SPECIAL REPORT

New Approach to Quantum Electrodynamics

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It is shown that a photon with a specific frequency can be identified with the Dirac magnetic monopole. When a Dirac-Wilson line forms a Dirac-Wilson loop, it is a photon. This loop model of photon is exactly solvable. From the winding numbers of this loop-form of photon, we derive the quantization properties of energy and electric charge. A new QED theory is presented that is free of ultraviolet divergences. The Dirac-Wilson line is as the quantum photon propagator of the new QED theory from which we can derive known QED effects such as the anomalous magnetic moment and the Lamb shift. The one-loop computation of these effects is simpler and is more accurate than that in the conventional QED theory. Furthermore, from the new QED theory, we have derived a new QED effect. A new formulation of the Bethe-Salpeter (BS) equation solves the difficulties of the BS equation and gives a modified ground state of the positronium. By the mentioned new QED effect and by the new formulation of the BS equation, a term in the orthopositronium decay rate that is missing in the conventional QED is found, resolving the orthopositronium lifetime puzzle completely. It is also shown that the graviton can be constructed from the photon, yielding a theory of quantum gravity that unifies gravitation and electromagnetism.

1 Introduction

It is well known that the quantum era of physics began with the quantization of energy of electromagnetic field, from which Planck derived the radiation formula. Einstein then introduced the light-quantum to explain the photoelectric effects. This light-quantum was regarded as a particle called photon [1–3]. Quantum mechanics was then developed, ushering in the modern quantum physics era. Subsequently, the quantization of the electromagnetic field and the theory of Quantum Electrodynamics (QED) were established.

In this development of quantum theory of physics, the photon plays a special role. While it is the beginning of quantum physics, it is not easy to understand as is the quantum mechanics of other particles described by the Schrödinger equation. In fact, Einstein was careful in regarding the light-quantum as a particle, and the acceptance of the light-quantum as a particle called photon did not come about until much later [1]. The quantum field theory of electromagnetic field was developed for the photon. However, such difficulties of the quantum field theory as the ultraviolet divergences are well known. Because of the difficulty of understanding the photon, Einstein once asked: "What is the photon?" [1].

On the other hand, based on the symmetry of the electric and magnetic field described by the Maxwell equation and on the complex wave function of quantum mechanics, Dirac derived the concept of the magnetic monopole, which is hypothetically considered as a particle with magnetic charge, in analogy to the electron with electric charge. An important feature of this magnetic monopole is that it gives the quantization of electric charge. Thus it is interesting and important to find such particles. However, in spite of much effort, no such particles have been found [4, 5].

In this paper we shall establish a mathematical model of photon to show that the magnetic monopole can be identified as a photon. Before giving the detailed model, let us discuss some thoughts for this identification in the following.

First, if the photon and the magnetic monopole are different types of elementary quantum particles in the electromagnetic field, it is odd that one type can be derived from the other. A natural resolution of this oddity is the identification of the magnetic monopole as a photon.

The quantum field theory of the free Maxwell equation is the basic quantum theory of photon [6]. This free field theory is a linear theory and the models of the quantum particles obtained from this theory are linear. However, a stable particle should be a soliton, which is of the nonlinear nature. Secondly, the quantum particles of the quantum theory of Maxwell equation are collective quantum effects in the same way the phonons which are elementary excitations in a statistical model. These phonons are usually considered as quasi-particles and are not regarded as real particles. Regarding the Maxwell equation as a statistical wave equation of electromagnetic field, we have that the quantum particles in the quantum theory of Maxwell equation are analogous to the phonons. Thus they should be regarded as quasi-photons and have properties of photons but not a complete description of photons.

In this paper, a nonlinear model of photon is established. In the model, we show that the Dirac magnetic monopole can be identified with the photon with some frequencies. We provide a $U(1)$ gauge theory of Quantum Electrodynamics (QED), from which we derive photon as a quantum Dirac-Wilson loop $W(z, z)$ of this model. This nonlinear loop model of the photon is exactly solvable and thus may be regarded as a quantum soliton. From the winding numbers of this loop model of the photon, we derive the quantization property of energy in Planck's formula of radiation and the quantization property of charge. We show that the quantization property of charge is derived from the quantization property of energy (in Planck's formula of radiation), when the magnetic monopole is identified with photon with certain frequencies. This explains why we cannot physically find a magnetic monopole. It is simply a photon with a specific frequency.

From this nonlinear model of the photon, we also construct a model of the electron which has a mass mechanism for generating mass of the electron. This mechanism of generating mass supersedes the conventional mechanism of generating mass (through the Higgs particles) and makes hypothesizing the existence of the Higgs particles unnecessary. This explains why we cannot physically find such Higgs particles.

The new quantum gauge theory is similar to the conventional QED theory except that the former is not based on the four dimensional space-time $(t, \mathbf{x})$ but is based on the proper time s in the theory of relativity. Only in a later stage in the new quantum gauge theory, the space-time variable $(t, \mathbf{x})$ is derived from the proper time s through the Lorentz metric $ds^2 = dt^2 - d\mathbf{x}^2$ to obtain space-time statistics and explain the observable QED effects.

The derived space variable $\mathbf{x}$ is a random variable in this quantum gauge theory. Recall that the conventional quantum mechanics is based on the space-time. Since the space variable $\mathbf{x}$ is actually a random variable as shown in the new quantum gauge theory, the conventional quantum mechanics needs probabilistic interpretation and thus has a most mysterious measurement problem, on which Albert Einstein once remarked: "God does not play dice with the universe." In contrast, the new quantum gauge theory does not involve the mentioned measurement problem because it is not based on the space-time and is deterministic. Thus this quantum gauge theory resolves the mysterious measurement problem of quantum mechanics.

Using the space-time statistics, we employ Feynman diagrams and Feynman rules to compute the basic QED effects such as the vertex correction, the photon self-energy and the electron self-energy. In this computation of the Feynman integrals, the dimensional regularization method in the conventional QED theory is also used. Nevertheless, while the conventional QED theory uses it to reduce the dimension 4 of space-time to a (fractional) number n to avoid the ultraviolet divergences in the Feynman integrals, the new QED theory uses it to increase the dimension 1 of the proper time to a number n less than 4, which is the dimension of the space-time, to derive the space-time statistics. In the new QED theory, there are no ultraviolet divergences, and the dimensional regularization method is not used for *regularization*.

After this increase of dimension, the renormalization method is used to derive the well-known QED effects. Unlike the conventional QED theory, the renormalization method is used in the new QED theory to compute the space-time statistics, but not to remove the ultraviolet divergences, since the ultraviolet divergences do not occur in the new QED theory. From these QED effects, we compute the anomalous magnetic moment and the Lamb shift [6]. The computation does not involve numerical approximation as does that of the conventional QED and is simpler and more accurate.

For getting these QED effects, the quantum photon propagator $W(z, z')$, which is like a line segment connecting two electrons, is used to derive the electrodynamic interaction. (When the quantum photon propagator $W(z, z')$ forms a closed circle with $z = z'$, it then becomes a photon $W(z, z)$.) From this quantum photon propagator, a photon propagator is derived that is similar to the Feynman photon propagator in the conventional QED theory.

