigma\mu} + \left(F_{\sigma\mu} u^\rho - F^\rho{}_\sigma u_\mu - \right. \\
&\left. - F^\rho{}_\mu u_\sigma\right)\Delta^\sigma_{\lambda\nu} - \left(F_{\lambda\mu} u^\sigma - F^\sigma{}_\lambda u_\mu - F^\sigma{}_\mu u_\lambda\right)\Delta^\rho_{\sigma\nu} -
\end{aligned}$$

$$- \left(F_{\sigma\nu} u^\rho - F^\rho_{\ \sigma} u_\nu - F^\rho_{\ \nu} u_\sigma\right) \Delta^\sigma_{\lambda\mu} + \frac{e}{2\,m\,c^2} \left(F_{\lambda\nu} F_{\sigma\mu} - \right.$$
$$\left. - F_{\lambda\mu} F_{\sigma\nu}\right) u^\sigma u^\rho + \frac{e}{2\,m\,c^2}\left(F_{\lambda\mu} F^\rho_{\ \sigma} - F_{\lambda\sigma} F^\rho_{\ \mu}\right) u_\nu u^\sigma +$$
$$+ \frac{e}{2\,m\,c^2}\left(F_{\lambda\sigma} F^\rho_{\ \nu} - F_{\lambda\nu} F^\rho_{\ \sigma}\right) u_\mu u^\sigma + \frac{e}{2mc^2}\left(F^\sigma_{\ \mu} F_{\sigma\nu} - \right.$$
$$\left. - F^\sigma_{\ \nu} F_{\sigma\mu}\right) u_\lambda u^\rho + \frac{e}{2mc^2}\left(F_{\sigma\nu} F^\rho_{\ \mu} - F_{\sigma\mu} F^\rho_{\ \nu}\right) u_\lambda u^\sigma +$$
$$+ \frac{e}{2mc^2} F^\sigma_{\ \lambda} F_{\sigma\nu} u_\mu u^\rho + \frac{e}{2mc^2} F^\sigma_{\ \nu} F^\rho_{\ \sigma} u_\lambda u_\mu -$$
$$- \frac{e}{2mc^2} F^\sigma_{\ \lambda} F_{\sigma\mu} u_\nu u^\rho - \frac{e}{2mc^2} F^\sigma_{\ \mu} F^\rho_{\ \sigma} u_\lambda u_\nu +$$
$$+ \frac{e}{2mc^2}\left(F_{\lambda\mu} F^\rho_{\ \nu} - F_{\lambda\nu} F^\rho_{\ \mu}\right)\Big\} + \Omega^\rho_{\ \lambda\mu\nu}\,,$$

where the tensor Ω consists of the remaining terms containing the material spin tensor ${}^2S$ (or ${}^3S$).

Now, keeping in mind that $\Gamma^\lambda_{(\mu\nu)} = \Delta^\lambda_{\mu\nu} - \frac{e}{2mc^2}\left(F^\lambda_{\ \mu} u_\nu + + F^\lambda_{\ \nu} u_\mu\right) + S^\lambda_{\ \mu} u_\nu + S^\lambda_{\ \nu} u_\mu$ and also $\gamma_\mu = \Gamma^\lambda_{\mu\lambda} = \Delta^\lambda_{\mu\lambda} + + \frac{e}{mc^2} F_{\mu\lambda} u^\lambda$, and decomposing the components of the generalized Ricci tensor as $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$, we see that

$$R_{(\mu\nu)} = \partial_\lambda \Gamma^\lambda_{(\mu\nu)} - \frac{1}{2}\left(\partial_\nu \gamma_\mu + \partial_\mu \gamma_\nu\right) + \Gamma^\lambda_{(\mu\nu)} \gamma_\lambda - $$
$$- \frac{1}{2}\left(\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\rho\nu} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}\right),$$

$$R_{[\mu\nu]} = \partial_\lambda \Gamma^\lambda_{[\mu\nu]} - \frac{1}{2}\left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right) + \Gamma^\lambda_{[\mu\nu]} \gamma_\lambda - $$
$$- \frac{1}{2}\left(\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}\right).$$

