SPECIAL REPORT

On a Geometric Theory of Generalized Chiral Elasticity with Discontinuities

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In this work we develop, in a somewhat extensive manner, a geometric theory of chiral elasticity which in general is endowed with *geometric discontinuities* (sometimes referred to as *defects*). By itself, the present theory generalizes both Cosserat and void elasticity theories to a certain extent via *geometrization* as well as by taking into account the action of the electromagnetic field, i.e., the incorporation of the electromagnetic field into the description of the so-called *microspin* (*chirality*) also forms the underlying structure of this work. As we know, the description of the electromagnetic field as a unified phenomenon requires four-dimensional space-time rather than three-dimensional space as its background. For this reason we embed the three-dimensional material space in four-dimensional space-time. This way, the electromagnetic spin is coupled to the non-electromagnetic microspin, both being parts of the complete microspin to be added to the macrospin in the full description of vorticity. In short, our objective is to generalize the existing continuum theories by especially describing microspin phenomena in a fully geometric way.

1 Introduction

Although numerous generalizations of the classical theory of elasticity have been constructed (most notably, perhaps, is the so-called Cosserat elasticity theory) in the course of its development, we are somewhat of the opinion that these generalizations simply lack geometric structure. In these existing theories, the introduced quantities supposedly describing microspin and irregularities (such as voids and cracks) seem to have been assumed from *without*, rather than from *within*. By our geometrization of microspin phenomena we mean exactly the description of microspin phenomena in terms of intrinsic geometric quantities of the material body such as its curvature and torsion. In this framework, we produce the microspin tensor and the anti-symmetric part of the stress tensor as intrinsic geometric objects rather than alien additions to the framework of classical elasticity theory. As such, the initial microspin variables are not to be freely chosen to be included in the potential energy functional as is often the case, but rather, at first we identify them with the internal properties of the geometry of the material body. In other words, we can not simply adhere to the simple way of adding external variables that are supposed to describe microspin and defects to those original variables of the classical elasticity theory in the construction of the potential energy functional without first discovering and unfolding their underlying internal geometric existence.

Since in this work we are largely concerned with the behavior of *material points* such as their translational and rotational motion, we need to primarily cast the field equations in a manifestly covariant form of the *Lagrangian* system of material coordinates attached to the material body. Due to the presence of geometric discontinuities (*geometric singularities*) and the *local non-orientability* of the material points, the full Lagrangian description is necessary. In other words, the compatibility between the spatial (*Eulerian*) and the material coordinate systems can not in general be directly invoked. This is because the smooth transitional transformation from the Lagrangian to the Eulerian descriptions and vice versa breaks down when geometric singularities and the non-orientability of the material points are taken into account. However, for the sake of accommodating the existence of all imaginable systems of coordinates, we shall assume, at least locally, that the material space lies within the three-dimensional space of spatial (Eulerian) coordinates, which can be seen as a (flat) hypersurface embedded in four-dimensional space-time. With respect to this embedding situation, we preserve the correspondence between the material and spatial coordinate systems in classical continuum mechanics, although not their equality since the field equations defined in the space of material points are in general not independent of the orientation of that local system of coordinates.

At present, due to the limits of space, we shall concentrate ourselves merely on the construction of the field equations of our geometric theory, from which the equations of motion shall follow. We shall not concern ourselves with the over-determination of the field equations and the extraction of their exact solutions. There is no doubt, however, that in the process of investigating particular solutions to the field equations, we might catch a glimpse into the initial states of the microspin field as well as the evolution of the field equations. We'd also like to comment that we have constructed our theory with a relatively small number of variables only, a characteristic which is important in order to prevent superfluous variables from encumbering the theory.

2 Geometric structure of the manifold $\Im_3$ of material coordinates

We shall briefly describe the local geometry of the manifold $\Im_3$ which serves as the space of material (Lagrangian) coordinates (material points)ξ^i $(i = 1, 2, 3)$. In general, in addition to the general non-orientability of its local points, the manifold $\Im_3$ may contain singularities or geometric defects which give rise to the existence of a local material curvature represented by a generally *non-holonomic* (path-dependent) curvature tensor, a consideration which is normally shunned in the standard continuum mechanics literature. This way, the manifold $\Im_3$ of material coordinates, may be defined either as a continuum or a discontinuum and can be seen as a three-dimensional hypersurface of non-orientable points, embedded in the physical four-dimensional space-time of spatial-temporal coordinates $\Re_4$. Consequently, we need to employ the language of general tensor analysis in which the local metric, the local connection, and the local curvature of the material body $\Im_3$ form the most fundamental structural objects of our consideration.

First, the material space $\Im_3$ is spanned by the three curvilinear, covariant (i.e., tangent) basis vectors g_i as $\Im_3$ is embedded in a four-dimensional space-time of physical events $\Re_4$ *for the sake of general covariance*, whose coordinates are represented by y^μ $(\mu = 1, 2, 3, 4)$ and whose covariant basis vectors are denoted by ω_μ. In a neighborhood of local coordinate points of $\Re_4$ we also introduce an enveloping space of spatial (Eulerian) coordinates x^A $(A = 1, 2, 3)$ spanned by locally constant orthogonal basis vectors e_A which form a three-dimensional Euclidean space $\mathbb{E}_3$. (From now on, it is to be understood that small and capital Latin indices run from 1 to 3, and that Greek indices run from 1 to 4.) As usual, we also define the dual, contravariant (i.e., cotangent) counterparts of the basis vectors g_i, e_A, and ω_μ, denoting them respectively as g^i, e^A, and ω^μ , according to the following relations:

$$\langle g^i, g_k \rangle = \delta^i_k,$$

$$\langle e^A, e_B \rangle = \delta^A_B,$$

$$\langle \omega^\mu, \omega_\nu \rangle = \delta^\mu_\nu,$$

where the brackets $\langle\rangle$ denote the so-called projection, i.e., the inner product and where δ denotes the Kronecker delta. From these basis vectors, we define their tetrad components as

$$\gamma^i_A = \langle g^i, e_A \rangle = \frac{\partial \xi^i}{\partial x^A},$$

$$\zeta^i_\mu = \langle g^i, \omega_\mu \rangle = \frac{\partial \xi^i}{\partial y^\mu},$$

$$e^A_\mu = \langle e^A, \omega_\mu \rangle = \frac{\partial x^A}{\partial y^\mu}.$$

Their duals are given in the following relations:

$$\gamma^i_A \gamma^A_k = \zeta^i_\mu \zeta^\mu_k = \delta^i_k \,,$$

$$e^A_\mu e^\mu_B = \delta^A_B \,.$$

(Einstein's summation convention is implied throughout this work.)

The distance between two infinitesimally adjacent points in the (initially undeformed) material body $\Im_3$ is given by the symmetric bilinear form (with $\otimes$ denoting the tensor product)

$$g = g_{ik}\, g^i \otimes g^k \,,$$

called the metric tensor of the material space, as

$$ds^2 = g_{ik}\, d\xi^i d\xi^k \,.$$

By means of projection, the components of the metric tensor of $\Im_3$ are given by

$$g_{ik} = \langle g_i, g_k \rangle \,.$$

Accordingly, for $a, b = 1, 2, 3$, they are related to the four-dimensional components of the metric tensor of $\Im_3$, i.e., $G_{\mu\nu} = \langle \omega_\mu, \omega_v \rangle$, by

$$g_{ik} = \zeta^\mu_i \zeta^\nu_k G_{\mu\nu} = \\ = \zeta^a_i \zeta^b_k G_{ab} + 2k_{(i} b_{k)} + \phi k_i k_k \,,$$

where the round brackets indicate symmetrization (in contrast to the square brackets denoting anti-symmetrization which we shall also employ later) and where we have set

$$k_i = \zeta^4_i = c \frac{\partial t}{\partial \xi^i} \,,$$

$$b_i = G_{4i} = \zeta^a_i G_{4a} \,,$$

$$\phi = G_{44} \,.$$

Here we have obviously put $y^4 = ct$ with $c$the speed of light in vacuum and t time.

Inversely, with the help of the following projective relations:

$$g_i = \zeta^\mu_i \omega_\mu \,,$$

$$\omega_\mu = \zeta^i_\mu g_i + \in n_\mu n \,,$$

we find that

$$G_{\mu\nu} = \zeta^i_\mu \zeta^k_\nu g_{ik} + \in n_\mu n_\nu \,,$$

or, calling the dual components of g, as shown in the relations

$$g_{ir}\, g^{kr} = \delta^k_i \,,$$

$$G_{\mu\rho} G^{\nu\rho} = \delta^\nu_\mu \,,$$

we have

$$\gamma_i^\mu \gamma_\nu^i = \delta_\nu^\mu - \in n^\mu n_\nu \,.$$

Here $\in = \pm 1$ and n^μ and n_ν respectively are the contravariant and covariant components of the unit vector field n normal to the hypersurface of material coordinates $\Im_3$, whose canonical form may be given as $\Phi\left(\xi^i, k\right) = 0$ where k is a parameter. (Note that the same 16 relations also hold for the inner product represented by $e_A^\mu e_\nu^A$.) We can write

$$n_\mu = \in^{1/2} \frac{\partial \Phi}{\partial y^\mu} \left(G^{\alpha\beta} \frac{\partial \Phi}{\partial y^\alpha} \frac{\partial \Phi}{\partial y^\beta} \right)^{-1/2} .$$

Note that

$$n_\mu \gamma_i^\mu = n_\mu e_i^\mu = 0 \,,$$

$$n_\mu n^\mu = \in \,.$$

Let now g denote the determinant of the three-dimensional components of the material metric tensor g_{ik}. Then the covariant and contravariant components of the totally anti-symmetric permutation tensor are given by

$$\in_{ijk} = g^{1/2} \varepsilon_{ijk} \,,$$

$$\in^{ijk} = g^{-1/2} \varepsilon^{ijk} \,,$$

where ε_{ijk} are the components of the usual *permutation tensor density*. More specifically, we note that

$$g_i \wedge g_j = \in_{ijk} g^k \,,$$

where the symbol $\wedge$ denotes exterior product, i.e., $g_i \wedge g_j = \left(\zeta_i^\mu \zeta_j^\nu - \zeta_j^\mu \zeta_i^\nu \right) \omega_\mu \otimes \omega_\nu$. In the same manner, we define the four-dimensional permutation tensor as one with components

$$\in_{\alpha\beta\rho\sigma} = G^{1/2} \varepsilon_{\alpha\beta\rho\sigma} \,,$$

$$\in^{\alpha\beta\rho\sigma} = G^{-1/2} \varepsilon^{\alpha\beta\rho\sigma} \,,$$

where $G = \det G_{\mu\nu}$. Also, we call the following simple transitive rotation group:

$$\omega_\alpha \wedge \omega_\beta = - \in \in_{\alpha\beta\rho\sigma} n^\rho \omega^\sigma \,,$$

where

$$\in_{ijk} n_\sigma = \zeta_i^\alpha \zeta_j^\beta \zeta_k^\rho \in_{\alpha\beta\rho\sigma} \,,$$

$$n_\sigma = \frac{1}{6} \zeta_i^\alpha \zeta_j^\beta \zeta_k^\rho \in^{ijk} \in_{\alpha\beta\rho\sigma} \,.$$

Note the following identities:

$$\in_{ijk} \in^{pqr} = \delta_{ijk}^{pqr} = \delta_i^p \left(\delta_j^q \delta_k^r - \delta_j^r \delta_k^q \right) + \delta_i^q \left(\delta_j^r \delta_k^p - \delta_j^p \delta_k^r \right) + \delta_i^r \left(\delta_j^p \delta_k^q - \delta_j^q \delta_k^p \right) ,$$

$$\in_{ijr} \in^{pqr} = \delta_{ij}^{pq} = \delta_i^p \delta_j^q - \delta_i^q \delta_j^p \,,$$

$$\in_{ijs} \in^{ijr} = \delta_s^r \,,$$

where δ_{ijk}^{pqr} and δ_{ij}^{pq} represent generalized Kronecker deltas. In the same manner, the four-dimensional components of the generalized Kronecker delta, i.e.,

$$\delta_{\mu\nu\gamma\lambda}^{\alpha\beta\rho\sigma} = \det \begin{pmatrix} \delta_\mu^\alpha & \delta_\mu^\beta & \delta_\mu^\rho & \delta_\mu^\sigma \\ \delta_\nu^\alpha & \delta_\nu^\beta & \delta_\nu^\rho & \delta_\nu^\sigma \\ \delta_\gamma^\alpha & \delta_\gamma^\beta & \delta_\gamma^\rho & \delta_\gamma^\sigma \\ \delta_\lambda^\alpha & \delta_\lambda^\beta & \delta_\lambda^\rho & \delta_\lambda^\sigma \end{pmatrix}$$

can be used to deduce the following identities:

$$\in_{\mu\nu\gamma\lambda} \in^{\alpha\beta\rho\sigma} = \in \delta_{\mu\nu\gamma\lambda}^{\alpha\beta\rho\sigma} \,,$$

$$\in_{\mu\nu\gamma\sigma} \in^{\alpha\beta\rho\sigma} = \in \delta_{\mu\nu\gamma}^{\alpha\beta\rho} \,,$$

$$\in_{\mu\nu\rho\sigma} \in^{\alpha\beta\rho\sigma} = 2 \in \delta_{\mu\nu}^{\alpha\beta} \,,$$

$$\in_{\mu\beta\rho\sigma} \in^{\alpha\beta\rho\sigma} = 6 \in \delta_\mu^\alpha \,.$$

Now, for the contravariant components of the material metric tensor, we have

$$g^{ik} = \zeta_a^i \zeta_b^k G^{ab} + 2k^{(i} b^{k)} + \bar{\phi} k^i k^k \,,$$

$$G^{\mu\nu} = \zeta_i^\mu \zeta_k^\nu g^{ik} + \in n^\mu n^\nu \,,$$

where

$$k^i = \frac{1}{c} \frac{\partial \xi^i}{\partial t} \,,$$

$$b^i = G^{4i} = \zeta_a^i G^{4a} \,,$$

$$\bar{\phi} = G^{44} \,.$$

Obviously, the quantities $\frac{\partial \xi^i}{\partial t}$ in k^i are the contravariant components of the local velocity vector field. If we choose an orthogonal coordinate system for the background space-time $\Re_4$, we simply have the following three-dimensional components of the material metric tensor:

$$g_{ik} = \zeta_i^a \zeta_k^b G_{ab} + \phi k_i k_k \,,$$

$$g^{ik} = \zeta_a^i \zeta_b^k G^{ab} + \bar{\phi} k^i k^k \,.$$

In a special case, if the space-time $\Re_4$ is (pseudo-)Euclidean, we may set $\phi = \bar{\phi} = \pm 1$. However, for the sake of generality, we shall not always need to assume the case just mentioned.