The photon-loop $W(z, z)$ leads to the renormalized electric charge e and the mass m of electron. In the conventional QED theory, the bare charge e_0 is of less importance than the renormalized charge e, in the sense that it is unobservable. In contrast, in this new theory of QED, the bare charge e_0 and the renormalized charge e are of equal importance. While the renormalized charge e leads to the physical results of QED, the bare charge e_0 leads to the universal gravitation constant G. It is shown that $e = n_e e_0$, where n_e is a very large winding number and thus e_0 is a very small number. It is further shown that the gravitational constant $G = 2e_0^2$ which is thus an extremely small number. This agrees with the fact that the experimental gravitational constant G is a very small number. The relationships, $e = n_e e_0$ and $G = 2e_0^2$, are a part of a theory unifying gravitation and electromagnetism. In this unified theory, the graviton propagator and the graviton are constructed from the quantum photon propagator. This construction leads to a theory of quantum gravity. In short, a new theory of quantum gravity is developed from the new QED theory in this paper, and unification of gravitation and electromagnetism is achieved.

In this paper, we also derive a new QED effect from the seagull vertex of the new QED theory. The conventional Bethe-Salpeter (BS) equation is reformulated to resolve its difficulties (such as the existence of abnormal solutions [7–32]) and to give a modified ground state wave function of the positronium. By the new QED effect and the reformulated BS equation, another new QED effect, a term in the orthopositronium decay rate that is missing in the conventional QED is discovered. Including the discovered term, the computed orthopositronium decay rate now agrees with the experimental rate, resolving the *orthopositronium lifetime puzzle* completely [33–52]. We note that the recent resolution of this orthopositronium lifetime puzzle resolves the puzzle only partially due to a special statistical nature of this new term in the orthopositronium decay rate.

This paper is organized as follows. In Section 2 we give a brief description of a new QED theory. With this theory, we introduce the classical Dirac-Wilson loop in Section 3. We show that the quantum version of this loop is a nonlinear exactly solvable model and thus can be regarded as a soliton. We identify this quantum Dirac-Wilson loop as a photon with the $U(1)$ group as the gauge group. To investigate the properties of this Dirac-Wilson loop, we derive a chiral symmetry from the gauge symmetry of this quantum model. From this chiral symmetry, we derive, in Section 4, a conformal field theory, which includes an affine Kac-Moody algebra and a quantum Knizhnik-Zamolodchikov (KZ) equation. A main point of our model on the quantum KZ equation is that we can derive two KZ equations which are dual to each other. This duality is the main point for the Dirac-Wilson loop to be exactly solvable and to have a winding property which explains properties of photon. This quantum KZ equation can be regarded as a quantum Yang-Mills equation.

In Sections 5 to 8, we solve the Dirac-Wilson loop in a form with a winding property, starting with the KZ equations. From the winding property of the Dirac-Wilson loop, we derive, in Section 9 and Section 10, the quantization of energy and the quantization of electric charge which are properties of photon and magnetic monopole. We then show that the quantization property of charge is derived from the quantization property of energy of Planck's formula of radiation, when we identify photon with the magnetic monopole for some frequencies. From this nonlinear model of photon, we also derive a model of the electron in Section 11. In this model of electron, we provide a mass mechanism for generating mass to electron. In Section 12, we show that the photon with a specific frequency can carry electric charge and magnetic charge, since an electron is formed from a photon with a specific frequency for giving the electric charge and magnetic charge. In Section 13, we derive the statistics of photons and electrons from the loop models of photons and electrons.

In Sections 14 to 22, we derive a new theory of QED, wherein we perform the computation of the known basic QED effects such as the photon self-energy, the electron self-energy and the vertex correction. In particular, we provide simpler and more accurate computation of the anomalous magnetic moment and the Lamb shift. Then in Section 23, we compute a new QED effect. Then from Section 24 to Section 25, we reformulate the Bethe-Salpeter (BS) equation. With this new version of the BS equation and the new QED effect, a modified ground state wave function of the positronium is derived. Then by this modified ground state of the positronium, we derive in Section 26 another new QED effect, a term missing in the theoretic orthopositronium decay rate of the conventional QED theory, and show that this new theoretical orthopositronium decay rate agrees with the experimental decay rate, completely resolving the orthopositronium life time puzzle [33–52].

In Section 27, the graviton is derived from the photon. This leads to a new theory of quantum gravity and a new unification of gravitation and electromagnetism. Then in Section 28, we show that the quantized energies of gravitons can be identified as dark energy. Then in a way similar to the construction of electrons by photons, we use gravitons to construct particles which can be regarded as dark matter. We show that the force among gravitons can be repulsive. This gives the diffusion phenomenon of dark energy and the accelerating expansion of the universe [53–57].

2 New gauge model of QED

Let us construct a quantum gauge model, as follows. In probability theory we have the Wiener measure ν which is a measure on the space $C[t_0, t_1]$ of continuous functions [58]. This measure is a well defined mathematical theory for the Brownian motion and it may be symbolically written in the following form:

$$d\nu = e^{-L_0}dx\,, \tag{1}$$

where $L_0 := \frac{1}{2}\int_{t_0}^{t_1}\left(\frac{dx}{dt}\right)^2 dt$ is the energy integral of the Brownian particle and $dx = \frac{1}{N}\prod_t dx(t)$ is symbolically a product of Lebesgue measures $dx(t)$ and N is a normalized constant.

Once the Wiener measure is defined we may then define other measures on $C[t_0, t_1]$, as follows [58]. Let a potential term $\frac{1}{2}\int_{t_0}^{t_1} V\,dt$ be added to L_0. Then we have a measure ν_1 on $C[t_0, t_1]$ defined by:

$$d\nu_1 = e^{-\frac{1}{2}\int_{t_0}^{t_1} V\,dt}\,d\nu\,. \tag{2}$$

Under some condition on V we have that ν_1 is well defined on $C[t_0, t_1]$. Let us call (2) as the Feynman-Kac formula [58].

Let us then follow this formula to construct a quantum model of electrodynamics, as follows. Then similar to the formula (2) we construct a quantum model of electrodynamics from the following energy integral:

$$\begin{aligned} -\int_{s_0}^{s_1} D\,ds := -\int_{s_0}^{s_1}\Big[\frac{1}{2}\left(\frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1}\right)^*\left(\frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1}\right) + \\ + \left(\frac{dZ^*}{ds} + ie_0\Big(\sum_{j=1}^2 A_j\frac{dx^j}{ds}\Big)Z^*\right) \times \\ \times \left(\frac{dZ}{ds} - ie_0\Big(\sum_{j=1}^2 A_j\frac{dx^j}{ds}\Big)Z\right)\Big]ds\,, \end{aligned} \tag{3}$$

where the complex variable $Z = Z(z(s))$ and the real variables $A_1 = A_1(z(s))$ and $A_2 = A_2(z(s))$ are continuous functions in a form that they are in terms of a (continuously differentiable) curve $z(s) = C(s) = (x^1(s), x^2(s))$, $s_0 \leqslant s \leqslant s_1$, $z(s_0) = z(s_1)$ in the complex plane where s is a parameter representing the proper time in relativity. (We shall also write

$z(s)$ in the complex variable form $C(s) = z(s) = x^1(s) + + ix^2(s)$, $s_0 \leqslant s \leqslant s_1$.) The complex variable $Z = Z(z(s))$ represents a field of matter, such as the electron (Z^* denotes its complex conjugate), and the real variables $A_1 = A_1(z(s))$ and $A_2 = A_2(z(s))$ represent a connection (or the gauge field of the photon) and e_0 denotes the (bare) electric charge.