In particular, we note that

$$R_{[\mu\nu]} = \partial_\lambda \Gamma^\lambda_{[\mu\nu]} - \frac{1}{2}\left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right) + \Gamma^\lambda_{[\mu\nu]} \gamma_\lambda - $$
$$- \frac{1}{2}\left(\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}\right) =$$
$$= \partial_\lambda \Gamma^\lambda_{[\mu\nu]} + \Gamma^\rho_{\lambda\rho} \Gamma^\lambda_{[\mu\nu]} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{[\nu\rho]} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{[\mu\rho]} +$$
$$+ \frac{1}{2}\left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right) = \nabla_\lambda \Gamma^\lambda_{[\mu\nu]} + \frac{1}{2}\left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right).$$

Hence we obtain the relation

$$R_{[\mu\nu]} = \frac{e}{2mc^2}\left(F_{\mu\nu} \nabla_\lambda u^\lambda + \frac{DF_{\mu\nu}}{Ds}\right) + \nabla_\lambda S^\lambda_{\ \mu\nu} +$$
$$+ \frac{1}{2}\left(\partial_\nu \gamma_\mu - \partial_\mu \gamma_\nu\right),$$

where $\frac{DF_{\mu\nu}}{Ds} = u^\lambda \nabla_\lambda F_{\mu\nu}$. More explicitly, we can write

$$R_{[\mu\nu]} = \frac{e}{2mc^2}\left(F_{\mu\nu} \nabla_\lambda u^\lambda + \frac{DF_{\mu\nu}}{Ds} + \left(\partial_\nu F_{\lambda\mu} - \right.\right.$$
$$\left.\left. - \partial_\mu F_{\lambda\nu}\right) u^\lambda + F_{\lambda\mu} \partial_\nu u^\lambda - F_{\lambda\nu} \partial_\mu u^\lambda\right) + \nabla_\lambda S^\lambda_{\ \mu\nu}.$$

It is therefore seen that, in general, the special identity

$$\partial_\lambda R_{[\mu\nu]} + \partial_\mu R_{[\nu\lambda]} + \partial_\nu R_{[\lambda\mu]} = 0$$

holds only when $\nabla_\mu u^\mu = 0$, $\frac{DF_{\mu\nu}}{Ds} = 0$, and $\nabla_\lambda S^\lambda_{\ \mu\nu} = 0$.

We are now in a position to generalize Einstein's field equation in the standard theory of general relativity. The usual Einstein's field equation is of course given by

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{R} = kT_{\mu\nu}\,,$$
$$\bar{\nabla}_\mu \bar{G}^{\mu\nu} = 0\,,$$

where $\bar{G}$ is the symmetric Einstein tensor, T is the energy-momentum tensor, and $k = \pm \frac{8\pi G}{c^4}$ is Einstein's coupling constant in terms of the Newtonian gravitational constant G. Taking $c = 1$ for convenience, in the absence of pressure, traditionally we write

$$\bar{G}^{\mu\nu} = k\left(\rho_m u^\mu u^\nu + \frac{1}{4\pi}\left(F^\mu_{\ \lambda} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right)\right),$$

where ρ_m is the material density and where the second term on the right-hand-side of the equation is widely regarded as representing the electromagnetic energy-momentum tensor.

Now, with the generalized Bianchi identity for the electromagnetic field, i.e., $\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = = -2\left(\Gamma^\rho_{[\mu\nu]} F_{\lambda\rho} + \Gamma^\rho_{[\nu\lambda]} F_{\mu\rho} + \Gamma^\rho_{[\lambda\mu]} F_{\nu\rho}\right)$, at hand, and assuming the "isochoric" condition $\frac{D\rho_m}{Ds} = -\rho_m \nabla_\mu u^\mu = 0$ ($\rho_m \neq 0$), we obtain

$$\nabla_\nu \bar{G}^{\mu\nu} = k g^{\mu\nu}\left(\Gamma^\lambda_{[\rho\sigma]} F_{\nu\lambda} + \Gamma^\lambda_{[\sigma\nu]} F_{\rho\lambda} + \Gamma^\lambda_{[\nu\rho]} F_{\sigma\lambda}\right) F^{\rho\sigma}.$$