Now, the components of the metric tensor of the local Euclidean space of spatial coordinates x^A, $h_{AB} = \langle e_A, e_B \rangle$, are just the components of the *Euclidean Kronecker delta*:

$$h_{AB} = \delta_{AB} \,.$$

Similarly, we have the following relations:

$$g_{ik} = \gamma_i^A \gamma_k^B h_{AB} = \gamma_i^A \gamma_k^A \,,$$

$$h_{AB} = \gamma_A^i \gamma_B^k g_{ik} \,.$$

Now we come to an important fact: from the structure of the material metric tensor alone, we can raise and lower the indices of arbitrary vectors and tensors defined in $\Im_3$, and hence in $\Re_4$, by means of its components, e.g.,

$$A^i = g^{ik} A_k,\, A_i = g_{ik} A^k,\, B^\mu = G^{\mu\nu} B_v,$$
$$B_\mu = G_{\mu\nu} B^v,\ etc.$$

Having introduced the metric tensor, let us consider the transformations among the physical objects defined as acting in the material space $\Im_3$. An arbitrary tensor field T of rank n in $\Im_3$ can in general be represented as

$$\begin{aligned} T &= T^{ij\dots}_{kl\dots}\, g_i \otimes g_j \otimes \dots \otimes g^k \otimes g^l \otimes \dots = \\ &= T'^{AB\dots}_{CD\dots}\, e_A \otimes e_B \otimes \dots \otimes e^C \otimes e^D \otimes \dots = \\ &= T''^{\alpha\beta\dots}_{\mu\nu\dots}\, \omega_\alpha \otimes \omega_\beta \otimes \dots \otimes \omega^\mu \otimes \omega^\nu \otimes \dots\,. \end{aligned}$$

In other words,

$$T^{ij\dots}_{kl\dots} = \gamma^i_A \gamma^j_B \dots \gamma^C_k \gamma^D_l \dots T'^{AB\dots}_{CD\dots} = \zeta^i_\alpha \zeta^j_\beta \dots \zeta^\mu_k \zeta^\nu_l \dots T''^{\alpha\beta\dots}_{\mu\nu\dots}\,,$$

$$T'^{AB\dots}_{CD\dots} = \gamma^A_i \gamma^B_j \dots \gamma^k_C \gamma^l_D \dots T^{ij\dots}_{kl\dots} = e^A_\alpha e^B_\beta \dots e^\mu_C e^\nu_D \dots T''^{\alpha\beta\dots}_{\mu\nu\dots}\,,$$

$$T''^{\alpha\beta\dots}_{\mu\nu\dots} = \zeta^k_\mu \zeta^l_\nu \dots \zeta^\alpha_i \zeta^\beta_j \dots T^{ij\dots}_{kl\dots} = e^C_\mu e^D_\nu \dots e^\alpha_A e^\beta_B T'^{AB\dots}_{CD\dots}\,.$$

For instance, the material line-element can once again be written as

$$ds^2 = g_{ik}\left(\xi^p\right) d\xi^i d\xi^k = \delta_{AB}\, dx^A dx^B = G_{\mu\nu}\left(y^\alpha\right) dy^\mu dy^\nu.$$

We now move on to the notion of *a covariant derivative* defined in the material space $\Im_3$. Again, for an arbitrary tensor field T of $\Im_3$, the covariant derivative of the components of T is given as

$$\begin{aligned} \nabla_p T^{ij\dots}_{kl\dots} = \frac{\partial T^{ij\dots}_{kl\dots}}{\partial \xi^p} &+ \Gamma^i_{rp} T^{rj\dots}_{kl\dots} + \Gamma^j_{rp} T^{ir\dots}_{kl\dots} + \dots - \\ &- \Gamma^r_{kp} T^{ij\dots}_{rl\dots} - \Gamma^r_{lp} T^{ij\dots}_{kr\dots} - \dots\,, \end{aligned}$$

such that

$$\nabla_p T = \frac{\partial T}{\partial \xi^p} = \nabla_p T^{ij\dots}_{kl\dots}\, g_i \otimes g_j \otimes \dots \otimes g^k \otimes g^l \otimes \dots\,,$$

where

$$\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik} g_r\,.$$

Here the $n^3 = 27$ quantities Γ^i_{jk} are the components of the connection field Γ, locally given by

$$\Gamma^i_{jk} = \gamma^i_A \frac{\partial \gamma^A_j}{\partial \xi^k}\,,$$

which, in our work, shall be non-symmetric in the pair of its lower indices (jk) in order to describe *both* torsion and discontinuities. If $\bar{\xi}^i$ represent another system of coordinates in the material space $\Im_3$, then locally the components of the connection field Γ are seen to transform *inhomogeneously* according to

$$\Gamma^i_{jk} = \frac{\partial \xi^i}{\partial \bar{\xi}^p} \frac{\partial \bar{\xi}^r}{\partial \xi^j} \frac{\partial \bar{\xi}^s}{\partial \xi^k} \bar{\Gamma}^p_{rs} + \frac{\partial \xi^i}{\partial \bar{\xi}^p} \frac{\partial^2 \bar{\xi}^p}{\partial \xi^k \partial \xi^j}\,,$$

i.e., the Γ^i_{jk} do not transform as components of a local tensor field. Before we continue, we shall note a few things regarding some *boundary conditions* of our material geometry. Because we have assumed that the hypersurface $\Im_3$ is embedded in the four-dimensional space-time $\Re_4$, we must in general have instead

$$\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik} g_r + \in K_{ik} n\,,$$

where $K_{ik} = \langle \nabla_k g_i, n\rangle = n_\mu \nabla_k \zeta^\mu_i$ are the covariant components of the extrinsic curvature of $\Im_3$. Then the scalar given by $\bar{K} = \in K_{ik} \frac{d\xi^i}{ds}\frac{d\xi^k}{ds}$, which is the Gaussian curvature of $\Im_3$, is arrived at. However our simultaneous embedding situation in which we have also defined an Euclidean space in $\Re_4$ as the space of spatial coordinates embedding the space of material coordinates $\Im_3$, means that the extrinsic curvature tensor, and hence also the Gaussian curvature of $\Im_3$, must vanish and we are left simply with $\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik} g_r$. This situation is analogous to the simple situation in which a plane (flat surface) is embedded in a three-dimensional space, where on that plane we define a family of curves which give rise to a system of curvilinear coordinates, however, with discontinuities in the transformation from the plane coordinates to the local curvilinear coordinates and vice versa.

Meanwhile, we have seen that the covariant derivative of the tensor field T is again a tensor field. As such, here we have

$$\nabla_p T^{ij\dots}_{kl\dots} = \gamma^i_A \gamma^j_B \dots \gamma^C_k \gamma^D_l \dots \gamma^E_p \frac{\partial T'^{AB\dots}_{CD\dots}}{\partial x^E}$$

Although a non-tensorial object, the connection field Γ is a fundamental geometric object that establishes comparison of local vectors at different points in $\Im_3$, i.e., in the Lagrangian coordinate system. Now, with the help of the material *metrical condition*

$$\nabla_p g_{ik} = 0\,,$$

i.e.,

$$\frac{\partial g_{ik}}{\partial \xi^p} = \Gamma_{ikp} + \Gamma_{kip}\,,$$

where $\Gamma_{ikp} = g_{ir}\ \Gamma^r_{kp}$, one solves for Γ^i_{jk} as follows:

$$\begin{aligned} \Gamma^i_{jk} = \frac{1}{2} g^{ri} \left(\frac{\partial g_{rj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^r} + \frac{\partial g_{kr}}{\partial \xi^j}\right) + \Gamma^i_{[jk]} - \\ - g^{ri}\left(g_{js}\Gamma^s_{[rk]} + g_{ks}\Gamma^s_{[rj]}\right)\,. \end{aligned}$$

From here, we define the following geometric objects:

1. The holonomic (path-independent) Christoffel or Levi-Civita connection, sometimes also called the *elastic connection*, whose components are symmetric in the pair of its lower indices (jk) and given by

$$\left\{{}^{i}_{jk}\right\} = \frac{1}{2} g^{ri} \left(\frac{\partial g_{rj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^r} + \frac{\partial g_{kr}}{\partial \xi^j} \right) .$$

2. The non-holonomic (path-dependent) object, a *chirality tensor* called the torsion tensor which describes *local rotation of material points* in $\Im_3$ and whose components are given by

$$\tau^i_{jk} = \Gamma^i_{[jk]} = \frac{1}{2} \gamma^i_A \left(\frac{\partial \gamma^A_j}{\partial \xi^k} - \frac{\partial \gamma^A_k}{\partial \xi^j} \right) .$$

3. The non-holonomic contorsion tensor, a linear combination of the torsion tensor, whose components are given by

$$\begin{aligned} T^i_{jk} &= \Gamma^i_{[jk]} - g^{ri} \left(g_{js} \Gamma^s_{[rk]} + g_{ks} \Gamma^s_{[rj]} \right) = \\ &= \gamma^i_A \tilde{\nabla}_k \gamma^A_j = \\ &= \gamma^i_A \left(\frac{\partial \gamma^A_j}{\partial \xi^k} - \left\{{}^{r}_{jk}\right\} \gamma^A_r \right) , \end{aligned}$$

which are actually anti-symmetric with respect to the first two indices $i$and$j$.

In the above, we have exclusively introduced a covariant derivative with respect to the holonomic connection alone, denoted by $\tilde{\nabla}_p$. Again, for an arbitrary tensor field T of $\Im_3$, we have

$$\begin{aligned} \tilde{\nabla}_p T^{ij\ldots}_{kl\ldots} &= \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial \xi^p} + \left\{{}^{i}_{rp}\right\} T^{rj\ldots}_{kl\ldots} + \left\{{}^{j}_{rp}\right\} T^{ir\ldots}_{kl\ldots} + \cdots - \\ &- \left\{{}^{r}_{kp}\right\} T^{ij\ldots}_{rl\ldots} - \left\{{}^{r}_{lp}\right\} T^{ij\ldots}_{kr\ldots} - \ldots . \end{aligned}$$

Now we can see that the metrical condition $\nabla_p g_{ik} = 0$ also implies that $\tilde{\nabla}_p g_{ik} = 0$, $\tilde{\nabla}_k \gamma^A_i = T^r_{ik} \gamma^A_r$, and $\nabla_k \gamma^A_i = 0$.