The integral (3) has the following gauge symmetry:

$$\begin{aligned} Z'(z(s)) &:= Z(z(s))\, e^{ie_0 a(z(s))} \\ A'_j(z(s)) &:= A_j(z(s)) + \frac{\partial a}{\partial x^j}\,, \quad j = 1, 2 \end{aligned} \tag{4}$$

where $a = a(z)$ is a continuously differentiable real-valued function of z.

We remark that this QED theory is similar to the conventional Yang-Mills gauge theories. A feature of (3) is that it is not formulated with the four-dimensional space-time but is formulated with the one dimensional proper time. This one dimensional nature let this QED theory avoid the usual ultraviolet divergence difficulty of quantum fields. As most of the theories in physics are formulated with the space-time let us give reasons of this formulation. We know that with the concept of space-time we have a convenient way to understand physical phenomena and to formulate theories such as the Newton equation, the Schrödinger equation, e.t.c. to describe these physical phenomena. However we also know that there are fundamental difficulties related to space-time such as the ultraviolet divergence difficulty of quantum field theory. To resolve these difficulties let us reexamine the concept of space-time. We propose that the space-time is a statistical concept which is not as basic as the proper time in relativity. Because a statistical theory is usually a convenient but incomplete description of a more basic theory this means that some difficulties may appear if we formulate a physical theory with the space-time. This also means that a way to formulate a basic theory of physics is to formulate it not with the space-time but with the proper time only as the parameter for evolution. This is a reason that we use (3) to formulate a QED theory. In this formulation we regard the proper time as an independent parameter for evolution. From (3) we may obtain the conventional results in terms of space-time by introducing the space-time as a statistical method.

Let us explain in more detail how the space-time comes out as a statistics. For statistical purpose when many electrons (or many photons) present we introduce space-time (t, x) as a statistical method to write ds^2 in the form

$$ds^2 = dt^2 - d\mathrm{x}^2. \tag{5}$$

We notice that for a given ds there may have many dt and $d\mathrm{x}$ which correspond to many electrons (or photons) such that (5) holds. In this way the space-time is introduced as a statistics. By (5) we shall derive statistical formulas for many electrons (or photons) from formulas obtained from (3). In this way we obtain the Dirac equation as a statistical equation for electrons and the Maxwell equation as a statistical equation for photons. In this way we may regard the conventional QED theory as a statistical theory extended from the proper-time formulation of this QED theory (From the proper-time formulation of this QED theory we also have a theory of space-time statistics which give the results of the conventional QED theory). This statistical interpretation of the conventional QED theory is thus an explanation of the mystery that the conventional QED theory is successful in the computation of quantum effects of electromagnetic interaction while it has the difficulty of ultraviolet divergence.

We notice that the relation (5) is the famous Lorentz metric. (We may generalize it to other metric in General Relativity.) Here our understanding of the Lorentz metric is that it is a statistical formula where the proper time s is more fundamental than the space-time (t, x) in the sense that we first have the proper time and the space-time is introduced via the Lorentz metric only for the purpose of statistics. This reverses the order of appearance of the proper time and the space-time in the history of relativity in which we first have the concept of space-time and then we have the concept of proper time which is introduced via the Lorentz metric. Once we understand that the space-time is a statistical concept from (3) we can give a solution to the quantum measurement problem in the debate about quantum mechanics between Bohr and Einstein. In this debate Bohr insisted that with the probability interpretation quantum mechanics is very successful. On the other hand Einstein insisted that quantum mechanics is incomplete because of probability interpretation. Here we resolve this debate by constructing the above QED theory which is a quantum theory as the quantum mechanics and unlike quantum mechanics which needs probability interpretation we have that this QED theory is deterministic since it is not formulated with the space-time.

Similar to the usual Yang-Mills gauge theory we can generalize this gauge theory with $U(1)$ gauge symmetry to non-abelian gauge theories. As an illustration let us consider $SU(2) \otimes U(1)$ gauge symmetry where $SU(2) \otimes U(1)$ denotes the direct product of the groups $SU(2)$ and $U(1)$.

Similar to (3) we consider the following energy integral:

$$\begin{aligned} L := \int_{s_0}^{s_1} \Big[\tfrac{1}{2}\, Tr\,(D_1 A_2 - D_2 A_1)^* (D_1 A_2 - D_2 A_1) + \\ + (D_0^* Z^*)(D_0 Z) \Big]\, ds\,, \end{aligned} \tag{6}$$

where $Z = (z_1, z_2)^T$ is a two dimensional complex vector; $A_j = \sum_{k=0}^{3} A_j^k t^k$ $(j = 1, 2)$ where A_j^k denotes a component of a gauge field A^k; $t^k = i\,T^k$ denotes a generator of $SU(2) \otimes U(1)$ where T^k denotes a self-adjoint generator of $SU(2) \otimes U(1)$ (here for simplicity we choose a convention that the complex i is absorbed by t^k and t^k is absorbed by A_j; and the notation A_j is with a little confusion with the notation A_j in the above formulation of (3) where $A_j, j = 1, 2$ are real valued); and $D_l = \frac{\partial}{\partial x_l} - e_0 (\sum_{j=1}^{2} A_j \frac{dx^j}{ds})$ for $l = 1, 2$; and $D_0 = \frac{d}{ds} - e_0 (\sum_{j=1}^{2} A_j \frac{dx^j}{ds})$ where e_0 is the bare electric charge for general interactions including the strong and weak interactions.

From (6) we can develop a nonabelian gauge theory as similar to that for the above abelian gauge theory. We have that (6) is invariant under the following gauge transformation:

$$Z'(z(s)) := U(a(z(s)))\, Z(z(s))$$

$$A'_j(z(s)) := U(a(z(s)))\, A_j(z(s))\, U^{-1}(a(z(s))) + \\ + U(a(z(s)))\frac{\partial U^{-1}}{\partial x^j}(a(z(s))), \quad j=1,2 \tag{7}$$

where $U(a(z(s))) = e^{a(z(s))}$; $a(z(s)) = \sum_k e_0 a^k(z(s))t^k$ for some functions a^k. We shall mainly consider the case that a is a function of the form $a(z(s)) = \sum_k \mathrm{Re}\,\omega^k(z(s))t^k$ where ω^k are analytic functions of z. (We let the function $\omega(z(s)) := \sum_k \omega^k(z(s))t^k$ and we write $a(z) = \mathrm{Re}\,\omega(z)$.)