In other words,

$$\nabla_\nu \bar{G}^{\mu\nu} = k\left(2g^{\mu\nu} \Gamma^\lambda_{[\sigma\nu]} F_{\rho\lambda} F^{\rho\sigma} - \frac{1}{4\pi} F^\mu_{\ \nu} J^\nu\right).$$

This is our first generalization of the standard Einstein's field equation, following the traditional ad hoc way of arbitrarily adding the electromagnetic contribution to the purely material part of the energy-momentum tensor.

Now, more generally and more naturally, using the generalized Bianchi identity $\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = = 2g^{\mu\nu} \Gamma^\lambda_{[\rho\mu]} R^\rho_{\ \lambda} + \Gamma^\lambda_{[\rho\sigma]} R^{\rho\sigma\nu}_{\ \ \ \ \lambda}$, we can obtain the following fundamental relation:

$$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = \frac{e}{mc^2}\left(F_\mu^{\ \nu} R^\mu_{\ \lambda} + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u^\lambda +$$
$$+ 2S_{\mu\lambda}^{\ \ \ \nu} R^{\mu\lambda} + S^\lambda_{\ \mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\,.$$

Alternatively, we can also write this as

$$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = \frac{e}{mc^2}\left(F_\mu^{\ \nu} R^\mu_{\ \lambda} + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u^\lambda +$$
$$+ S_{\mu\rho} R^{\rho\mu} u^\nu - S^\lambda_{\ \rho} u_\mu R^{\mu\rho\nu}_{\ \ \ \ \lambda} + \left(S^\nu_{\ \mu} R^{\rho\mu} - S^\lambda_{\ \mu} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u_\rho\,.$$

Now, as a special consideration, let Σ be the "area" of a three-dimensional space-like hypersurface representing matter in $\mathbb{S}_4$. Then, if we make the following traditional choice for the third-rank material spin tensor ${}^3S$:

$$S^{\mu\nu\lambda} = \iiint_\Sigma \rho_m \left(x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} \right) d\Sigma \,,$$

where now T is the total asymmetric energy-momentum tensor in our theory, we see that, in the presence of matter, the condition $S^{\mu\nu\lambda} = 0$ implies that

$$T^{[\mu\nu]} = -\frac{1}{2} \left(x^\mu \nabla_\lambda T^{\lambda\nu} - x^\nu \nabla_\lambda T^{\lambda\mu} \right).$$

In this special case, we obtain the simplified expression

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \frac{e}{mc^2} \left(F_\mu{}^\nu R^\mu_{\ \lambda} + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}{}_\lambda \right) u^\lambda.$$

If we further assume that the sectional curvature $\Psi = \frac{1}{12} R$ of $\mathbb{S}_4$ is everywhere constant in a space-time region where the electromagnetic field (and hence the torsion) is absent, we may consider writing $R_{\mu\nu\rho\sigma} = \Psi \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$ such that $\mathbb{S}_4$ is conformally flat ($C_{\mu\nu\rho\sigma} = 0$), and hence $R_{\mu\nu} = 3\Psi g_{\mu\nu}$ and $R_{[\mu\nu]} = 0$. In this case, we are left with the simple expression

$$\nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = -\frac{eR}{6mc^2} F^\mu_{\ \nu} u^\nu.$$

This is equivalent to the equation of motion

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = -\frac{6}{R} \nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right).$$

4 The minimal Lagrangian density of the theory

Using the general results from the preceding section, we obtain

$$R = \bar{R} + \frac{e^2}{4m^2c^4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{m^2c^4} F_{\mu\lambda} F^\mu_{\ \nu} u^\lambda u^\nu - \frac{2e}{mc^2} \bar{\nabla}_\mu f^\mu + 2 S_{\mu\nu} S^{\mu\nu} - K_{\mu(\nu\lambda)} K^{\mu(\nu\lambda)} \,,$$

for the curvature scalar of $\mathbb{S}_4$. Here $f^\mu = F^\mu_{\ \nu} u^\nu$ can be said to be the components of the so-called Lorentz force.