Finally, with the help of the connection field Γ , we derive the third fundamental geometric objects of $\Im_3$, i.e., the local fourth-order curvature tensor of the material space

$$R = R^i_{.jkl} g_i \otimes g^j \otimes g^k \otimes g^l ,$$

where

$$R^i_{.jkl} = \frac{\partial \Gamma^i_{jl}}{\partial \xi^k} - \frac{\partial \Gamma^i_{jk}}{\partial \xi^l} + \Gamma^r_{jl} \Gamma^i_{rk} - \Gamma^r_{jk} \Gamma^i_{rl} .$$

These are given in the relations

$$\left(\nabla_k \nabla_j - \nabla_j \nabla_k \right) F_i = R^r_{.ijk} F_r - 2 \Gamma^r_{[jk]} \nabla_r F_i ,$$

where F_i are the covariant components of an arbitrary vector field F of $\Im_3$. Correspondingly, for the contravariant components F^i we have

$$\left(\nabla_k \nabla_j - \nabla_j \nabla_k \right) F^i = - R^i_{.rjk} F^r - 2 \Gamma^r_{[jk]} \nabla_r F^i .$$

The *Riemann-Christoffel curvature tensor* $\tilde{R}$ here then appears as the part of the curvature tensor R built from the symmetric, holonomic Christoffel connection alone, whose components are given by

$$\tilde{R}^i_{.jkl} = \frac{\partial}{\partial \xi^k} \left\{{}^{i}_{jl}\right\} - \frac{\partial}{\partial \xi^l} \left\{{}^{i}_{jk}\right\} + \left\{{}^{r}_{jl}\right\} \left\{{}^{i}_{rk}\right\} - \left\{{}^{r}_{jk}\right\} \left\{{}^{i}_{rl}\right\} .$$

Correspondingly, the components of the symmetric Ricci tensor are given by

$$\begin{aligned} \tilde{R}_{ik} = \tilde{R}^r_{.irk} &= \frac{\partial}{\partial \xi^r} \left\{{}^{r}_{ik}\right\} - \frac{\partial^2 \, {}^e\log \, (g)^{1/2}}{\partial \xi^k \partial \xi^i} + \\ &+ \left\{{}^{s}_{ik}\right\} \frac{\partial^e \log \, (g)^{1/2}}{\partial \xi^s} - \left\{{}^{s}_{ir}\right\} \left\{{}^{r}_{sk}\right\} , \end{aligned}$$

where we have used the relations

$$\left\{{}^{k}_{ik}\right\} = \frac{\partial^e \log \, (g)^{1/2}}{\partial \xi^i} = \Gamma^k_{ki} .$$

Then the Ricci scalar is simply $\tilde{R} = \tilde{R}^i_{.i}$, an important geometric object which shall play the role of the *microspin* (*chirality*) *potential* in our generalization of classical elasticity theory developed here.

Now, it is easily verified that

$$\left(\tilde{\nabla}_k \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_k \right) F_i = \tilde{R}^r_{.ijk} F_r$$

and

$$\left(\tilde{\nabla}_k \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_k \right) F^i = - \tilde{R}^i_{.rjk} F^r .$$

The remaining parts of the curvature tensor R are then the remaining non-holonomic objects J and Q whose components are given as

$$J^i_{.jkl} = \frac{\partial T^i_{jl}}{\partial \xi^k} - \frac{\partial T^i_{jk}}{\partial \xi^l} + T^r_{jl} T^i_{rk} - T^r_{jk} T^i_{rl}$$

and

$$Q^i_{.jkl} = \left\{{}^{r}_{jl}\right\} T^i_{rk} + T^r_{jl} \left\{{}^{i}_{rk}\right\} - \left\{{}^{r}_{jk}\right\} T^i_{rl} - T^r_{jk} \left\{{}^{i}_{rl}\right\} .$$

Hence, we write

$$R^i_{.jkl} = \tilde{R}^i_{.jkl} + J^i_{.jkl} + Q^i_{.jkl} .$$

More explicitly,

$$R^i_{.jkl} = \tilde{R}^i_{.jkl} + \tilde{\nabla}_k T^i_{jl} - \tilde{\nabla}_l T^i_{jk} + T^r_{jl} T^i_{rk} - T^r_{jk} T^i_{rl} .$$

From here, we define the two important contractions of the components of the curvature tensor above. We have the generalized Ricci tensor whose components are given by

$$R_{ik} = R^r_{.irk} = \tilde{R}_{ik} + \tilde{\nabla}_r T^r_{ik} - T^r_{is} T^s_{rk} - \tilde{\nabla}_k \omega_i + T^r_{ik} \omega_r ,$$

where the $n = 3$ quantities

$$\omega_i = T^k_{ik} = 2 \Gamma^k_{[ik]}$$

define the components of the microspin vector. Furthermore, with the help of the relations $g^{rs}T^{i}_{rs} = -2g^{ik}\Gamma^{s}_{[ks]} = -\omega^{i}$, the generalized Ricci scalar is

$$R = R^{i}_{.i} = \tilde{R} - 2\tilde{\nabla}_{i}\omega^{i} - \omega_{i}\omega^{i} - T_{ijk}T^{ikj}\,.$$

It is customary to give the fully covariant components of the Riemann-Christoffel curvature tensor. They can be expressed somewhat more conveniently in the following form (when the g_{ik} are continuous):

$$\tilde{R}_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial\xi^k\partial\xi^j} + \frac{\partial^2 g_{jk}}{\partial\xi^l\partial\xi^i} - \frac{\partial^2 g_{ik}}{\partial\xi^l\partial\xi^j} - \frac{\partial^2 g_{jl}}{\partial\xi^k\partial\xi^i}\right) + $$
$$+ g_{rs}\left(\left\{{r \atop il}\right\}\left\{{s \atop jk}\right\} - \left\{{r \atop ik}\right\}\left\{{s \atop jl}\right\}\right).$$

In general, when the g_{ik} are continuous, all the following symmetries are satisfied:

$$\tilde{R}_{ijkl} = -\tilde{R}_{jikl} = -\tilde{R}_{ijlk}\,,$$

$$\tilde{R}_{ijkl} = \tilde{R}_{klij}\,.$$

However, for the sake of generality, we may as well drop the condition that the g_{ik} are continuous in their second derivatives, i.e., with respect to the material coordinates ξ^i such that we can define further more non-holonomic, anti-symmetric objects extracted from R such as the tensor field V whose components are given by

$$V_{ik} = R^{r}_{.rik} = -\gamma^{l}_{A}\left(\frac{\partial}{\partial\xi^k}\left(\frac{\partial\gamma^{A}_{l}}{\partial\xi^i}\right) - \frac{\partial}{\partial\xi^i}\left(\frac{\partial\gamma^{A}_{l}}{\partial\xi^k}\right)\right).$$

The above relations are equivalent to the following $\frac{1}{2}\,n\,(n-1) = 3$ equations for the components of the material metric tensor:

$$\frac{\partial}{\partial\xi^l}\left(\frac{\partial g_{ij}}{\partial\xi^k}\right) - \frac{\partial}{\partial\xi^k}\left(\frac{\partial g_{ij}}{\partial\xi^l}\right) = -\left(R_{ijkl} + R_{jikl}\right),$$

which we shall denote simply by $\|g_{ij,kl}\|$. When the g_{ik} possess such discontinuities, we may define the *discontinuity potential* by

$$\eta_i = \left\{{k \atop ik}\right\} = \frac{\partial\, ^{e}\log\,(g)^{1/2}}{\partial\xi^i}\,.$$

Hence we have

$$V_{ik} = \frac{\partial\eta_k}{\partial\xi^i} - \frac{\partial\eta_i}{\partial\xi^k}\,.$$

From the expression of the determinant of the material metric tensor, i.e.,

$$g = \varepsilon_{ijk}\,g_{1i}\,g_{2j}\,g_{3k}$$

we see, more specifically, that a discontinuum with arbitrary geometric singularities is characterized by the following discontinuity equations:

$$\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right) g =$$
$$= \varepsilon_{ijk}\,g_{2j}\,g_{3k}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right) g_{1i} +$$
$$+ \varepsilon_{ijk}\,g_{1i}\,g_{3k}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right) g_{2j} +$$
$$+ \varepsilon_{ijk}\,g_{1i}\,g_{2j}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right) g_{3k}\,.$$

In other words,

$$\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right) g = -\varepsilon_{ijk}\left(R_{1irs} + R_{i1rs}\right) g_{2j}\,g_{3k} -$$
$$- \varepsilon_{ijk}\left(R_{2jrs} + R_{j2rs}\right) g_{1i}\,g_{3k} -$$
$$- \varepsilon_{ijk}\left(R_{3krs} + R_{k3rs}\right) g_{1i}\,g_{2j}\,.$$

It is easy to show that in three dimensions the components of the curvature tensor R obey the following decomposition:

$$R_{ijkl} = W_{ijkl} + g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il} +$$
$$+ \frac{1}{2}\left(g_{il}\,g_{jk} - g_{ik}\,g_{jl}\right) R\,,$$

i.e.,

$$R^{ij}_{..kl} = W^{ij}_{..kl} + \delta^{i}_{k}R_{jl} + \delta^{j}_{l}R^{i}_{.k} - \delta^{i}_{l}R_{jk} - \delta^{j}_{k}R^{i}_{.l} +$$
$$+ \frac{1}{2}\left(\delta^{i}_{l}\delta^{j}_{k} - \delta^{i}_{k}\delta^{j}_{l}\right) R\,,$$

where W_{ijkl} (and $W^{ij}_{..kl}$) are the components of the Weyl tensor W satisfying $W^{r}_{.irk} = 0$, whose symmetry properties follow exactly those of R_{ijkl}. Similarly, for the components of the Riemann-Christoffel curvature tensor $\tilde{R}$ we have

$$\tilde{R}^{ij}_{..kl} = \tilde{W}^{ij}_{..kl} + \delta^{i}_{k}\tilde{R}_{jl} + \delta^{j}_{l}\tilde{R}^{i}_{.k} - \delta^{i}_{l}\tilde{R}_{jk} - \delta^{j}_{k}\tilde{R}^{i}_{.l} +$$
$$+ \frac{1}{2}\left(\delta^{i}_{l}\delta^{j}_{k} - \delta^{i}_{k}\delta^{j}_{l}\right) \tilde{R}\,.$$

Later, the above equations shall be needed to generalize the components of the elasticity tensor of classical continuum mechanics, i.e., by means of the components

$$\frac{1}{2}\left(\delta^{i}_{k}\delta^{j}_{l} - \delta^{i}_{l}\delta^{j}_{k}\right)\tilde{R}\,.$$

Furthermore with the help of the relations

$$R^{i}_{.jkl} + R^{i}_{.klj} + R^{i}_{.ljk} = -2\left(\frac{\partial\Gamma^{i}_{[jk]}}{\partial\xi^l} + \frac{\partial\Gamma^{i}_{[kl]}}{\partial\xi^j} + \frac{\partial\Gamma^{i}_{[lj]}}{\partial\xi^k} + \right.$$
$$\left. + \Gamma^{i}_{rj}\Gamma^{r}_{[kl]} + \Gamma^{i}_{rk}\Gamma^{r}_{[lj]} + \Gamma^{i}_{rl}\Gamma^{r}_{[jk]}\right),$$

we derive the following identities:

$$\nabla_p R_{ijkl} + \nabla_k R_{ijlp} + \nabla_l R_{ijpk} = 2\left(\Gamma^{r}_{[kl]}R_{ijrp} + \right.$$
$$\left. + \Gamma^{r}_{[lp]}R_{ijrk} + \Gamma^{r}_{[pk]}R_{ijrl}\right),$$

$$\nabla_i \left(R^{ik} - \frac{1}{2} g^{ik} R \right) = 2 g^{ik} \Gamma^s_{[ri]} R^r_{.s} + \Gamma^r_{[ij]} R^{ijk}_{\ldots r} \,.$$

From these more general identities, we then derive the simpler and more specialized identities:

$$\tilde{R}_{ijkl} + \tilde{R}_{iklj} + \tilde{R}_{iljk} = 0 \,,$$

$$\tilde{\nabla}_p \tilde{R}_{ijkl} + \tilde{\nabla}_k \tilde{R}_{ijlp} + \tilde{\nabla}_l \tilde{R}_{ijpk} = 0 \,,$$

$$\tilde{\nabla}_i \left(\tilde{R}^{ik} - \frac{1}{2} g^{ik} \tilde{R} \right) = 0 \,,$$

often referred to as the Bianchi identities.

We are now able to state the following about the sources of the curvature of the material space $\Im_3$: there are actually two sources that generate the curvature which can actually be sufficiently represented by the Riemann-Christoffel curvature tensor alone. The first source is the torsion represented by $\Gamma^i_{[jk]}$ which makes the hypersurface $\Im_3$ *non-orientable* as any field shall in general depend on the twisted path it traces therein. As we have said, this torsion is the source of microspin, i.e., point-rotation. The torsion tensor enters the curvature tensor as an integral part and hence we can equivalently attribute the source of microspin to the Riemann-Christoffel curvature tensor as well. The second source is the possible discontinuities in regions of $\Im_3$ which, as we have seen, render the components of the material metric tensor $g_{ik} = \gamma^A_i \gamma^B_k \delta_{AB}$ discontinuous at least in their second derivatives with respect to the material coordinates ξ^i. This is explicitly shown in the following relations:

$$R^i_{.jkl} = -\gamma^i_A \left(\frac{\partial}{\partial \xi^l} \left(\frac{\partial \gamma^A_j}{\partial \xi^k} \right) - \frac{\partial}{\partial \xi^k} \left(\frac{\partial \gamma^A_j}{\partial \xi^l} \right) \right) =$$

$$= -\gamma^i_A \left(\nabla_l K^A_{jk} - \nabla_k K^A_{jl} \right) + \Omega^i_{.jkl} \,,$$

where

$$K^A_{ij} = \frac{\partial \gamma^A_i}{\partial \xi^j} = \frac{1}{2} \left(\frac{\partial \gamma^A_i}{\partial \xi^j} + \frac{\partial \gamma^A_j}{\partial \xi^i} \right) + \gamma^A_k \Gamma^k_{[ij]}$$

and

$$\Omega^i_{.jkl} = \gamma^i_A \left(\Gamma^r_{jk} K^A_{rl} - \Gamma^r_{jl} K^A_{rk} - 2 \Gamma^r_{[kl]} K^A_{jr} \right) \,.$$

Another way to cognize the existence of the curvature in the material space $\Im_3$ is as follows: let us inquire into the possibility of "parallelism" in the material space $\Im_3$. Take now a "parallel" vector field ${}^pB$ such that

$$\nabla_k {}^pB_i = 0 \,,$$

i.e.,

$$\frac{\partial {}^pB_i}{\partial \xi^k} = \Gamma^r_{ik} \, {}^pB_r \,.$$

Then in general we obtain the following non-integrable equations of the form

$$\frac{\partial}{\partial \xi^l} \left(\frac{\partial {}^pB_i}{\partial \xi^k} \right) - \frac{\partial}{\partial \xi^k} \left(\frac{\partial {}^pB_i}{\partial \xi^l} \right) = -R^r_{.ikl} \, {}^pB_r$$

showing that not even the "parallel" vector field ${}^pB$ is path-independent. Hence even though parallelism may be possibly defined in our geometry, *absolute parallelism* is obtained if and only if the integrability condition $R^i_{.jkl} = 0$ holds, i.e., if the components of the Riemann-Christoffel curvature tensor are given by

$$\tilde{R}^i_{.jkl} = \tilde{\nabla}_l T^i_{jk} - \tilde{\nabla}_k T^i_{jl} + T^r_{jk} T^i_{rl} - T^r_{jl} T^i_{rk} \,.$$

In other words, in the presence of torsion (microspin) the above situation concerning absolute parallelism is only possible if the material body is free of geometric defects, also known as singularities.