The above gauge theory is based on the Banach space X of continuous functions $Z(z(s))$, $A_j(z(s))$, $j=1,2$, $s_0 \leqslant s \leqslant s_1$ on the one dimensional interval $[s_0, s_1]$.

Since L is positive and the theory is one dimensional (and thus is simpler than the usual two dimensional Yang-Mills gauge theory) we have that this gauge theory is similar to the Wiener measure except that this gauge theory has a gauge symmetry. This gauge symmetry gives a degenerate degree of freedom. In the physics literature the usual way to treat the degenerate degree of freedom of gauge symmetry is to introduce a gauge fixing condition to eliminate the degenerate degree of freedom where each gauge fixing will give equivalent physical results [59]. There are various gauge fixing conditions such as the Lorentz gauge condition, the Feynman gauge condition, etc. We shall later in the Section on the Kac-Moody algebra adopt a gauge fixing condition for the above gauge theory. This gauge fixing condition will also be used to derive the quantum KZ equation in dual form which will be regarded as a quantum Yang-Mill equation since its role will be similar to the classical Yang-Mill equation derived from the classical Yang-Mill gauge theory.

3 Classical Dirac-Wilson loop

Similar to the Wilson loop in quantum field theory [60] from our quantum theory we introduce an analogue of Wilson loop, as follows. (We shall also call a Wilson loop as a Dirac-Wilson loop.)

Definition A classical Wilson loop $W_R(C)$ is defined by:

$$W_R(C) := W(z_0, z_1) := Pe^{e_0 \int_C A_j dx^j}, \tag{8}$$

where R denotes a representation of $SU(2)$; $C(\cdot) = z(\cdot)$ is a fixed closed curve where the quantum gauge theories are based on it as specific in the above Section. As usual the notation P in the definition of $W_R(C)$ denotes a path-ordered product [60–62].

Let us give some remarks on the above definition of Wilson loop, as follows.

1. We use the notation $W(z_0, z_1)$ to mean the Wilson loop $W_R(C)$ which is based on the whole closed curve $z(\cdot)$. Here for convenience we use only the end points z_0 and z_1 of the curve $z(\cdot)$ to denote this Wilson loop (We keep in mind that the definition of $W(z_0, z_1)$ depends on the whole curve $z(\cdot)$ connecting z_0 and z_1).

Then we extend the definition of $W_R(C)$ to the case that $z(\cdot)$ is not a closed curve with $z_0 \neq z_1$. When $z(\cdot)$ is not a closed curve we shall call $W(z_0, z_1)$ as a Wilson line.

2. In constructing the Wilson loop we need to choose a representation R of the $SU(2)$ group. We shall see that because a Wilson line $W(z_0, z_1)$ is with two variables z_0 and z_1 a natural representation of a Wilson line or a Wilson loop is the tensor product of the usual two dimensional representation of the $SU(2)$ for constructing the Wilson loop. ⋄

A basic property of a Wilson line $W(z_0, z_1)$ is that for a given continuous path A_j, $j=1,2$ on $[s_0, s_1]$ the Wilson line $W(z_0, z_1)$ exists on this path and has the following transition property:

$$W(z_0, z_1) = W(z_0, z)W(z, z_1) \tag{9}$$

where $W(z_0, z_1)$ denotes the Wilson line of a curve $z(\cdot)$ which is with z_0 as the starting point and z_1 as the ending point and z is a point on $z(\cdot)$ between z_0 and z_1.

This property can be proved as follows. We have that $W(z_0, z_1)$ is a limit (whenever exists) of ordered product of $e^{A_j \triangle x^j}$ and thus can be written in the following form:

$$W(z_0, z_1) = I + \int_{s'}^{s''} e_0 A_j(z(s))\frac{dx^j(s)}{ds}\,ds + \\ + \int_{s'}^{s''} e_0 A_j(z(s_2))\frac{dx^j(s_2)}{ds} \times \\ \times \left[\int_{s'}^{s_2} e_0 A_j(z(s_3))\frac{dx^j(s_3)}{ds}ds_3\right]ds_2 + \cdots \tag{10}$$

where $z(s') = z_0$ and $z(s'') = z_1$. Then since A_i are continuous on $[s', s'']$ and $x^i(z(\cdot))$ are continuously differentiable on $[s', s'']$ we have that the series in (10) is absolutely convergent. Thus the Wilson line $W(z_0, z_1)$ exists. Then since $W(z_0, z_1)$ is the limit of ordered product we can write $W(z_0, z_1)$ in the form $W(z_0, z)W(z, z_1)$ by dividing $z(\cdot)$ into two parts at z. This proves the basic property of Wilson line. ⋄

Remark (classical and quantum versions of Wilson loop) From the above property we have that the Wilson line $W(z_0, z_1)$ exists in the classical pathwise sense where A_i are as classical paths on $[s_0, s_1]$. This pathwise version of the Wilson line $W(z_0, z_1)$; from the Feynman path integral point of view; is as a partial description of the quantum version of the Wilson line $W(z_0, z_1)$ which is as an operator when A_i are as operators. We shall in the next Section derive and define a quantum generator J of $W(z_0, z_1)$ from the quantum gauge theory. Then by using this generator J we shall compute the quantum version of the Wilson line $W(z_0, z_1)$.

We shall denote both the classical version and quantum version of Wilson line by the same notation $W(z_0, z_1)$ when there is no confusion. ⋄

By following the usual approach of deriving a chiral symmetry from a gauge transformation of a gauge field we have the following chiral symmetry which is derived by applying an analytic gauge transformation with an analytic function ω for the transformation:

$$\begin{aligned} W(z_0,z_1) \mapsto \ \ W'(z_0,z_1) \ = \\ = U(\omega(z_1))W(z_0,z_1)U^{-1}(\omega(z_0))\,, \end{aligned} \tag{11}$$

where $W'(z_0,z_1)$ is a Wilson line with gauge field:

$$A'_\mu = \frac{\partial U(z)}{\partial x^\mu}\,U^{-1}(z) + U(z)\,A_\mu\,U^{-1}(z)\,. \tag{12}$$

This chiral symmetry is analogous to the chiral symmetry of the usual guage theory where U denotes an element of the gauge group [61]. Let us derive (11) as follows. Let $U(z) := := U(\omega(z(s)))$ and $U(z+dz) \approx U(z) + \frac{\partial U(z)}{\partial x^\mu}\,dx^\mu$ where $dz = (dx^1, dx^2)$. Following [61] we have

$$\begin{aligned} &U(z+dz)(1+e_0dx^\mu A_\mu)U^{-1}(z) \ = \\ &= U(z+dz)U^{-1}(z) + e_0dx^\mu U(z+dz)A_\mu U^{-1}(s) \\ &\approx\ 1+\tfrac{\partial U(z)}{\partial x^\mu}U^{-1}(z)dx^\mu + e_0dx^\mu U(z+dz)A_\mu U^{-1}(s) \\ &\approx\ 1+\tfrac{\partial U(z)}{\partial x^\mu}U^{-1}(z)dx^\mu + e_0dx^\mu U(z)A_\mu U^{-1}(z) \\ &=:\ 1+\tfrac{\partial U(z)}{\partial x^\mu}U^{-1}(z)dx^\mu + e_0dx^\mu U(z)A_\mu U^{-1}(z) \\ &=:\ 1+e_0dx^\mu A'_\mu\,. \end{aligned} \tag{13}$$

From (13) we have that (11) holds.