Furthermore, we see that

$$K_{\mu(\nu\lambda)} K^{\mu(\nu\lambda)} = \frac{e^2}{m^2c^4} F_{\mu\nu} F^{\mu\nu} + 2 S_{\mu\nu} S^{\mu\nu} - \frac{2e}{mc^2} F_{\mu\nu} S^{\mu\nu} - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_{\ \nu} u^\lambda u^\nu.$$

Hence we obtain

$$R = \bar{R} - \frac{e^2}{2m^2c^4} F_{\mu\nu} F^{\mu\nu} - \frac{2e}{mc^2} \left(\bar{\nabla}_\mu f^\mu + F_{\mu\nu} S^{\mu\nu} \right) - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_{\ \nu} u^\lambda u^\nu.$$

The last two terms on the right-hand-side of the expression can then be grouped into a single scalar source as follows:

$$\phi = -\frac{2e}{mc^2} \left(\bar{\nabla}_\mu f^\mu + F_{\mu\nu} S^{\mu\nu} \right) - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_{\ \nu} u^\lambda u^\nu.$$

Assuming that ϕ accounts for both the total (material-electromagnetic) charge density as well as the total energy density, our unified field theory may be described by the following action integral (where the $L = R\sqrt{\det(g)}$ is the minimal Lagrangian density):

$$I = \iiiint R \sqrt{\det(g)}\, d^4x = \iiiint \left(\bar{R} - \frac{e^2}{2m^2c^4} F_{\mu\nu} F^{\mu\nu} + \phi \right) \sqrt{\det(g)}\, d^4x \,.$$

In this minimal fashion, gravity (described by $\bar{R}$) appears as an emergent phenomenon whose intrinsic nature is of electromagnetic and purely material origin since, in our theory, the electromagnetic and material spin fields are nothing but components of a single torsion field.

5 The non-Abelian Yang-Mills gauge field as a sub-torsion field in $\mathbb{S}_4$

In $\mathbb{S}_4$, let there exist a space-like three-dimensional hypersurface Θ_3, with local coordinates X^i (Latin indices shall run from 1 to 3). From the point of view of projective differential geometry alone, we may say that Θ_3i s embedded (immersed) in $\mathbb{S}_4$. Then, the tetrad linking the embedded space Θ_3 to the enveloping space-time $\mathbb{S}_4$ is readily given by

$$\omega^i_\mu = \frac{\partial X^i}{\partial x^\mu} \,, \qquad \omega^\mu_i = \left(\omega^i_\mu \right)^{-1} = \frac{\partial x^\mu}{\partial X^i} \,.$$

Furthermore, let N be a unit vector normal to the hypersurface Θ_3. We may write the parametric equation of the hypersurface Θ_3 as $H(x^\mu, d) = 0$, where d is constant. Hence

$$N^\mu = \frac{g^{\mu\nu} \partial_\nu H}{\sqrt{g^{\alpha\beta} (\partial_\alpha H)(\partial_\beta H)}} \,,$$

$$N_\mu N^\mu = 1 \,.$$

In terms of the axial unit vectors a, b, and c spanning the hypersurface Θ_3, we may write

$$N_\mu = \frac{\varepsilon_{\mu\nu\rho\sigma} a^\nu b^\rho c^\sigma}{\varepsilon_{\alpha\beta\lambda\eta} N^\alpha a^\beta b^\lambda c^\eta} \,,$$

where $\varepsilon_{\mu\nu\rho\sigma}$ are the components of the completely anti-symmetric four-dimensional Levi-Civita permutation tensor density.