The relations we have been developing so far of course account for arbitrary nonorientability conditions as well as geometric discontinuities of the material space $\Im_3$. Consequently, we see that the holonomic field equations of classical continuum mechanics shall be obtained whenever we drop the assumptions of non-orientability of points and geometric discontinuities of the material body. We also emphasize that *geometric non-linearity* of the material body has been fully taken into account. A material body then becomes linear if and only if we neglect any quadratic and higher-order terms involving the connection field Γ of the material space $\Im_3$.

3 Elements of the generalized kinematics: deformation analysis

Having described the internal structure of the material space $\Im_3$, i.e., the material body, we now move on to the dynamics of the continuum/discontinuum $\Im_3$ when it is subject to an external displacement field. Our goal in this kinematical section is to generalize the notion of a material derivative with respect of the material motion. We shall deal with the external displacement field in the direction of motion of $\Im_3$ which brings $\Im_3$ from its initially undeformed configuration to the deformed configuration ${}^*\Im_3$. We need to generalize the structure of the external displacement (i.e., *external diffeomorphism*) to include two kinds of microspin of material points: the *non-electromagnetic microspin* as well as the *electromagnetic microspin* which is generated, e.g., by electromagnetic polarization.

In this work, in order to geometrically describe the mechanics of the so-called Cosserat continuum as well as other generalized continua, we define the external displacement field ψ as being generally complex according to the decomposition

$$\psi^i = u^i + i \varphi^i \,,$$

where the diffeomorphism $\Im_3 \xrightarrow{\psi} {}^*\Im_3$ is given by

$${}^*\xi^i = \xi^i + \psi^i \,.$$

Here u^i are the components of the usual displacement field u in the neighborhood of points in $\Im_3$, and φ^i are the components of the microspin "point" displacement field φ satisfying

$$\nabla_k \varphi_i + \nabla_i \varphi_k = 0\,,$$

which can be written as an exterior ("Lie") derivative:

$$L_\varphi g_{ik} = 0\,.$$

This just says that the components of the material metric tensor remain invariant with respect to the action of the field φ. We shall elaborate on the notion of exterior differentiation in a short while.

The components of the displacement gradient tensor D are then

$$\begin{aligned} D_{ik} &= \nabla_k \psi_i = \\ &= \frac{1}{2}\left(\nabla_k \psi_i + \nabla_i \psi_k\right) + \frac{1}{2}\left(\nabla_k \psi_i - \nabla_i \psi_k\right) = \\ &= \frac{1}{2}\left(\nabla_k u_i + \nabla_i u_k\right) + \frac{1}{2}\left(\nabla_k u_i - \nabla_i u_k\right) + \\ &+ \frac{1}{2}\, i \left(\nabla_k \varphi_i - \nabla_i \varphi_k\right) = \\ &= \varepsilon_{ik} + \omega_{ik}\,. \end{aligned}$$

Accordingly,

$$\varepsilon_{ik} = \frac{1}{2}\left(\nabla_k u_i + \nabla_i u_k\right) = \frac{1}{2} L_u g_{ik} \stackrel{lin}{=} \frac{1}{2}\left({}^*g_{ik} - g_{ik}\right)$$

are the components of the *linear strain tensor* and

$$\omega_{ik} = \Omega_{ik} + \phi_{ik}$$

are the components of the *generalized spin* (*vorticity*) *tensor*, where

$$\Omega_{ik} = \frac{1}{2}\left(\nabla_k u_i - \nabla_i u_k\right)$$

are the components of the *ordinary macrospin tensor*, and

$$\phi_{ik} = \frac{1}{2}\, i \left(\nabla_k \varphi_i - \nabla_i \varphi_k\right)$$

are the components of the microspin tensor describing rotation of material points on their own axes due to torsion, or, in the literature, the so-called *distributed moment*. At this point, it may be that the internal rotation of material points is analogous to the spin of electrons if the material point themselves are seen as charged point-particles. However, we know that electrons possess internal spin due to internal structural reasons while the material points also rotate partly due to *externally induced couple stress* giving rise to torsion. For this reason we split the components of the microspin φ into two parts:

$$\varphi_i = \phi_i + e A_i\,,$$

where ϕ_i describe non-electromagnetic microspin and eA_i describe pure electron spin with e being the electric charge and A_i, up to a constant of proportionality, being the material components of the electromagnetic vector potential A:

$$A_i = q\omega^\mu_i A_\mu\,,$$

where q is a parametric constant and A_μ are the components of the four-dimensional electromagnetic vector potential in the sense of Maxwellian electrodynamics. Inversely, we have

$$A_\mu = \frac{1}{q}\left(\omega^i_\mu A_i + \in N n_\mu\right)\,,$$

where $N = q n_\mu A^\mu$. The correspondence with classical electrodynamics becomes complete if we link the *electromagnetic microspin tensor* f represented by the components

$$f_{ik} = \frac{1}{2}\, ie\left(\nabla_k A_i - \nabla_i A_k\right)$$

to the electromagnetic field tensor $F = F_{\mu\nu}\, \omega^\mu \otimes \omega^\nu$ through

$$f_{ik} = \frac{\bar\gamma}{e}\, \omega^\mu_i \omega^\nu_k F_{\mu\nu}\,,$$

where $\bar\gamma = \frac{1}{2} iqe^2$. The four-dimensional components of the electromagnetic field tensor in canonical form are

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial y^\nu} - \frac{\partial A_\nu}{\partial y^\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix},$$

where $E = \left(E^1, E^2, E^3\right)$ and $B = \left(B^1, B^2, B^3\right)$ are the electric and magnetic fields, respectively. In three-dimensional vector notation, $E = -\frac{1}{c}\frac{\partial \vec A}{\partial t} - \vec\nabla\phi$ and $B = \mathrm{curl}\vec A$, where $\bar{\mathrm{A}} = A^\mu \omega_\mu = \left(\vec A, \phi\right)$. They satisfy Maxwell's equations in the Lorentz gauge $\mathrm{div}\,\vec A = 0$, i.e.,

$$\frac{1}{c}\frac{\partial E}{\partial t} = \mathrm{curl}B - \frac{4\pi}{c}\, j\,,$$

$$\mathrm{div}E = -\nabla^2\phi = 4\pi\rho_e\,,$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\mathrm{curl}E\,,$$

$$\mathrm{div}B = 0\,,$$

where j is the electromagnetic current density vector and ρ_e is the electric charge density. In addition, we can write

$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c}\, j^\mu\,,$$

i.e., $\nabla_\nu F^{A\nu} = \frac{4\pi}{c} j^A$ and $j^4 = \rho_e$. The inverse transformation relating f_{ik} to F_{ik} is then given by

$$F_{\mu\nu} = \frac{e}{\bar\gamma}\, \omega^i_\mu \omega^k_\nu f_{ik} + \hat F_{\mu\nu}\,,$$

where

$$\hat F_{\mu\nu} = -\in\left(n_\mu F_{\nu\sigma} - n_\nu F_{\mu\sigma}\right) n^\sigma\,,$$

$$\hat F_{\mu\nu} n^\nu = \in^2 F_{\mu\nu} n^\nu = F_{\mu\nu} n^\nu\,.$$

This way, the components of the generalized vorticity tensor once again are

$$\omega_{ik} = \frac{1}{2}\left(\nabla_k u_i - \nabla_k u_i\right) + \frac{1}{2} i\left(\nabla_k \phi_i - \nabla_i \phi_k\right) + \\ + \frac{1}{2} i e\left(\nabla_k A_i - \nabla_i A_k\right) = \\ = \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi^k} - \frac{\partial u_k}{\partial \xi^i}\right) + \frac{1}{2} i\left(\frac{\partial \phi_i}{\partial \xi^k} - \frac{\partial \phi_k}{\partial \xi^i}\right) + \\ + \frac{1}{2} i e\left(\frac{\partial A_i}{\partial \xi^k} - \frac{\partial A_k}{\partial \xi^i}\right) - \Gamma^r_{[ik]}\left(u_r + i\phi_r + ieA_r\right) = \\ = \Omega_{ik} + \varpi_{ik} + \frac{\bar{\gamma}}{e}\,\omega^\mu_i \omega^\nu_k F_{\mu\nu}\,,$$

where

$$\varpi_{ik} = \frac{1}{2} i\left(\nabla_k \phi_i - \nabla_i \phi_k\right)$$

are the components of the non-electromagnetic microspin tensor. Thus we have now seen, in our generalized deformation analysis, how the microspin field is incorporated into the vorticity tensor.

Finally, we shall now produce some basic framework for equations of motion applicable to arbitrary tensor fields in terms of exterior derivatives. We define the exterior derivative of an arbitrary vector field (i.e., a rank-one tensor field) of $\Im_3$, say W, with respect to the so-called *Cartan basis* as the totally anti-symmetric object

$$L_U W = 2U_{[i}\, W_{k]}\, g^i \otimes g^k\,,$$

where U is the velocity vector in the direction of motion of the material body $\Im_3$, i.e., $U^i = \frac{\partial \psi^i}{\partial t}$. If we now take the local basis vectors as *directional derivatives*, i.e., the *Cartan coordinate basis vectors* $g_i = \frac{\partial}{\partial \xi^i} = \partial_i$ and $g^i = d\xi^i$, we obtain for instance, in component notation,

$$\left(L_U W\right)_i = L_U W_i = U^k \partial_k W_i + W_k \partial_i U^k\,.$$

Using the exterior product, we actually see that

$$L_U W = U \wedge W = U \otimes W - W \otimes U\,.$$

Correspondingly, for W^i, we have

$$\left(L_U W\right)^i = L_U W^i = U^k \partial_k W^i - W^k \partial_k U^i\,.$$

The *exterior material derivative* is then a direct generalization of the ordinary material derivative (e.g., as we know, for a scalar field ρ it is given by $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \xi^i} U^i$) as follows:

$$\frac{DW_i}{Dt} = \frac{\partial W_i}{\partial t} + L_U W_i = \frac{\partial W_i}{\partial t} + \left(U \wedge V\right)_i = \frac{\partial W_i}{\partial t} + \\ + U^k \partial_k W_i + W_k \partial_i U^k\,,$$

$$\frac{DW^i}{Dt} = \frac{\partial W^i}{\partial t} + L_U W^i = \frac{\partial W^i}{\partial t} + \left(U \wedge V\right)^i = \frac{\partial W^i}{\partial t} + \\ + U^k \partial_k W^i - W^k \partial_i U^k\,.$$

Finally, we obtain the generalized material derivative of the components of an arbitrary tensor field T of $\Im_3$ as

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + U^m \partial_m T^{ij\ldots}_{kl\ldots} + T^{ij\ldots}_{ml\ldots} \partial_k U^m + \\ + T^{ij\ldots}_{km\ldots} \partial_l U^m + \cdots - T^{mj\ldots}_{kl\ldots} \partial_m U^i - T^{im\ldots}_{kl\ldots} \partial_m U^j - \ldots\,,$$

or alternatively as

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + U^m \nabla_m T^{ij\ldots}_{kl\ldots} + T^{ij\ldots}_{ml\ldots} \nabla_k U^m + \\ + T^{ij\ldots}_{km\ldots} \nabla_l U^m + \cdots - T^{mj\ldots}_{kl\ldots} \nabla_m U^i - T^{im\ldots}_{kl\ldots} \nabla_m U^j - \\ - \cdots + 2\Gamma^m_{[kp]} T^{ij\ldots}_{ml\ldots} U^p + 2\Gamma^m_{[lp]} T^{ij\ldots}_{km\ldots} U^p + \cdots - \\ - 2\Gamma^i_{[mp]} T^{mj\ldots}_{kl\ldots} U^p - 2\Gamma^j_{[mp]} T^{im\ldots}_{kl\ldots} U^p - \ldots\,.$$

Written more simply,

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + L_U T^{ij\ldots}_{kl\ldots} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + \left(U \wedge T\right)^{ij\ldots}_{kl\ldots}\,.$$

For a scalar field Θ, we have simply

$$\frac{D\Theta}{Dt} = \frac{\partial \Theta}{\partial t} + U^k \partial_k \Theta\,,$$

which is just the ordinary material derivative.