As analogous to the WZW model in conformal field theory [65, 66] from the above symmetry we have the following formulas for the variations $\delta_\omega W$ and $\delta_{\omega'} W$ with respect to this symmetry (see [65] p.621):

$$\delta_\omega W(z,z') = W(z,z')\omega(z) \tag{14}$$

and

$$\delta_{\omega'} W(z,z') = -\omega'(z')W(z,z')\,, \tag{15}$$

where z and z' are independent variables and $\omega'(z') = \omega(z)$ when $z' = z$. In (14) the variation is with respect to the z variable while in (15) the variation is with respect to the z' variable. This two-side-variations when $z \neq z'$ can be derived as follows. For the left variation we may let ω be analytic in a neighborhood of z and extended as a continuously differentiable function to a neighborhood of z' such that $\omega(z') = 0$ in this neighborhood of z'. Then from (11) we have that (14) holds. Similarly we may let ω' be analytic in a neighborhood of z' and extended as a continuously differentiable function to a neighborhood of z such that $\omega'(z) = 0$ in this neighborhood of z. Then we have that (15) holds.

4 Gauge fixing and affine Kac-Moody algebra

This Section has two related purposes. One purpose is to find a gauge fixing condition for eliminating the degenerate degree of freedom from the gauge invariance of the above quantum gauge theory in Section 2. Then another purpose is to find an equation for defining a generator J of the Wilson line $W(z,z')$. This defining equation of J can then be used as a gauge fixing condition. Thus with this defining equation of J the construction of the quantum gauge theory in Section 2 is then completed.

We shall derive a quantum loop algebra (or the affine Kac-Moody algebra) structure from the Wilson line $W(z,z')$ for the generator J of $W(z,z')$. To this end let us first consider the classical case. Since $W(z,z')$ is constructed from $SU(2)$ we have that the mapping $z \to W(z,z')$ (We consider $W(z,z')$ as a function of z with z' being fixed) has a loop group structure [63, 64]. For a loop group we have the following generators:

$$J^a_n = t^a z^n \qquad n = 0\,, \pm 1, \pm 2, \ldots \tag{16}$$

These generators satisfy the following algebra:

$$[J^a_m, J^b_n] = if_{abc}J^c_{m+n}\,. \tag{17}$$

This is the so called loop algebra [63, 64]. Let us then introduce the following generating function J:

$$J(w) = \sum_a J^a(w) = \sum_a j^a(w)\,t^a, \tag{18}$$

where we define

$$J^a(w) = j^a(w)t^a := \sum_{n=-\infty}^{\infty} J^a_n(z)(w-z)^{-n-1}. \tag{19}$$

From J we have

$$J^a_n = \frac{1}{2\pi i}\oint_z dw\,(w-z)^n J^a(w)\,, \tag{20}$$

where $\oint_z$ denotes a closed contour integral with center z. This formula can be interpreted as that J is the generator of the loop group and that J^a_n is the directional generator in the direction $\omega^a(w) = (w-z)^n$. We may generalize (20) to the following directional generator:

$$\frac{1}{2\pi i}\oint_z dw\,\omega(w)\,J(w)\,, \tag{21}$$

where the analytic function $\omega(w) = \sum_a \omega^a(w)t^a$ is regarded as a direction and we define

$$\omega(w)J(w) := \sum_a \omega^a(w)J^a. \tag{22}$$

Then since $W(z,z') \in SU(2)$, from the variational formula (21) for the loop algebra of the loop group of $SU(2)$ we have that the variation of $W(z,z')$ in the direction $\omega(w)$ is given by

$$W(z,z')\frac{1}{2\pi i}\oint_z dw\,\omega(w)\,J(w)\,. \tag{23}$$

Now let us consider the quantum case which is based on the quantum gauge theory in Section 2. For this quantum case we shall define a quantum generator J which is analogous to the J in (18). We shall choose the equations (34) and (35) as the equations for defining the quantum generator J. Let us first give a formal derivation of the equation (34), as follows. Let us consider the following formal functional integration:

$$\begin{aligned}&\langle W(z,z')A(z)\rangle := \\ &:= \int dA_1 dA_2 dZ^* dZ e^{-L} W(z,z')A(z),\end{aligned} \tag{24}$$

where $A(z)$ denotes a field from the quantum gauge theory. (We first let z' be fixed as a parameter.)

Let us do a calculus of variation on this integral to derive a variational equation by applying a gauge transformation on (24) as follows. (We remark that such variational equations are usually called the Ward identity in the physics literature.)

Let (A_1, A_2, Z) be regarded as a coordinate system of the integral (24). Under a gauge transformation (regarded as a change of coordinate) with gauge function $a(z(s))$ this coordinate is changed to another coordinate (A_1', A_2', Z'). As similar to the usual change of variable for integration we have that the integral (24) is unchanged under a change of variable and we have the following equality:

$$\begin{aligned}&\int dA_1' dA_2' dZ'^* dZ' e^{-L'} W'(z,z')A'(z) = \\ &= \int dA_1 dA_2 dZ^* dZ e^{-L} W(z,z')A(z)\,,\end{aligned} \tag{25}$$

where $W'(z,z')$ denotes the Wilson line based on A_1' and A_2' and similarly $A'(z)$ denotes the field obtained from $A(z)$ with (A_1, A_2, Z) replaced by (A_1', A_2', Z').

Then it can be shown that the differential is unchanged under a gauge transformation [59]:

$$dA_1' dA_2' dZ'^* dZ' = dA_1 dA_2 dZ^* dZ\,. \tag{26}$$

Also by the gauge invariance property the factor e^{-L} is unchanged under a gauge transformation. Thus from (25) we have

$$0 = \langle W'(z,z')A'(z)\rangle - \langle W(z,z')A(z)\rangle\,, \tag{27}$$

where the correlation notation $\langle\cdot\rangle$ denotes the integral with respect to the differential

$$e^{-L} dA_1 dA_2 dZ^* dZ \tag{28}$$

We can now carry out the calculus of variation. From the gauge transformation we have the formula:

$$W'(z,z') = U(a(z))W(z,z')U^{-1}(a(z'))\,, \tag{29}$$

where $a(z) = \mathrm{Re}\,\omega(z)$. This gauge transformation gives a variation of $W(z,z')$ with the gauge function $a(z)$ as the variational direction a in the variational formulas (21) and (23). Thus analogous to the variational formula (23) we have that the variation of $W(z,z')$ under this gauge transformation is given by

$$W(z,z')\frac{1}{2\pi i}\oint_z dw\,a(w)\,J(w)\,, \tag{30}$$

where the generator J for this variation is to be specific. This J will be a quantum generator which generalizes the classical generator J in (23).