Now, the tetrad satisfies the following projective relations:

$$\omega^i_\mu N^\mu = 0 \,, \qquad \omega^i_\mu \omega^\mu_k = \delta^i_k \,,$$

$$\omega^\mu_i \omega^i_\nu = \delta^\mu_\nu - N^\mu N^\nu.$$

If we denote the local metric tensor of Θ_3 by h, we obtain the following relations:

$$h_{ik} = \omega^\mu_i \omega^\nu_k g_{\mu\nu} ,$$

$$g_{\mu\nu} = \omega^i_\mu \omega^k_\nu h_{ik} + N_\mu N_\nu .$$

Furthermore, in the hypersurface Θ_3, let us set $\nabla_i = \omega^\mu_i \nabla_\mu$ and $\partial_i = \frac{\partial}{\partial X^i} = \omega^\mu_i \partial_\mu$. Then we have the following fundamental expressions:

$$\nabla_\nu \omega^i_\mu = Z^i{}_k \omega^k_\nu N_\mu = \partial_\nu \omega^i_\mu - \omega^i_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^i_{kl} \omega^k_\mu \omega^l_\nu ,$$

$$\nabla_k \omega^\mu_i = Z_{ik} N^\mu = \partial_k \omega^\mu_i - \omega^\mu_p \Gamma^p_{ik} + \Gamma^\mu_{\rho\sigma} \omega^\rho_i \omega^\sigma_k ,$$

$$\omega^\mu_i \nabla_\nu \omega^k_\mu = 0 ,$$

$$\nabla_i N^\mu = -Z^k{}_i \omega^\mu_k ,$$

where Z is the extrinsic curvature tensor of the hypersurface Θ_3, which is generally asymmetric in our theory.

The connection of the hypersurface Θ_3 is linked to that of the space-time $\mathbb{S}_4$ via

$$\Gamma^p_{ik} = \omega^p_\mu \partial_k \omega^\mu_i + \omega^p_\lambda \Gamma^\lambda_{\mu\nu} \omega^\mu_i \omega^\nu_k .$$

After some algebra, we obtain

$$\Gamma^\lambda_{\mu\nu} = \omega^\lambda_i \partial_\nu \omega^i_\mu + \omega^\lambda_p \Gamma^p_{ik} \omega^i_\mu \omega^k_\nu + N^\lambda \partial_\nu N_\mu + N^\lambda Z_{ik} \omega^i_\mu \omega^k_\nu - N_\mu Z^i{}_k \omega^\lambda_i \omega^k_\nu .$$

The fundamental geometric relations describing our embedding theory are then given by the following expressions (see [4] for instance):

$$R_{ijkl} = Z_{ik} Z_{jl} - Z_{il} Z_{jk} + R_{\mu\nu\rho\sigma} \omega^\mu_i \omega^\nu_j \omega^\rho_k \omega^\sigma_l - \omega^\mu_i \Lambda_{\mu jkl} ,$$

$$\nabla_l Z_{ik} - \nabla_k Z_{il} = -R_{\mu\nu\rho\sigma} N^\mu \omega^\nu_i \omega^\rho_k \omega^\sigma_l - 2\Gamma^p_{[kl]} Z_{ip} + N^\mu \Lambda_{\mu ikl} ,$$

$$\Lambda^\mu{}_{ijk} = \left(\partial_k \partial_j - \partial_j \partial_k\right) \omega^\mu_i + \omega^\sigma_i \Gamma^\mu_{\sigma\rho} \left(\partial_k \omega^\rho_j - \partial_j \omega^\rho_k\right) .$$

Actually, these relations are just manifestations of the following single expression:

$$\left(\nabla_k \nabla_j - \nabla_j \nabla_k\right) \omega^\mu_i = R^p{}_{ijk} \omega^\mu_p - R^\mu{}_{\nu\rho\sigma} \omega^\nu_i \omega^\rho_j \omega^\sigma_k - 2\Gamma^p_{[jk]} Z_{ip} N^\mu + \Lambda^\mu{}_{ijk} .$$

We may note that $\Gamma^p_{[ik]}$ and

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^p_{jl} \Gamma^i_{pk} - \Gamma^p_{jk} \Gamma^i_{pl}$$

are the components of the torsion tensor and the intrinsic curvature tensor of the hypersurface Θ_3, respectively.