Now, with the help of the Cartan basis vectors, the torsion tensor can be expressed directly in terms of the permutation tensor as

$$\Gamma^i_{[jk]} = -\frac{1}{2} g^{ip}\, \epsilon_{pjk}\,.$$

Hence from the generalized material derivative for the components of the material metric tensor g (defined with respect to the Cartan basis), i.e.,

$$\frac{Dg_{ik}}{Dt} = \frac{\partial g_{ik}}{\partial t} + L_U\, g_{ik} = \frac{\partial g_{ik}}{\partial t} + \left(U \wedge g\right)_{ik}\,,$$

we find especially that

$$\frac{Dg_{ik}}{Dt} = \frac{\partial g_{ik}}{\partial t} + \nabla_k U_i + \nabla_i U_k\,,$$

with the help of the metrical condition $\nabla_p g_{ik} = 0$. Similarly, we also find

$$\frac{Dg^{ik}}{Dt} = \frac{\partial g^{ik}}{\partial t} - \left(\nabla^k U^i + \nabla^i U^k\right)\,.$$

Note also that

$$\frac{D\delta^i_k}{Dt} = 0\,.$$

The components of the velocity gradient tensor are given by

$$L_{ik} = \nabla_k U_i = \frac{1}{2}\left(\nabla_k U_i + \nabla_i U_k\right) + \frac{1}{2}\left(\nabla_k U_i - \nabla_i U_k\right)\,,$$

where, following the so-called Helmholtz decomposition theorem, we can write

$$U_i = \nabla_i \alpha + \frac{1}{2} g_{il} \in^{ljk} \left(\nabla_j \beta_k - \nabla_k \beta_j\right),$$

for a scalar field α and a vector field β. However, note that in our case we obtain the following generalized identities:

$$\text{div curl}\, U = -\frac{1}{2} \in^{ijk} \left(R^{l}_{.kij} U_l - 2\Gamma^{l}_{[ij]} \nabla_l A_k \right),$$

$$\text{curl grad}\, \alpha = \in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \alpha ,$$

which must hold throughout unless a constraint is invoked. We now define the generalized shear scalar by

$$\theta = \nabla_i U^i = \nabla_i \nabla^i \alpha + \frac{1}{2} \in^{ijk} \left(\nabla_i \nabla_j - \nabla_j \nabla_i\right) \beta_k =$$
$$= \nabla^2 \alpha - \frac{1}{2} \in^{ijk} R^{l}_{.kij} \beta_l + \in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \beta_k .$$

In other words, the shear now depends on the *microspin field* generated by curvature and torsion tensors.

Meanwhile, we see that the "contravariant" components of the *local acceleration vector* will simply be given by

$$a^i = \frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + U^k \partial_k U^i - U^k \partial_k U^i =$$
$$= \frac{\partial U^i}{\partial t} .$$

However, we also have

$$a_i = \frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + \left(\nabla_k U_i + \nabla_i U_k\right) U^k ,$$

for the "covariant" components.

Furthermore, we have

$$a_i = \frac{\partial U_i}{\partial t} + \left(\frac{Dg_{ik}}{Dt} - \frac{\partial g_{ik}}{\partial t} \right) U^k .$$

Now, define the *local acceleration covector* through

$$\hat{a}^i = g^{ik} a_i =$$
$$= g^{ik} \frac{\partial U_k}{\partial t} + \left(\nabla_k U^i + \nabla^i U_k\right) U^k =$$
$$= \frac{\partial U^i}{\partial t} - U_k \frac{Dg^{ik}}{Dt} ,$$

such that we have

$$a^i - \hat{a}^i = U_k \frac{Dg^{ik}}{Dt} .$$

Hence we see that the sufficient condition for the two local acceleration vectors to coincide is

$$\frac{Dg_{ik}}{Dt} = 0 .$$

In other words, in such a situation we have

$$\frac{\partial g_{ik}}{\partial t} = -\left(\nabla_k U_i + \nabla_i U_k\right) .$$

In this case, a purely rotational motion is obtained only when the material motion is *rigid*, i.e., when $\frac{\partial g_{ik}}{\partial t} = 0$ or, in other words, when the condition

$$L_U g_{ik} = L_{(ik)} = \nabla_k U_i + \nabla_i U_k = 0$$

is satisfied identically. Similarly, a purely translational motion is obtained when $L_{[ik]} = 0$, which describes a potential motion, where we have $U_i = \nabla_i \alpha$. However, as we have seen, in the presence of torsion even any potential motion of this kind is still obviously path-dependent as the relations $\in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \alpha \neq 0$ hold in general.

We now consider the path-dependent displacement field δ tracing a loop ℓ, say, from point P_1 to point P_2 in $\Re_4$ with components:

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \oint_{P_1 - P_2} \left(\varepsilon^{i}_{.k} + \omega^{i}_{.k} - \psi^l \Gamma^{i}_{lk} \right) d\xi^{.k}$$

Let us observe that

$$\psi^k \Gamma^{i}_{kl} = \psi^k \gamma^i_A \frac{\partial \gamma^A_k}{\partial \xi^l} = -\psi^k \gamma^A_k \frac{\partial \gamma^i_A}{\partial \xi^l} =$$
$$= -\psi^A \frac{\partial \gamma^i_A}{\partial \xi^l} = -\left(\frac{\partial}{\partial \xi^l} \left(\gamma^i_A \psi^A\right) - \gamma^i_A \frac{\partial \psi^A}{\partial \xi^l} \right) =$$
$$= \gamma^i_A \frac{\partial \psi^A}{\partial \xi^l} - \frac{\partial \psi^i}{\partial \xi^l} .$$

Now since $\psi^i = \delta \xi^i$, and using $\frac{\partial \delta f}{\partial \xi^i} = \delta \frac{\partial f}{\partial \xi^i}$ for an arbitrary function f, we have

$$\psi^k \Gamma^{i}_{kl} = \gamma^i_A \delta \left(\frac{\partial x^A}{\partial x^B} \right) \gamma^B_l - \delta \left(\frac{\partial \xi^i}{\partial \xi^l} \right) = 0 ,$$

and we are left with

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \oint_{P_1 - P_2} \left(\varepsilon^{i}_{.k} + \omega^{i}_{.k} \right) d\xi^k .$$

Assuming that the ε_{ik} are continuous, we can now derive the following relations:

$$D\psi^i = \nabla_k \psi^i d\xi^k = \frac{\partial \psi^i}{\partial \xi^k} d\xi^k ,$$

$$\nabla_l \omega^{i}_{.k} d\xi^k = \left(\nabla_l \varepsilon^{i}_{.k} - \nabla^i \varepsilon_{lk} \right) d\xi^k = \left(\frac{\partial \varepsilon^{i}_{.k}}{\partial \xi^l} - \frac{\partial \varepsilon_{lk}}{\partial \xi_i} \right) d\xi^k .$$

With the help of the above relations and by direct partial integration, we then have

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \omega^{i}_{.k} \xi^k \Big|_{P_1}^{P_2} - \oint_{P_1 - P_2} \left(\nabla_l \psi^{\,i}_{.k} - \nabla_k \psi^{i}_{.l} \right) \delta \xi^k d\xi^l ,$$

where

$$\psi^i_{.k} = \varepsilon^i_{.k} - \xi^j \left(\nabla_j \varepsilon^i_{.k} - \nabla^i \varepsilon_{jk}\right) .$$

It can be seen that

$$\nabla_l \psi^i_{.k} - \nabla_k \psi^i_{.l} = -Z^i_{.jkl}\,\xi^j ,$$

where we have defined another non-holonomic tensor Z with the components

$$Z^i_{.jkl} = \nabla_l \nabla_j \varepsilon^i_{.k} + \nabla_k \nabla^i \varepsilon_{jl} - \nabla_k \nabla_j \varepsilon^i_{.l} - \nabla_l \nabla^i \varepsilon_{jk} .$$

Now, the linearized components of the Riemann-Christoffel curvature tensor are given by

$$\tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial\xi^k \partial\xi^j} + \frac{\partial^2 g_{jk}}{\partial\xi^l \partial\xi^i} - \frac{\partial^2 g_{ik}}{\partial\xi^l \partial\xi^j} - \frac{\partial^2 g_{jl}}{\partial\xi^k \partial\xi^i}\right) .$$

Direct calculation gives

$$\delta_\psi \tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2}\left(\nabla_k \nabla_j \delta_\psi g_{il} + \nabla_l \nabla_i \delta_\psi g_{jk} - \nabla_l \nabla_j \delta_\psi g_{ik} - \right.$$
$$\left. - \nabla_k \nabla_i \delta_\psi g_{jl}\right) .$$

However, $\delta_\psi g_{ik} = \varepsilon_{ik}$, and hence we obtain

$$\delta_\psi \tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2}\left(\nabla_k \nabla_j \varepsilon_{il} + \nabla_l \nabla_i \varepsilon_{jk} - \nabla_l \nabla_j \varepsilon_{ik} - \right.$$
$$\left. - \nabla_k \nabla_i \varepsilon_{jl}\right) .$$

In other words,

$$Z_{ijkl} \overset{lin}{=} -2\delta_\psi \tilde{R}_{ijkl} .$$

Obviously the Z_{ijkl} possess almost the same fundamental symmetries as the components of the Riemann-Christoffel curvature tensor, i.e., $Z_{ijkl} = -\,Z_{jikl} = -\,Z_{ijlk}$ as well as the general asymmetry $Z_{ijkl} \neq Z_{klij}$ as

$$Z_{ijkl} - Z_{klij} = \left(R^r_{.ijl} + R^r_{.jli}\right)\varepsilon_{rk} + \left(R^r_{.klj} + R^r_{.lkj}\right)\varepsilon_{ir} +$$
$$+ \left(R^r_{.jik} + R^r_{.ikj}\right)\varepsilon_{rl} + \left(R^r_{.lik} + R^r_{.kli}\right)\varepsilon_{jr} -$$
$$- 2\left(\Gamma^r_{[jl]}\nabla_r \varepsilon_{ik} + \Gamma^r_{[ik]}\nabla_r \varepsilon_{jl} + \Gamma^r_{[kj]}\nabla_r \varepsilon_{il} + \right.$$
$$\left. + \Gamma^r_{[li]}\nabla_r \varepsilon_{jk}\right) .$$

When the tensor Z vanishes we have, of course, a set of integrable equations giving rise to the integrability condition for the components of the strain tensor, which is equivalent to the vanishing of the field δ. That is, to the first order in the components of the strain tensor, if the condition

$$\delta_\psi \tilde{R}_{ijkl} = 0$$

is satisfied identically.

Finally, we can write (still to the first order in the components of the strain tensor)

$$\delta^i = \oint\limits_{P_1-P_2} d\psi^i = \left(\omega^i_{.k}\xi^k\right)\Big|^{P_2}_{P_1} + \frac{1}{2}\oint\limits_{P_1-P_2} Z^i_{.jkl}\,\xi^j\, dS^{kl} =$$
$$= \left(\omega^i_{.k}\xi^k\right)\Big|^{P_2}_{P_1} - \oint\limits_{P_1-P_2} \delta_\psi \tilde{R}^i_{.jkl}\,\xi^j\, dS^{kl} ,$$

where

$$dS^{ik} = d\xi^i \delta\xi^k - d\xi^k \delta\xi^i$$

are the components of an infinitesimal closed surface in $\Im_3$ spanned by the displacements $d\xi$ and $\delta\xi$ in 2 preferred directions.