Thus under a gauge transformation with gauge function $a(z)$ from (27) we have the following variational equation:

$$\begin{aligned}0 = \Big\langle W(z,z')\Big[&\delta_a A(z) + \\ &+ \frac{1}{2\pi i}\oint_z dw\,a(w)J(w)A(z)\Big]\Big\rangle,\end{aligned} \tag{31}$$

where $\delta_a A(z)$ denotes the variation of the field $A(z)$ in the direction $a(z)$. From this equation an ansatz of J is that J satisfies the following equation:

$$W(z,z')\Big[\delta_a A(z) + \frac{1}{2\pi i}\oint_z dw\,a(w)J(w)A(z)\Big] = 0\,. \tag{32}$$

From this equation we have the following variational equation:

$$\delta_a A(z) = \frac{-1}{2\pi i}\oint_z dw\,a(w)J(w)A(z)\,. \tag{33}$$

This completes the formal calculus of variation. Now (with the above derivation as a guide) we choose the following equation (34) as one of the equation for defining the generator J:

$$\delta_\omega A(z) = \frac{-1}{2\pi i}\oint_z dw\,\omega(w)J(w)A(z)\,, \tag{34}$$

where we generalize the direction $a(z) = \mathrm{Re}\,\omega(z)$ to the analytic direction $\omega(z)$. (This generalization has the effect of extending the real measure of the pure gauge part of the gauge theory to include the complex Feynman path integral since it gives the transformation $ds \to -i\,ds$ for the integral of the Wilson line $W(z,z')$.)

Let us now choose one more equation for determine the generator J in (34). This choice will be as a gauge fixing condition. As analogous to the WZW model in conformal field theory [65–67] let us consider a J given by

$$J(z) := -k_0 W^{-1}(z,z')\,\partial_z W(z,z')\,, \tag{35}$$

where we define $\partial_z = \partial_{x^1} + i\partial_{x^2}$ and we set $z' = z$ after the differentiation with respect to z; $k_0 > 0$ is a constant which is fixed when the J is determined to be of the form (35) and the minus sign is chosen by convention. In the WZW model [65, 67] the J of the form (35) is the generator of the chiral symmetry of the WZW model. We can write the J in (35) in the following form:

$$J(w) = \sum_a J^a(w) = \sum_a j^a(w)\, t^a. \tag{36}$$

We see that the generators t^a of $SU(2)$ appear in this form of J and this form is analogous to the classical J in (18). This shows that this J is a possible candidate for the generator J in (34).

Since $W(z,z')$ is constructed by gauge field we need to have a gauge fixing for the computations related to $W(z,z')$. Then since the J in (34) and (35) is constructed by $W(z,z')$ we have that in defining this J as the generator J of $W(z,z')$ we have chosen a condition for the gauge fixing. In this paper we shall always choose this defining equations (34) and (35) for J as the gauge fixing condition.

In summary we introduce the following definition.

Definition The generator J of the quantum Wilson line $W(z,z')$ whose classical version is defined by (8), is an operator defined by the two conditions (34) and (35). ⋄

Remark We remark that the condition (35) first defines J classically. Then the condition (34) raises this classical J to the quantum generator J. ⋄

Now we want to show that this generator J in (34) and (35) can be uniquely solved. (This means that the gauge fixing condition has already fixed the gauge that the degenerate degree of freedom of gauge invariance has been eliminated so that we can carry out computation.)

Let us now solve J. From (11) and (35) the variation $\delta_\omega J$ of the generator J in (35) is given by [65, p. 622] and [67]:

$$\delta_\omega J = [J,\omega] - k_0 \partial_z \omega\,. \tag{37}$$

From (34) and (37) we have that J satisfies the following relation of current algebra [65–67]:

$$J^a(w)J^b(z) = \frac{k_0\delta_{ab}}{(w-z)^2} + \sum_c i f_{abc}\frac{J^c(z)}{(w-z)}\,, \tag{38}$$

where as a convention the regular term of $J^a(w)J^b(z)$ is omitted. Then by following [65–67] from (38) and (36) we can show that the J^a_n in (18) for the corresponding Laurent series of the quantum generator J satisfy the following Kac-Moody algebra:

$$[J^a_m, J^b_n] = i f_{abc} J^c_{m+n} + k_0\, m\, \delta_{ab}\delta_{m+n,0}\,, \tag{39}$$

where k_0 is usually called the central extension or the level of the Kac-Moody algebra.

Remark Let us also consider the other side of the chiral symmetry. Similar to the J in (35) we define a generator J' by:

$$J'(z') = k_0 \partial_{z'} W(z,z') W^{-1}(z,z')\,, \tag{40}$$

where after differentiation with respect to z' we set $z = z'$. Let us then consider the following formal correlation:

$$\begin{aligned}&\langle A(z')W(z,z')\rangle := \\ &:= \int dA_1 dA_2 dZ^* dZ A(z')\, W(z,z')\, e^{-L},\end{aligned} \tag{41}$$

where z is fixed. By an approach similar to the above derivation of (34) we have the following variational equation:

$$\delta_{\omega'} A(z') = \frac{-1}{2\pi i}\oint_{z'} dw A(z') J'(w)\omega'(w)\,, \tag{42}$$

where as a gauge fixing we choose the J' in (42) be the J' in (40). Then similar to (37) we also have

$$\delta_{\omega'} J' = [J',\omega'] - k_0 \partial_{z'}\omega'. \tag{43}$$

Then from (42) and (43) we can derive the current algebra and the Kac-Moody algebra for J' which are of the same form of (38) and (39). From this we have $J' = J$. ⋄

Now with the above current algebra J and the formula (34) we can follow the usual approach in conformal field theory to derive a quantum Knizhnik-Zamolodchikov (KZ) equation for the product of primary fields in a conformal field theory [65–67]. We derive the KZ equation for the product of n Wilson lines $W(z,z')$. Here an important point is that from the two sides of $W(z,z')$ we can derive two quantum KZ equations which are dual to each other. These two quantum KZ equations are different from the usual KZ equation in that they are equations for the quantum operators $W(z,z')$ while the usual KZ equation is for the correlations of quantum operators. With this difference we can follow the usual approach in conformal field theory to derive the following quantum Knizhnik-Zamolodchikov equation [65, 66, 68]:

$$\begin{aligned}&\partial_{z_i} W(z_1,z_1')\cdots W(z_n,z_n') = \\ &= \tfrac{-e_0^2}{k_0+g_0}\textstyle\sum_{j\neq i}^n \frac{\sum_a t_i^a\otimes t_j^a}{z_i - z_j} W(z_1,z_1')\cdots W(z_n,z_n')\,,\end{aligned} \tag{44}$$

for $i = 1,\dots,n$ where g_0 denotes the dual Coxeter number of a group multiplying with e_0^2 and we have $g_0 = 2e_0^2$ for the group $SU(2)$ (When the gauge group is $U(1)$ we have $g_0 = 0$). We remark that in (44) we have defined $t_i^a := t^a$ and:

$$\begin{aligned}&t_i^a \otimes t_j^a W(z_1,z_1')\cdots W(z_n,z_n') := W(z_1,z_1')\cdots \\ &\cdots [t^a W(z_i,z_i')]\cdots[t^a W(z_j,z_j')]\cdots W(z_n,z_n')\,.\end{aligned} \tag{45}$$