Now, let us observe that

$$\partial_\nu \omega^i_\mu - \partial_\mu \omega^i_\nu = 2 \left(\omega^i_\lambda \Gamma^\lambda_{[\mu\nu]} - \Gamma^i_{[kl]} \omega^k_\mu \omega^l_\nu + Z^i{}_k \omega^k_{[\nu} N_{\mu]}\right) .$$

Hence letting

$$F^i_{\mu\nu} = 2\omega^i_\lambda \Gamma^\lambda_{[\mu\nu]} ,$$

we arrive at the expression

$$F^i_{\mu\nu} = \partial_\nu \omega^i_\mu - \partial_\mu \omega^i_\nu + 2\Gamma^i_{[kl]} \omega^k_\mu \omega^l_\nu + 2 Z^i{}_k \omega^k_{[\mu} N_{\nu]} .$$

In addition, we also see that

$$\Gamma^i_{[kl]} = \frac{1}{2}\, \omega^\mu_k \omega^\nu_l F^i_{\mu\nu} - \frac{1}{2}\, \omega^\mu_k \omega^\nu_l \left(\partial_\nu \omega^i_\mu - \partial_\mu \omega^i_\nu\right) .$$

Now, with respect to the local coordinate transformation given by $X^i = X^i\left(\bar{X}^A\right)$ in Θ_3, let us invoke the following Cartan-Lie algebra:

$$\left[e_i, e_k\right] = e_i \otimes e_k - e_k \otimes e_i = C^p_{ik} e_p ,$$

$$C_{ikl} = h_{ip} C^p_{kl} = -2\Gamma_{i[kl]} = -\, i\hat{g} \in_{ikl} ,$$

where $e_i = e^A_i \frac{\partial}{\partial \bar{X}^A}$ are the elements of the basis vector spanning Θ_3, C^p_{ik} are the spin coefficients, $i = \sqrt{-1}$, $\hat{g}$ is a coupling constant, and $\in_{ikl} = \sqrt{\det(h)}\,\varepsilon_{ikl}$ (where ε_{ikl} are the components of the completely anti-symmetric three-dimensional Levi-Civita permutation tensor density).

Hence we obtain

$$F^i_{\mu\nu} = \partial_\nu \omega^i_\mu - \partial_\mu \omega^i_\nu + i\hat{g} \in^i{}_{kl}\, \omega^k_\mu \omega^l_\nu + 2 Z^i{}_k \omega^k_{[\mu} N_{\nu]} .$$

At this point, our key insight is to define the gauge field potential as the tetrad itself, i.e.,

$$B^i_\mu = \omega^i_\mu .$$

Hence, at last, we arrive at the following important expression:

$$F^i_{\mu\nu} = \partial_\nu B^i_\mu - \partial_\mu B^i_\nu + i\hat{g} \in^i{}_{kl}\, B^k_\mu B^l_\nu + 2 Z^i{}_k B^k_{[\mu} N_{\nu]} .$$

Clearly, $F^i_{\mu\nu}$ are the components of the generalized Yang-Mills gauge field strength. To show this, consider the hypersurface $\mathbb{E}_3$ of rigid frames (where the metric tensor is strictly constant) which is a reduction (or, in a way, local infinitesimal representation) of the more general hypersurface Θ_3. We shall call this an "isospace". In it, we have

$$h_{ik} = \delta_{ik} ,$$

$$\det(h) = 1 ,$$

$$\Gamma^i_{kl} = \Gamma_{ikl} = \Gamma_{i[kl]} - \Gamma_{l[ik]} - \Gamma_{k[il]} = \frac{1}{2}\, i\hat{g}\varepsilon_{ikl} ,$$

$$Z_{ik} = 0 .$$

Hence we arrive at the familiar expression

$$F_{i\mu\nu} = \partial_\nu B_{\mu i} - \partial_\mu B_{\nu i} + i\hat{g}\varepsilon_{ikl} B_{\mu k} B_{\nu l} .$$

In other words, setting $\vec{F}_{\mu\nu} = F_{i\mu\nu} e_i$ and $\vec{B}_\mu = B_{\mu i} e_i$, we obtain

$$\vec{F}_{\mu\nu} = \partial_\nu \vec{B}_\mu - \partial_\mu \vec{B}_\nu - \left[\vec{B}_\mu, \vec{B}_\nu\right] .$$