Ending this section, let us give further in-depth investigation of the local translational-rotational motion of points on the material body. Define the unit velocity vector by

$$\hat{U}^i = \frac{\zeta^i_4}{\sqrt{g_{kl}\,\zeta^k_4\zeta^l_4}} = \frac{d\xi^i}{ds} ,$$

such that

$$\zeta^i_4 = \frac{\partial\xi^i}{\partial t} = \left(g_{kl}\,\zeta^k_4\zeta^l_4\right)^{1/2}\frac{d\xi^i}{ds} ,$$

i.e.,

$$ds = \left(g_{ik}\,\zeta^i_4\zeta^k_4\right)^{1/2}\frac{\partial t}{\partial\xi^l}\,d\xi^l = \left(U_i U^i\right)^{1/2} dt = U\,dt .$$

Then the local equations of motion along arbitrary curves on the hypersurface of material coordinates $\Im_3 \subset \Re_4$ can be described by the quadruplet of unit space-time vectors $(\hat{U}, \hat{V}, \hat{W}, \in n)$ orthogonal to each other where the first three unit vectors (i.e., $\hat{U}, \hat{V}, \hat{W}$) are exclusively defined as local tangent vectors in the hypersurface $\Im_3$ and n is the unit normal vector to the hypersurface $\Im_3$. These equations of motion are derived by generalizing the ordinary Frenet equations of orientable points of a curve in three-dimensional Euclidean space to four-dimensions as well as to include effects of microspin generated by geometric torsion. Setting

$$\hat{U} = u^\mu \omega_\mu = \hat{U}^i g_i ,$$
$$\hat{V} = v^\mu \omega_\mu = \hat{V}^i g_i ,$$
$$\hat{W} = w^\mu \omega_\mu = \hat{W}^i g_i ,$$
$$n = n^\mu \omega_\mu ,$$

we obtain, in general, the following set of equations of motion of the material points on the material body:

$$\frac{\delta u^\mu}{\delta s} = k v^\mu ,$$
$$\frac{\delta v^\mu}{\delta s} = \tau w^\mu - k u^\mu ,$$
$$\frac{\delta w^\mu}{\delta s} = \tau v^\mu + \lambda n^\mu ,$$

$$\frac{\delta n^\mu}{\delta s} = \lambda w^\mu ,$$

where the operator

$$\frac{\delta}{\delta s} = \hat{U}^i \nabla_i = u^\mu \nabla_\mu$$

represents the *absolute covariant derivative* in $\Im_3 \subset \Re_4$. In the above equations we have defined the following invariants:

$$k = \left(G_{\mu\nu} \frac{\delta u^\mu}{\delta s} \frac{\delta u^\nu}{\delta s} \right)^{1/2} = \left(g_{ik} \frac{\delta \hat{U}^i}{\delta s} \frac{\delta \hat{U}^k}{\delta s} \right)^{1/2} ,$$

$$\tau = \in_{\mu\nu\rho\sigma} u^\mu v^\nu \frac{\delta v^\rho}{\delta s} n^\sigma = \in \in_{ijk} \hat{U}^i \hat{V}^j \frac{\delta \hat{V}^k}{\delta s} ,$$

$$\lambda = \left(G_{\mu\nu} \frac{\delta n^\mu}{\delta s} \frac{\delta n^\nu}{\delta s} \right)^{1/2} .$$

In our case, however, the vanishing of the extrinsic curvature of the hypersurface $\Im_3$ means that the direction of the unit normal vector n is fixed. Consequently, we have

$$\lambda = 0 ,$$

and our equations of motion can be written as

$$\hat{U}^k \tilde{\nabla}_k \hat{U}^i = k \hat{V}^i - T^i_{kl} \hat{U}^k \hat{U}^l ,$$

$$\hat{U}^k \tilde{\nabla}_k \hat{V}^i = \tau \hat{W}^i - k \hat{U}^i - T^i_{kl} \hat{V}^k \hat{U}^l ,$$

$$\hat{U}^k \tilde{\nabla}_k \hat{W}^i = \tau \hat{V}^i - T^i_{kl} \hat{W}^k \hat{U}^l$$

in three-dimensional notation. In particular, we note that, just as the components of the contorsion tensor T^i_{jk}, the scalar τ measures the twist of any given curve in $\Im_3$ due to microspin.

Furthermore, it can be shown that the gradient of the unit velocity vector can be decomposed accordingly as

$$\nabla_k \hat{U}_i = \alpha_{ik} + \beta_{ik} + \frac{1}{4} h_{ik} \hat{\theta} + \hat{U}_k \hat{A}_i ,$$

where

$$h_{ik} = g_{ik} - \hat{U}_i \hat{U}_k ,$$

$$\alpha_{ik} = \frac{1}{4} h^r_i h^s_k \left(\nabla_r \hat{U}_s + \nabla_s \hat{U}_r \right) = \frac{1}{4} h^r_i h^s_k \left(\tilde{\nabla}_r \hat{U}_s + \tilde{\nabla}_s \hat{U}_r \right) - \frac{1}{2} h^r_i h^s_k T^l_{(rs)} \hat{U}_l ,$$

$$\beta_{ik} = \frac{1}{4} h^r_i h^s_k \left(\nabla_r \hat{U}_s - \nabla_s \hat{U}_r \right) = \frac{1}{4} h^r_i h^s_k \left(\tilde{\nabla}_r \hat{U}_s - \tilde{\nabla}_s \hat{U}_r \right) - \frac{1}{2} h^r_i h^s_k T^l_{[rs]} \hat{U}_l ,$$

$$\hat{\theta} = \nabla_i \hat{U}^i ,$$

$$\hat{A}_i = \frac{\delta \hat{U}_i}{\delta s} .$$

Note that

$$h_{ik} \hat{U}^k = \alpha_{ik} \hat{U}^k = \beta_{ik} \hat{U}^k = 0 .$$

Setting $\bar{\lambda} = \left(g_{ik} \zeta^i_4 \zeta^k_4 \right)^{-1/2}$ such that $\hat{U}^i = \bar{\lambda} U^i$, we obtain in general

$$\bar{\lambda} \nabla_k U_i = \frac{1}{4} \bar{\lambda} h^r_i h^s_k \left(\nabla_r U_s + \nabla_s U_r \right) + \frac{1}{4} \bar{\lambda} h^r_i h^s_k \left(\nabla_r U_s - \nabla_s U_r \right) + \frac{1}{2} \bar{\lambda} \nabla_i U_k + \frac{1}{4} g_{ik} \frac{\delta \bar{\lambda}}{\delta s} + \frac{1}{4} \bar{\lambda} g_{ik} \nabla_l U^l + \bar{\lambda} U_i U_k \frac{\delta \bar{\lambda}}{\delta s} + \bar{\lambda}^2 U_k \frac{\delta U_i}{\delta s} - \frac{1}{4} \bar{\lambda}^2 U_i U_k \frac{\delta \bar{\lambda}}{\delta s} - \frac{1}{2} \bar{\lambda}^3 U_i \frac{\delta U_k}{\delta s} - \frac{1}{4} \bar{\lambda}^3 U_i U_k \nabla_l U^l .$$

Again, the vanishing of the extrinsic curvature of the hypersurface $\Im_3$ gives $\frac{\delta \bar{\lambda}}{\delta s} = 0$. Hence we have

$$\nabla_k U_i = \frac{1}{4} h^r_i h^s_k \left(\nabla_r U_s + \nabla_s U_r \right) + \frac{1}{4} h^r_i h^s_k \left(\nabla_r U_s - \nabla_s U_r \right) + \frac{1}{2} \nabla_i U_k + \frac{1}{4} g_{ik} \nabla_l U^l + \bar{\lambda} U_k \frac{\delta U_i}{\delta s} - \frac{1}{2} \bar{\lambda}^2 U_i \frac{\delta U_k}{\delta s} - \frac{1}{4} \bar{\lambda}^2 U_i U_k \nabla_l U^l ,$$

for the components of the velocity gradient tensor.

Meanwhile, with the help of the identities

$$\hat{U}^j \nabla_k \nabla_j \hat{U}_i = \nabla_k \left(\hat{U}^j \nabla_j \hat{U}_i \right) - \left(\nabla_k \hat{U}_j \right) \left(\nabla^j \hat{U}_i \right) = \nabla_k \hat{A}_i - \left(\nabla_k \hat{U}_j \right) \left(\nabla^j \hat{U}_i \right) ,$$

$$\hat{U}^j \left(\nabla_k \nabla_j - \nabla_j \nabla_k \right) \hat{U}_i = R^l_{.ijk} \hat{U}_l \hat{U}^j - 2 \Gamma^l_{[jk]} \hat{U}^j \nabla_l \hat{U}_i ,$$

we can derive the following equation:

$$\frac{\delta \hat{\theta}}{\delta s} = \nabla_i \left(\frac{\delta \hat{U}^i}{\delta s} \right) - \left(\nabla_i \hat{U}^k \right) \left(\nabla_k \hat{U}^i \right) - R_{ik} \hat{U}^i \hat{U}^k + 2 \Gamma^l_{[ik]} \hat{U}^i \nabla_l \hat{U}^k .$$

Hence we obtain

$$\frac{\delta \theta}{\delta s} = \nabla_i \left(\frac{\delta U^i}{\delta s} \right) - \bar{\lambda} \left(\nabla_i U^k \right) \left(\nabla_k U^i \right) - 2 \frac{\delta U^i}{\delta s} \nabla_i \left({}^e \log \lambda \right) - \bar{\lambda} R_{ik} U^i U^k + 2 \bar{\lambda} \Gamma^l_{[ik]} U^i \nabla_l U^k ,$$

for the rate of shear with respect to the local arc length of the material body.

4 Generalized components of the elasticity tensor of the material body $\Im_3$ in the presence of microspin and geometric discontinuities (defects)

As we know, the most general form of the components of a fourth-rank isotropic tensor is given in terms of spatial coordinates by

$$I_{ABCD} = C_1 \delta_{AB} \delta_{CD} + C_2 \delta_{AC} \delta_{BD} + C_3 \delta_{AD} \delta_{BC} ,$$

where C_1, C_2, and C_3 are constants. In the case of anisotropy, C_1, C_2, andC_3 are no longer constant but still remain invariant with respect to the change of the coordinate system. Transforming these to material coordinates, we have

$$I^{ij}_{..kl} = C_1 g^{ij} g_{kl} + C_2 \delta^i_k \delta^j_l + C_3 \delta^i_l \delta^j_k \,.$$

On reasonably relaxing the ordinary symmetries, we now generalize the components of the fourth-rank elasticity tensor with the addition of a geometrized part describing microspin and geometric discontinuities as follows:

$$C^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right) + \gamma \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right)\,,$$

where

$$\alpha = \frac{2}{15} A^{ik}_{.i.k} - \frac{1}{15} A^{ki}_{.i.k}\,,$$

$$\beta = \frac{1}{10} A^{ki}_{..ik} - \frac{1}{10} A^{ik}_{.i.k}\,,$$

$$\gamma = \frac{1}{2}\eta \tilde{R}\,,$$

where η is a non-zero constant, and where

$$A^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right)$$

are, of course, the components of the ordinary, non-microspin (non-micropolar) elasticity tensor obeying the symmetries $A^{ij}_{..kl} = A^{ji}_{..kl} = A^{ij}_{..lk} = A_{kl..}^{ij}$. Now if we define the remaining components by

$$B^{ij}_{..kl} = \frac{1}{2}\eta \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) \tilde{R}\,,$$

with $B^{ij}_{..kl} = -B^{ji}_{..kl} = -B^{ij}_{..lk} = B_{kl..}^{ij}$, then we have relaxed the ordinary symmetries of the elasticity tensor. Most importantly, we note that our choice of the Ricci curvature scalar $\tilde{R}$ (rather than the more general curvature scalar R of which $\tilde{R}$ is a component) to enter our generalized elasticity tensor is meant to accommodate very general situations such that in the absence of geometric discontinuities the above equations will in general still hold. This corresponds to the fact that the existence of the Ricci curvature tensor $\tilde{R}$ is primarily due to microspin while geometric discontinuities are described by the full curvature tensor R as we have seen in Section 2.

Now with the help of the decomposition of the Riemann-Christoffel curvature tensor, we obtain

$$C^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right) + \eta \Big(\tilde{W}^{ij}_{..kl} + \delta^i_k \tilde{R}^j_{.l} + \\ + \delta^j_l \tilde{R}^i_{.k} - \delta^i_l \tilde{R}^j_{.k} - \delta^j_k \tilde{R}^i_{.l} - \tilde{R}^{ij}_{..kl}\Big)$$

for the components of the generalized elasticity tensor. Hence for linear elastic continua/discontinua, with the help of the potential energy functional $\bar{F}$, i.e., the one given by

$$\bar{F} = \frac{1}{2} C_{ij..}^{\;\;kl} D^{ij} D_{kl}\,,$$

such that

$$\sigma_{ij} = \frac{\partial \bar{F}}{\partial D^{ij}}\,,$$

i.e.,

$$\sigma_{(ij)} = \frac{\partial \bar{F}}{\partial \varepsilon^{ij}}\,,$$

$$\sigma_{[ij]} = \frac{\partial \bar{F}}{\partial \omega^{ij}}\,,$$

we obtain the following constitutive relations:

$$\sigma_{ij} = C_{ij..}^{\;\;kl} D_{kl}\,,$$

relating the components of the stress tensor σ to the components of the displacement gradient tensor D. Then it follows, as we have expected, that the stress tensor becomes asymmetric. Since $B_{(ij)..}^{\;\;\;\;kl} = 0$, we obtain

$$\sigma_{(ij)} = C_{(ij)..}^{\;\;\;\;kl} D_{kl} = A_{ij..}^{\;\;kl} \varepsilon_{kl} = \alpha g_{ij} \varepsilon^k_{.k} + 2\beta \varepsilon_{ij}\,,$$

for the components of the symmetric part of the stress tensor, in terms of the components strain tensor and the dilation scalar $\kappa = \varepsilon^i_{.i}$. Correspondingly, since $A_{[ij]..}^{\;\;\;\;kl} = 0$, the components of the anti-symmetric part of the stress tensor are then given by

$$\begin{aligned}\sigma_{[ij]} &= C_{[ij]..}^{\;\;\;\;kl} D_{kl} = B_{ij..}^{\;\;kl} \omega_{kl} = \\ &= \eta \left(\tilde{W}_{ij..}^{\;\;kl} - \tilde{R}_{ij..}^{\;\;kl}\right) + \eta \Big(D_{i.}^{\;k} \tilde{R}_{jk} + D^k_{.j} \tilde{R}_{ik} - \\ &- D^k_{.i} \tilde{R}_{jk} - D_{j.}^{\;k} \tilde{R}_{ik}\Big) = \\ &= \eta \left(\tilde{W}_{ij..}^{\;\;kl} - \tilde{R}_{ij..}^{\;\;kl}\right) \omega_{kl} + 2\eta \left(\omega_{i.}^{\;k} \tilde{R}_{jk} - \omega_{j.}^{\;k} \tilde{R}_{ik}\right) = \\ &= \eta \omega_{ij} \tilde{R}\,,\end{aligned}$$

in terms of the components of the generalized vorticity tensor. We can now define the *geometrized microspin potential* by the scalar

$$S = \eta \tilde{R} = \eta \left(R + 2\tilde{\nabla}_i \omega^i + \omega_i \omega^i + T_{ijk} T^{ikj}\right)\,.$$

Then, more specifically, we write

$$\sigma_{[ij]} = S \left(\Omega_{ij} + \varpi_{ij} + \frac{\bar{\gamma}}{e} \omega^\mu_i \omega^\nu_j F_{\mu\nu}\right)\,.$$

From the above relations, we see that when the electromagnetic contribution vanishes, we arrive at a *geometrized Cosserat elasticity theory*. As we know, the standard Cosserat elasticity theory does not consider effects generated by the electromagnetic field. Various continuum theories which can be described as conservative theories often take into consideration electrostatic phenomena since the electric field is simply described by a gradient of a scalar potential which corresponds to their conservative description of force and stress. But that proves to be a limitation especially because magnetic effects are still neglected.