It is interesting and important that we also have the following quantum Knizhnik-Zamolodchikov equation with respect to the z_i' variables which is dual to (44):

$$\begin{aligned}&\partial_{z_i'} W(z_1,z_1')\cdots W(z_n,z_n') = \\ &= \tfrac{-e_0^2}{k_0+g_0}\textstyle\sum_{j\neq i}^n W(z_1,z_1')\cdots W(z_n,z_n')\frac{\sum_a t_i^a\otimes t_j^a}{z_j' - z_i'}\end{aligned} \tag{46}$$

for $i = 1, \dots, n$ where we have defined:

$$W(z_1, z_1') \cdots W(z_n, z_n') t_i^a \otimes t_j^a := W(z_1, z_1') \cdots \cdots [W(z_i, z_i') t^a] \cdots [W(z_j, z_j') t^a] \cdots W(z_n, z_n') . \quad (47)$$

Remark From the quantum gauge theory we derive the above quantum KZ equation in dual form by calculus of variation. This quantum KZ equation in dual form may be considered as a quantum Euler-Lagrange equation or as a quantum Yang-Mills equation since it is analogous to the classical Yang-Mills equation which is derived from the classical Yang-Mills gauge theory by calculus of variation. ⋄

5 Solving quantum KZ equation in dual form

Let us consider the following product of two quantum Wilson lines:

$$G(z_1, z_2, z_3, z_4) := W(z_1, z_2) W(z_3, z_4) , \quad (48)$$

where the quantum Wilson lines $W(z_1, z_2)$ and $W(z_3, z_4)$ represent two pieces of curves starting at z_1 and z_3 and ending at z_2 and z_4 respectively.

We have that this product $G(z_1, z_2, z_3, z_4)$ satisfies the KZ equation for the variables z_1, z_3 and satisfies the dual KZ equation for the variables z_2 and z_4. Then by solving the two-variables-KZ equation in (44) we have that a form of $G(z_1, z_2, z_3, z_4)$ is given by [69–71]:

$$e^{-\hat{t} \log[\pm(z_1 - z_3)]} C_1 , \quad (49)$$

where $\hat{t} := \frac{e_0^2}{k_0 + g_0} \sum_a t^a \otimes t^a$ and C_1 denotes a constant matrix which is independent of the variable $z_1 - z_3$.

We see that $G(z_1, z_2, z_3, z_4)$ is a multi-valued analytic function where the determination of the $\pm$ sign depended on the choice of the branch.

Similarly by solving the dual two-variable-KZ equation in (46) we have that G is of the form

$$C_2 \, e^{\hat{t} \log[\pm(z_4 - z_2)]} , \quad (50)$$

where C_2 denotes a constant matrix which is independent of the variable $z_4 - z_2$.

From (49), (50) and letting:

$$C_1 = A e^{\hat{t} \log[\pm(z_4 - z_2)]}, \quad C_2 = e^{-\hat{t} \log[\pm(z_1 - z_3)]} A , \quad (51)$$

where A is a constant matrix we have that $G(z_1, z_2, z_3, z_4)$ is given by:

$$G(z_1, z_2, z_3, z_4) = e^{-\hat{t} \log[\pm(z_1 - z_3)]} A e^{\hat{t} \log[\pm(z_4 - z_2)]} , \quad (52)$$

where at the singular case that $z_1 = z_3$ we define $\log[\pm(z_1 - z_3)] = 0$. Similarly for $z_2 = z_4$.

Let us find a form of the initial operator A. We notice that there are two operators $\Phi_\pm(z_1 - z_3) := e^{-\hat{t} \log[\pm(z_1 - z_3)]}$ and $\Psi_\pm(z_4 - z_2) = e^{\hat{t} \log[\pm(z_4 - z_2)]}$ acting on the two sides of A respectively where the two independent variables z_1, z_3 of $\Phi_\pm$ are mixed from the two quantum Wilson lines $W(z_1, z_2)$ and $W(z_3, z_4)$ respectively and the the two independent variables z_2, z_4 of $\Psi_\pm$ are mixed from the two quantum Wilson lines $W(z_1, z_2)$ and $W(z_3, z_4)$ respectively. From this we determine the form of A as follows.

Let D denote a representation of $SU(2)$. Let $D(g)$ represent an element g of $SU(2)$ and let $D(g) \otimes D(g)$ denote the tensor product representation of $SU(2)$. Then in the KZ equation we define

$$[t^a \otimes t^a][D(g_1) \otimes D(g_1)] \otimes [D(g_2) \otimes D(g_2)] := \\ := [t^a D(g_1) \otimes D(g_1)] \otimes [t^a D(g_2) \otimes D(g_2)] \quad (53)$$

and

$$[D(g_1) \otimes D(g_1)] \otimes [D(g_2) \otimes D(g_2)][t^a \otimes t^a] := \\ := [D(g_1) \otimes D(g_1) t^a] \otimes [D(g_2) \otimes D(g_2) t^a] . \quad (54)$$

Then we let $U(\mathbf{a})$ denote the universal enveloping algebra where $\mathbf{a}$ denotes an algebra which is formed by the Lie algebra $su(2)$ and the identity matrix.

Now let the initial operator A be of the form $A_1 \otimes A_2 \otimes A_3 \otimes A_4$ with $A_i, i = 1, \dots, 4$ taking values in $U(\mathbf{a})$. In this case we have that in (52) the operator $\Phi_\pm(z_1 - z_3)$ acts on A from the left via the following formula:

$$t^a \otimes t^a A = [t^a A_1] \otimes A_2 \otimes [t^a A_3] \otimes A_4 . \quad (55)$$

Similarly the operator $\Psi_\pm(z_4 - z_2)$ in (52) acts on A from the right via the following formula:

$$A t^a \otimes t^a = A_1 \otimes [A_2 t^a] \otimes A_3 \otimes [A_4 t^a] . \quad (56)$$

We may generalize the above tensor product of two quantum Wilson lines as follows. Let us consider a tensor product of n quantum Wilson lines: $W(z_1, z_1') \cdots W(z_n, z_n')$ where the variables z_i, z_i' are all independent. By solving the two KZ equations we have that this tensor product is given by:

$$W(z_1, z_1') \cdots W(z_n, z_n') = \\ = \prod_{ij} \Phi_\pm(z_i - z_j) A \prod_{ij} \Psi_\pm(z_i' - z_j') , \quad (57)$$

where $\prod_{ij}$ denotes a product of $\Phi_\pm(z_i - z_j)$ or $\Psi_\pm(z_i' - z_j')$ for $i, j = 1, \dots, n$ where $i \neq j$. In (57) the initial operator A is represented as a tensor product of operators $A_{iji'j'}$, $i, j, i', j' = 1, \dots, n$ where each $A_{iji'j'}$ is of the form of the initial operator A in the above tensor product of two-Wilson-lines case and is acted by $\Phi_\pm(z_i - z_j)$ or $\Psi_\pm(z_i' - z_j')$ on its two sides respectively.