Finally, let us define the gauge field potential of the second kind via

$$\omega_{\mu ik} = \varepsilon_{ikp} B^p_\mu ,$$

such that

$$B^i_\mu = \frac{1}{2}\,\varepsilon_{ikl}\,\omega_{\mu kl}\,.$$

Let us then define the gauge field strength of the second kind via

$$R_{ik\mu\nu} = \in_{ikp}\, F^p_{\mu\nu}\,,$$

such that

$$F^p_{\mu\nu} = \frac{1}{2}\, \in^{pik}\, R_{ik\mu\nu}\,.$$

Hence we obtain the general expression

$$\begin{aligned} R_{ik\mu\nu} &= i\hat{g}\,\sqrt{\det(h)}\Big\{\partial_\nu\omega_{\mu ik} - \partial_\mu\omega_{\nu ik} + \\ &+ \frac{1}{\sqrt{\det(h)}}\left(\omega_{\mu ip}\omega_{\nu kp} - \omega_{\mu kp}\omega_{\nu ip}\right)\Big\} + \\ &+ \sqrt{\det(h)}\;\varepsilon_{ikp}\,Z^p_r\,B^r_{[\mu}\,N_{\nu]}\,. \end{aligned}$$

We may regard the object given by this expression as the curvature of the local gauge spin connection of the hypersurface Θ_3.

Again, if we refer this to the isospace $\mathbb{E}_3$ instead of the more general hypersurface Θ_3, we arrive at the familiar relation

$$R_{ik\mu\nu} = i\hat{g}\left(\partial_\nu\omega_{\mu ik} - \partial_\mu\omega_{\nu ik} + \omega_{\mu ip}\omega_{\nu kp} - \omega_{\mu kp}\omega_{\nu ip}\right).$$

6 Conclusion

We have just completed our program of building the structure of a unified field theory in which gravity, electromagnetism, material spin, and the non-Abelian Yang-Mills gauge field (which is also capable of describing the weak force in the standard model particle physics) are all geometrized only in four dimensions. As we have seen, we have also generalized the expression for the Yang-Mills gauge field strength.

In our theory, the (generalized) Yang-Mills gauge field strength is linked to the electromagnetic field tensor via the relation

$$F_{\mu\nu} = 2\frac{m\,c^2}{e}\,\Gamma^\lambda_{[\mu\nu]}\,u_\lambda = \frac{m\,c^2}{e}\,F^i_{\mu\nu}\,u_i\,,$$

where $u^i = \omega^i_\mu u^\mu$. This enables us to express the connection in terms of the Yang-Mills gauge field strength instead of the electromagnetic field tensor as follows:

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\,g^{\lambda\rho}\left(\partial_\nu\,g_{\rho\mu} - \partial_\rho\,g_{\mu\nu} + \partial_\mu\,g_{\nu\rho}\right) + \frac{1}{2}\,u_i\left(F^i_{\mu\nu}\,u^\lambda - \right. \\ &\left. - F^{i\lambda}{}_\mu\,u_\nu - F^{i\lambda}{}_\nu\,u_\mu\right) + S^\lambda{}_{\mu\nu} - g^{\lambda\rho}\left(S_{\mu\rho\nu} + S_{\nu\rho\mu}\right), \end{aligned}$$

i.e., the Yang-Mills gauge field is nothing but a sub-torsion field in the space-time manifold $\mathbb{S}_4$.

The results which we have obtained in this work may subsequently be quantized simply by following the method given in our previous work [1] since, in a sense, the present work is but a further in-depth classical consideration of the fundamental method of geometrization outlined in the previous theory.

Dedication

I dedicate this work to my patron and source of inspiration, Albert Einstein (1879–1955), from whose passion for the search of the ultimate physical truth I have learned something truly fundamental of the meaning of being a true scientist and independent, original thinker, even amidst the adversities often imposed upon him by the world and its act of scientific institutionalization.

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