As usual, should we consider thermal effects, then we would define the components of the thermal stress t by

$$t_{ik} = -\mu_T g_{ik} \Delta T ,$$

where μ_T is the thermal coefficient and ΔT is the temperature increment. Hence the components of the generalized stress tensor become

$$\sigma_{ik} = C_{ik..}^{\;\;\;\;rs} D_{rs} - \mu_T g_{ik} \Delta T .$$

Setting now

$$\bar{\mu}_T = -\frac{1}{3} \frac{\mu_T}{(\alpha + 2\beta)} ,$$

we can alternatively write

$$\sigma_{ik} = C_{ik..}^{\;\;\;\;rs} \left(D_{rs} + \bar{\mu}_T g_{rs} \Delta T \right) .$$

Finally, we shall obtain

$$\sigma_{ik} = \left(\alpha \varepsilon^{r}_{.r} - \mu_T \Delta T \right) g_{ik} + 2\beta \varepsilon_{ik} + \eta \omega_{ik} \tilde{R} ,$$

i.e., as before

$$\sigma_{(ij)} = \left(\alpha \varepsilon^{r}_{.r} - \mu_T \Delta T \right) g_{ij} + 2\beta \varepsilon_{ij} ,$$

$$\sigma_{[ij]} = \eta \omega_{ij} \tilde{R} = \frac{1}{2} \eta \in_{ijk} S^k \tilde{R} ,$$

where S^k are the components of the generalized vorticity vector S.

We note that, as is customary, in order to accord with the standard physical description of continuum mechanics, we need to set

$$\alpha = G = \frac{E}{2(1+\nu)} ,$$

$$\beta = G \left(\frac{2\nu}{1 - 2\nu} \right) ,$$

where G is the shear modulus, E is Young's modulus, and ν is Poisson's ratio.

Extending the above description, we shall have a glimpse into the more general non-linear constitutive relations given by

$$\sigma_{ij} = C_{ij..}^{\;\;\;\;kl} D_{kl} + K_{ij..mn}^{\;\;\;\;kl} D_{kl} D^{mn} + \ldots ,$$

where the dots represent terms of higher order. Or, up to the second order in the displacement gradient tensor, we have

$$\sigma_{ij} = C_{ij..}^{\;\;\;\;kl} D_{kl} + K_{ij..mn}^{\;\;\;\;kl} D_{kl} D^{mn} .$$

Here the K_{ijklmn} are the components of the sixth-rank, isotropic, non-linear elasticity tensor whose most general form appears to be given by

$$\begin{aligned} K_{ijklmn} = {} & A_1 g_{ij} g_{kl} g_{mn} + A_2 g_{ij} g_{km} g_{nl} + A_3 g_{ij} g_{kn} g_{lm} + \\ & + A_4 g_{kl} g_{im} g_{jn} + A_5 g_{kl} g_{in} g_{jm} + A_6 g_{mn} g_{ik} g_{jl} + \\ & + A_7 g_{mn} g_{il} g_{jk} + A_8 g_{im} g_{jk} g_{nl} + A_9 g_{im} g_{jl} g_{kn} + \\ & + A_{10} g_{in} g_{jk} g_{lm} + A_{11} g_{in} g_{jl} g_{km} + A_{12} g_{jm} g_{ik} g_{nl} + \\ & + A_{13} g_{jm} g_{il} g_{kn} + A_{14} g_{jn} g_{ik} g_{lm} + A_{15} g_{jn} g_{il} g_{km} , \end{aligned}$$

where $A_1, A_2, \ldots, A_{15}$ are invariants. In a similar manner as in the generalized linear case, we shall call the following symmetries:

$$K_{ijklmn} = K_{klijmn} = K_{klmnij} = K_{mnijkl} .$$

Hence, we can bring the K_{ijklmn} into the form

$$\begin{aligned} K_{ijklmn} = {} & B_1 g_{ij} g_{kl} g_{mn} + B_2 g_{ij} \left(g_{km} g_{nl} + g_{kn} g_{lm} \right) + \\ & + B_3 g_{ij} \left(g_{km} g_{nl} - g_{kn} g_{lm} \right) + B_4 g_{kl} \left(g_{im} g_{jn} + \right. \\ & \left. + g_{in} g_{jm} \right) + B_5 g_{kl} \left(g_{im} g_{jn} - g_{in} g_{jm} \right) + \\ & + B_6 g_{mn} \left(g_{ik} g_{jl} + g_{il} g_{jk} \right) + B_7 g_{mn} \left(g_{ik} g_{jl} - \right. \\ & \left. - g_{il} g_{jk} \right) + B_8 g_{im} \left(g_{jk} g_{nl} + g_{jl} g_{kn} \right) + \\ & + B_9 g_{im} \left(g_{jk} g_{nl} - g_{jl} g_{kn} \right) + B_{10} g_{jm} \left(g_{ik} g_{nl} + \right. \\ & \left. + g_{il} g_{kn} \right) + B_{11} g_{jm} \left(g_{ik} g_{nl} - g_{il} g_{kn} \right) , \end{aligned}$$

where, again, $B_1, B_2, \ldots, B_{11}$ are invariants. As in the generalized linear case, relating the coefficients B_3, B_5, B_7, B_9, and B_{11} to the *generator of microspin* in our theory, i.e., the Riemann-Christoffel curvature tensor, we obtain

$$\begin{aligned} K_{ijklmn} = {} & \lambda_1 g_{ij} g_{kl} g_{mn} + \lambda_2 g_{ij} \left(g_{km} g_{nl} + g_{kn} g_{lm} \right) + \\ & + \lambda_3 g_{kl} \left(g_{im} g_{jn} + g_{in} g_{jm} \right) + \lambda_4 g_{mn} \left(g_{ik} g_{jl} + \right. \\ & \left. + g_{il} g_{jk} \right) + \lambda_5 g_{im} \left(g_{jk} g_{nl} + g_{jl} g_{kn} \right) + \\ & + \lambda_6 g_{jm} \left(g_{ik} g_{nl} + g_{il} g_{kn} \right) + \frac{1}{2} \kappa_1 g_{ij} \left(g_{km} g_{nl} - \right. \\ & \left. - g_{kn} g_{lm} \right) \tilde{R} + \frac{1}{2} \kappa_2 g_{kl} \left(g_{im} g_{jn} - g_{in} g_{jm} \right) \tilde{R} + \\ & + \frac{1}{2} \kappa_3 g_{mn} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \tilde{R} + \frac{1}{2} \kappa_4 g_{im} \left(g_{jk} g_{nl} - \right. \\ & \left. - g_{jl} g_{kn} \right) \tilde{R} + \frac{1}{2} \kappa_5 g_{jm} \left(g_{ik} g_{nl} - g_{il} g_{kn} \right) \tilde{R} , \end{aligned}$$

where we have set $B_1 = \lambda_1$, $B_2 = \lambda_2$, $B_4 = \lambda_3$, $B_6 = \lambda_4$, $B_8 = \lambda_5$, $B_{10} = \lambda_6$ and where, for constant $\kappa_1, \kappa_2, \ldots, \kappa_5$, the five quantities

$$K_1 = B_3 = \frac{1}{2} \kappa_1 \tilde{R} ,$$

$$K_2 = B_5 = \frac{1}{2} \kappa_2 \tilde{R} ,$$

$$K_3 = B_7 = \frac{1}{2} \kappa_3 \tilde{R} ,$$

$$K_4 = B_9 = \frac{1}{2} \kappa_4 \tilde{R} ,$$

$$K_5 = B_{11} = \frac{1}{2} \kappa_5 \tilde{R}$$

form a set of *additional microspin potentials*. Hence we see that in the non-linear case, at least there are in general six microspin potentials instead of just one as in the linear case.

Then the constitutive equations are readily derivable by means of the third-order potential functional

$$^{*}\bar{F} = \frac{1}{2} C_{ij..}^{\;\;kl} D^{ij} D_{kl} + \frac{1}{3} K_{ij..mn}^{\;\;kl} D^{ij} D_{kl} D^{mn}$$

through

$$\sigma_{ij} = \frac{\partial\, ^{*}\bar{F}}{\partial D^{ij}} = {}^{(1)}\sigma_{ij} + {}^{(2)}\sigma_{ij}\,,$$

where (1) indicates the linear part and (2) indicates the non-linear part. Note that this is true whenever the K_{ijklmn} in general possess the above mentioned symmetries. Direct, but somewhat lengthy, calculation gives

$$\begin{aligned} {}^{(2)}\sigma_{ij} &= K_{ij..mn}^{\;\;kl} D_{kl} D^{mn} = \\ &= \left(\lambda_1 \left(\varepsilon^{k}_{.k}\right)^2 + 2\lambda_2 \varepsilon_{kl}\varepsilon^{kl} + \kappa_1 \omega_{kl}\omega^{kl}\tilde{R}\right) g_{ij} + \\ &+ \varepsilon^{k}_{.k} \left(\left(2\lambda_3 + 2\lambda_4\right)\varepsilon_{ij} + \left(\kappa_2 + \kappa_3\right)\omega_{ij}\tilde{R}\right) + \\ &+ D^{k}_{i.} \left(2\lambda_5 \varepsilon_{jk} + \kappa_4 \omega_{jk}\tilde{R}\right) + D^{k}_{j.} \left(2\lambda_6 \varepsilon_{ik} + \kappa_5 \omega_{ik}\tilde{R}\right). \end{aligned}$$

Overall, we obtain, for the components of the stress tensor, the following:

$$\begin{aligned} \sigma_{ij} &= \left(\alpha\varepsilon^{k}_{.k} - \mu_T \Delta T\right) g_{ij} + 2\beta\varepsilon_{ij} + \eta\omega_{ij}\tilde{R} + \\ &+ \left(\lambda_1 \left(\varepsilon^{k}_{.k}\right)^2 + 2\lambda_2 \varepsilon_{kl}\varepsilon^{kl} + \kappa_1 \omega_{kl}\omega^{kl}\tilde{R}\right) g_{ij} + \\ &+ \varepsilon^{k}_{.k} \left(\left(2\lambda_3 + 2\lambda_4\right)\varepsilon_{ij} + \left(\kappa_2 + \kappa_3\right)\omega_{ij}\tilde{R}\right) + \\ &+ D^{k}_{i.} \left(2\lambda_5 \varepsilon_{jk} + \kappa_4 \omega_{jk}\tilde{R}\right) + D^{k}_{j.} \left(2\lambda_6 \varepsilon_{ik} + \kappa_5 \omega_{ik}\tilde{R}\right). \end{aligned}$$

5 Variational derivation of the field equations. Equations of motion

We shall now see that our theory can best be described, in the linear case, independently by 2 Lagrangian densities. We give the first Lagrangian density as

$$\begin{aligned} \bar{L} = \sqrt{g}\Big(&\sigma^{ik}\left(\nabla_k \psi_i - D_{ik}\right) + \frac{1}{2} C^{ij}_{..kl} D_{ij} D^{kl} - \\ &- \mu_T D^{i}_{.i} \Delta T + U^i \left(\nabla_i \psi_k\right) \left(f\psi^k - \rho_m U^k\right)\Big), \end{aligned}$$

where ρ_m is the material density and f is a scalar potential. From here we then arrive at the following invariant integral:

$$\begin{aligned} I = \int_{vol} \Big(&\sigma^{ik}\left(\nabla_{(k}\psi_{i)} - \varepsilon_{ik}\right) + \sigma^{ik}\left(\nabla_{[k}\psi_{i]} - \omega_{ik}\right) + \\ &+ \frac{1}{2} A^{ij}_{..kl}\varepsilon_{ij}\varepsilon^{kl} + \frac{1}{2} B^{ij}_{..kl}\omega_{ij}\omega^{kl} - \mu_T \varepsilon^{i}_{.i}\Delta T + \\ &+ U^i\left(\nabla_i\psi_k\right)\left(f\psi^k - \rho_m U^k\right)\Big)\, dV\,, \end{aligned}$$

where $dV = \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3$.

Writing $\bar{L} = \sqrt{g}\, L$, we then have

$$\begin{aligned} \delta I = \int_{vol} \bigg(&\frac{\partial L}{\partial\sigma^{ik}}\delta\sigma^{ik} + \frac{\partial L}{\partial\varepsilon^{ik}}\delta\varepsilon^{ik} + \frac{\partial L}{\partial\omega^{ik}}\delta\omega^{ik} + \\ &+ \frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\delta\left(\nabla_i\psi_k\right)\bigg)\, dV = 0\,. \end{aligned}$$

Now

$$\begin{aligned} \int_{vol} \frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\delta\left(\nabla_i\psi_k\right) dV &= \int_{vol} \nabla_i\left(\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\delta\psi_k\right) dV - \\ &- \int_{vol} \nabla_i\left(\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\right)\delta\psi_k\, dV = \\ &= -\int_{vol} \nabla_i\left(\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\right)\delta\psi_k\, dV, \end{aligned}$$

since the first term on the right-hand-side of the first line is an absolute differential that can be transformed away on the boundary of integration by means of the divergence theorem. Hence we have

$$\begin{aligned} \delta I = \int_{vol} \bigg(&\frac{\partial L}{\partial\sigma^{ik}}\delta\sigma^{ik} + \frac{\partial L}{\partial\varepsilon^{ik}}\delta\varepsilon^{ik} + \frac{\partial L}{\partial\omega^{ik}}\delta\omega^{ik} - \\ &- \nabla_i\left(\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\right)\delta\psi_k\bigg)\, dV = 0\,, \end{aligned}$$

where each term in the integrand is independent of the others. Note also that the variations $\delta\sigma^{ik}$, $\delta\varepsilon^{ik}$, $\delta\omega^{ik}$, and$\delta\psi_k$ are arbitrary.

From $\frac{\partial L}{\partial\sigma^{ik}} = 0$, we obtain

$$\varepsilon_{ik} = \nabla_{(k}\,\psi_{i)}\,,$$

$$\omega_{ik} = \nabla_{[k}\,\psi_{i]}\,,$$

i.e., the components of the strain and vorticity tensors, respectively.

From $\frac{\partial L}{\partial\varepsilon^{ik}} = 0$, we obtain

$$\sigma^{(ik)} = A^{ik}_{..rs}\varepsilon^{rs} - \mu_T g^{ik}\Delta T\,,$$

i.e., the symmetric components of the stress tensor.

From $\frac{\partial L}{\partial\omega^{ik}} = 0$, we obtain

$$\sigma^{[ik]} = B^{ik}_{..rs}\omega^{rs} = \eta\omega^{ik}\tilde{R}\,,$$

i.e., the anti-symmetric components of the stress tensor.

Finally, from the fourth variation we now show in detail that it yields the equations of motion. We first see that

$$\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)} = \sigma^{ik} + U^i\left(f\psi^k - \rho_m U^k\right)\,.$$

Hence

$$\begin{aligned} \nabla_i\left(\frac{\partial L}{\partial\left(\nabla_i\psi_k\right)}\right) &= \nabla_i\sigma^{ik} + \nabla_i\left(fU^i\right)\psi^k + fU^i\nabla_i\psi^k - \\ &- \nabla_i\left(\rho_m U^i\right)U^k - \rho_m U^i\nabla_i U^k\,. \end{aligned}$$

Define the "extended" shear scalar and the mass current density vector, respectively, through

$$l = \nabla_i \left(f U^i \right) ,$$

$$J^i = \rho_m U^i .$$

Now we readily identify the force per unit mass f and the body force per unit mass b, respectively, by

$$f^i = U^k \nabla_k U^i = \frac{\delta U^i}{\delta t} ,$$

$$b^i = \frac{1}{\rho_m} \left(l\psi^i + f \left(1 - \nabla_k J^k\right) U^i \right) = = \frac{1}{\rho_m} \left(l\psi^i + f \left(1 + \frac{\partial \rho_m}{\partial t}\right) U^i \right) ,$$

where we have used the relation

$$\frac{D\rho_m}{Dt} = -\rho_m \nabla_i U^i ,$$

i.e., $\frac{\partial \rho_m}{\partial t} + \nabla_i \left(\rho_m U^i\right) = 0$, derivable from the four-dimensional conservation law $\nabla_\mu \left(\rho_m \, {}^*U^\mu\right) = 0$ where ${}^*U^\mu = \left(U^i, c\right)$.

Hence we have (for arbitrary $\delta\psi_k$)

$$\int\limits_{vol} \left(\nabla_i \sigma^{ik} + \rho_m b^k - \rho_m f^k\right) \delta\psi_k dV = 0 ,$$

i.e., the equations of motion

$$\nabla_i \sigma^{ik} = \rho_m \left(f^k - b^k\right) .$$

Before we move on to the second Lagrangian density, let's discuss briefly the so-called couple stress, i.e., the couple per unit area also known as the distributed moment. We denote the couple stress tensor by the second-rank tensor field M. In analogy to the linear constitutive relations relating the stress tensor σ to displacement gradient tensor D, we write

$$M_{ik} = D_{ik..}^{\;\;\;rs} N_{rs} ,$$

where

$$D_{ijkl} = E_{ijkl} + F_{ijkl}$$

are assumed to possess the same symmetry properties as C_{ijkl} (i.e., E_{ijkl} have the same symmetry properties as A_{ijkl} while F_{ijkl}, representing the chirality part, have the same symmetry properties as B_{ijkl}).

Likewise,

$$N_{ik} = N_{(ik)} + N_{[ik]} = X_{ik} + Y_{ik}$$

are comparable to $D_{ik} = D_{(ik)} + D_{[ik]} = \varepsilon_{ik} + \omega_{ik}$.

As a boundary condition, let us now define a completely anti-symmetric third-rank spin tensor as follows:

$$J^{ikl} = J^{[ikl]} = \frac{1}{2} \in^{ikl} \psi ,$$

where ψ is a scalar function such that the spin tensor of our theory (which contains both the macrospin and microspin tensors) can be written as a gradient, i.e.,

$$S_i = \eta \in_{ijk} \tilde{R}\omega^{jk} = \nabla_i \psi ,$$

such that whenever we desire to subject the above to the integrability condition $\in_{ijk} \nabla^j S^k = 0$, we have $\in^{ijk} \Gamma^l_{[jk]} S_l = 0$, resulting in $Y_{ik} = 0$.

In other words,

$$\psi = \psi_0 + \eta \int \in_{ijk} \tilde{R}\omega^{ij} d\xi^k ,$$

where ψ_0 is constant, acts as a scalar generator of spin.

As a consequence, we see that

$$\begin{aligned} \nabla_l J^{ikl} &= \frac{1}{2} \in^{ikl} \nabla_l \psi = \frac{1}{2} \in^{ikl} S_l = \\ &= \frac{1}{2} \eta \tilde{R} \in^{ikl} \in_{pql} \omega^{pq} \\ &= \frac{1}{2} \eta \tilde{R} \left(\delta^i_p \delta^k_q - \delta^i_q \delta^k_p\right) \omega^{pq} \\ &= \eta \tilde{R} \omega^{ik} , \end{aligned}$$

i.e.,

$$\nabla_l J^{ikl} = \sigma^{[ik]} .$$

Taking the divergence of the above equations and using the relations $2\nabla_{[k} \nabla_{i]} \psi = -2\Gamma^l_{[ik]} \nabla_l \psi$, we obtain the following divergence equations:

$$\nabla_k \sigma^{[ik]} = \frac{1}{2} \in^{ikl} \Gamma^r_{[kl]} S_r ,$$

coupling the components of the spin vector to the components of the torsion tensor. Furthermore, we obtain

$$\nabla_k \omega^{ik} = \frac{1}{2} \left(\eta \tilde{R}\right) \in^{ikl} \Gamma^r_{[kl]} S_r - \omega^{ik} \frac{\partial^{\,e} \log \left(\eta \tilde{R}\right)}{\partial \xi^k} .$$

We now form the second Lagrangian density of our theory as

$$\begin{aligned} \bar{H} = \sqrt{g} \Big(& M^{ik} \left(\nabla_k S_i - N_{ik}\right) + \frac{1}{2} D^{ij}_{\;\;..kl} N_{ij} N^{kl} - \\ & - \in^k_{.rs} \left(\nabla_i S_k\right) J^{rsi} + U^i \nabla_i S_k \left(h S^k - I \rho_m V^k\right) \Big), \end{aligned}$$

where h is a scalar function (not to be confused with the scalar function f), I is the moment of inertia, and V^i are the components of the angular velocity field.

Hence the action integral corresponding to this is

$$\begin{aligned} J = \int\limits_{vol} \Big(& M^{ik} \left(\nabla_{(k} S_{i)} - X_{ik}\right) + M^{ik} \left(\nabla_{[k} S_{i]} - Y_{ik}\right) + \\ & + \frac{1}{2} E^{ij}_{\;\;..kl} X_{ij} X^{kl} + \frac{1}{2} F^{ij}_{\;\;..kl} Y_{ij} Y^{kl} - \in^k_{.rs} \left(\nabla_i S_k\right) J^{rsi} + \\ & + U^i \left(\nabla_i S_k\right) \left(h S^k - I \rho_m V^k\right) \Big) dV . \end{aligned}$$

As before, writing $\bar{H} = \sqrt{g}H$ and performing the variation $\delta J = 0$, we have

$$\delta J = \int_{vol} \left(\frac{\partial H}{\partial M^{ik}}\,\delta M^{ik} + \frac{\partial H}{\partial X^{ik}}\,\delta X^{ik} + \frac{\partial H}{\partial Y^{ik}}\,\delta Y^{ik} - \right.$$

$$\left. - \nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k\right)} \right) \delta S_k \right) dV = 0\,,$$

with arbitrary variations δM^{ik}, δX^{ik}, δY^{ik}, and δS_k.

From $\frac{\partial H}{\partial M^{ik}} = 0$, we obtain

$$X_{ik} = \nabla_{(k}\, S_{i)}\,,$$

$$Y_{ik} = \nabla_{[k}\, S_{i]}\,.$$

From $\frac{\partial H}{\partial X^{ik}} = 0$, we obtain

$$M^{(ik)} = E^{ik}_{..rs} X^{rs}\,.$$

From $\frac{\partial H}{\partial Y^{ik}} = 0$, we obtain

$$M^{[ik]} = F^{ik}_{..rs} Y^{rs}\,.$$

Again, we shall investigate the last variation

$$-\int_{vol} \nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k\right)} \right) \delta S_k dV = 0$$

in detail.

Firstly,

$$\frac{\partial H}{\partial \left(\nabla_i S_k\right)} = M^{ik} - \in^{k}_{.rs} J^{rsi} + U^i \left(hS^k - I\rho V^k \right)\,.$$

Then we see that

$$\nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k\right)} \right) = \nabla_i M^{ik} - \in^{k}_{.rs} \sigma^{[rs]} + \nabla_i \left(hU^i \right) S^k +$$

$$+ hU^i \nabla_i S^k - I \nabla_i \left(\rho_m U^i \right) V^k -$$

$$- I \rho_m U^i \nabla_i V^k\,.$$

We now define the angular force per unit mass α by

$$\alpha^i = U^k \nabla_k V^i = \frac{\delta V^i}{\delta t}\,,$$

and the angular body force per unit mass β by

$$\beta^i = \frac{1}{\rho_m} \left(\bar{l} S^i + h \frac{\delta S^i}{\delta t} - I \left(\nabla_k J^k \right) V^i \right)\,,$$

where $\bar{l} = \nabla_i \left(h\; U^i \right)$.

We have

$$\int_{vol} \left(\nabla_i M^{ik} - \in^{k}_{.rs} \sigma^{[rs]} + \rho_m \beta^k - I \rho_m \alpha^k \right) \delta S_k dV = 0\,.$$

Hence we obtain the equations of motion

$$\nabla_i M^{ik} = \in^{k}_{.rs} \sigma^{[rs]} + \rho_m \left(I \alpha^k - \beta^k \right)\,.$$

6 Concluding remarks

At this point we see that we have reproduced the field equations and the equations of motion of Cosserat elasticity theory by our variational method, and hence we have succeeded in showing parallels between the fundamental equations of Cosserat elasticity theory and those of our present theory. However we must again emphasize that our field equations as well as our equations of motion involving chirality are *fully geometrized*. In other words, we have succeeded in generalizing various extensions of the classical elasticity theory, especially the Cosserat theory and the so-called void elasticity theory by ascribing both microspin phenomena and geometric defects to the action of geometric torsion and to the source of local curvature of the material space. As we have seen, it is precisely this curvature that plays the role of a fundamental, intrinsic differential invariant which explains microspin and defects throughout the course of our work.

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