6 Computation of quantum Wilson lines

Let us consider the following product of two quantum Wilson lines:

$$G(z_1, z_2, z_3, z_4) := W(z_1, z_2) W(z_3, z_4) , \quad (58)$$

where the quantum Wilson lines $W(z_1, z_2)$ and $W(z_3, z_4)$ represent two pieces of curves starting at z_1 and z_3 and ending at z_2 and z_4 respectively. As shown in the above Section we have that $G(z_1, z_2, z_3, z_4)$ is given by the following formula:

$$G(z_1, z_2, z_3, z_4) = e^{-\hat{t}\log[\pm(z_1-z_3)]} A e^{\hat{t}\log[\pm(z_4-z_2)]}, \quad (59)$$

where the product is a 4-tensor.

Let us set $z_2 = z_3$. Then the 4-tensor $W(z_1, z_2)W(z_3, z_4)$ is reduced to the 2-tensor $W(z_1, z_2)W(z_2, z_4)$. By using (59) the 2-tensor $W(z_1, z_2)W(z_2, z_4)$ is given by:

$$W(z_1, z_2)W(z_2, z_4) = e^{-\hat{t}\log[\pm(z_1-z_2)]} A_{14} e^{\hat{t}\log[\pm(z_4-z_2)]}, \quad (60)$$

where $A_{14} = A_1 \otimes A_4$ is a 2-tensor reduced from the 4-tensor $A = A_1 \otimes A_2 \otimes A_3 \otimes A_4$ in (59). In this reduction the $\hat{t}$ operator of $\Phi = e^{-\hat{t}\log[\pm(z_1-z_2)]}$ acting on the left side of A_1 and A_3 in A is reduced to acting on the left side of A_1 and A_4 in A_{14}. Similarly the $\hat{t}$ operator of $\Psi = e^{\hat{t}\log[\pm(z_4-z_2)]}$ acting on the right side of A_2 and A_4 in A is reduced to acting on the right side of A_1 and A_4 in A_{14}.

Then since $\hat{t}$ is a 2-tensor operator we have that $\hat{t}$ is as a matrix acting on the two sides of the 2-tensor A_{14} which is also as a matrix with the same dimension as $\hat{t}$. Thus Φ and Ψ are as matrices of the same dimension as the matrix A_{14} acting on A_{14} by the usual matrix operation. Then since $\hat{t}$ is a Casimir operator for the 2-tensor group representation of $SU(2)$ we have that Φ and Ψ commute with A_{14} since Φ and Ψ are exponentials of $\hat{t}$. (We remark that Φ and Ψ are in general not commute with the 4-tensor initial operator A.) Thus we have

$$e^{-\hat{t}\log[\pm(z_1-z_2)]} A_{14} e^{\hat{t}\log[\pm(z_4-z_2)]} = e^{-\hat{t}\log[\pm(z_1-z_2)]} e^{\hat{t}\log[\pm(z_4-z_2)]} A_{14}. \quad (61)$$

We let $W(z_1, z_2)W(z_2, z_4)$ be as a representation of the quantum Wilson line $W(z_1, z_4)$:

$$W(z_1, z_4) := W(z_1, w_1)W(w_1, z_4) = e^{-\hat{t}\log[\pm(z_1-w_1)]} e^{\hat{t}\log[\pm(z_4-w_1)]} A_{14}. \quad (62)$$

This representation of the quantum Wilson line $W(z_1, z_4)$ means that the line (or path) with end points z_1 and z_4 is specific that it passes the intermediate point $w_1 = z_2$. This representation shows the quantum nature that the path is not specific at other intermediate points except the intermediate point $w_1 = z_2$. This unspecification of the path is of the same quantum nature of the Feynman path description of quantum mechanics.

Then let us consider another representation of the quantum Wilson line $W(z_1, z_4)$. We consider the three-product $W(z_1, w_1)W(w_1, w_2)W(w_2, z_4)$ which is obtained from the three-tensor $W(z_1, w_1)W(u_1, w_2)W(u_2, z_4)$ by two reductions where z_j, w_j, u_j, $j = 1, 2$ are independent variables. For this representation we have:

$$W(z_1, w_1)W(w_1, w_2)W(w_2, z_4) = e^{-\hat{t}\log[\pm(z_1-w_1)]} \times e^{-\hat{t}\log[\pm(z_1-w_2)]} e^{\hat{t}\log[\pm(z_4-w_1)]} e^{\hat{t}\log[\pm(z_4-w_2)]} A_{14}. \quad (63)$$

This representation of the quantum Wilson line $W(z_1, z_4)$ means that the line (or path) with end points z_1 and z_4 is specific that it passes the intermediate points w_1 and w_2. This representation shows the quantum nature that the path is not specific at other intermediate points except the intermediate points w_1 and w_2. This unspecification of the path is of the same quantum nature of the Feynman path description of quantum mechanics.

Similarly we may represent $W(z_1, z_4)$ by path with end points z_1 and z_4 and is specific only to pass at finitely many intermediate points. Then we let the quantum Wilson line $W(z_1, z_4)$ as an equivalent class of all these representations. Thus we may write:

$$W(z_1, z_4) = W(z_1, w_1)W(w_1, z_4) = W(z_1, w_1)W(w_1, w_2)W(w_2, z_4) = \cdots \quad (64)$$

Remark Since A_{14} is a 2-tensor we have that a natural group representation for the Wilson line $W(z_1, z_4)$ is the 2-tensor group representation of the group $SU(2)$.

7 Representing braiding of curves by quantum Wilson lines

Consider again the $G(z_1, z_2, z_3, z_4)$ in (58). We have that $G(z_1, z_2, z_3, z_4)$ is a multi-valued analytic function where the determination of the $\pm$ sign depended on the choice of the branch.

Let the two pieces of curves represented by $W(z_1, z_2)$ and $W(z_3, z_4)$ be crossing at w. In this case we write $W(z_i, z_j)$ as $W(z_i, z_j) = W(z_i, w)W(w, z_j)$ where $i = 1, 3$, $j = 2, 4$. Thus we have

$$W(z_1, z_2)W(z_3, z_4) = W(z_1, w)W(w, z_2)W(z_3, w)W(w, z_4). \quad (65)$$

If we interchange z_1 and z_3, then from (65) we have the following ordering:

$$W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4). \quad (66)$$

Now let us choose a branch. Suppose these two curves are cut from a knot and that following the orientation of a knot the curve represented by $W(z_1, z_2)$ is before the curve represented by $W(z_3, z_4)$. Then we fix a branch such that the product in (59) is with two positive signs:

$$W(z_1, z_2)W(z_3, z_4) = e^{-\hat{t}\log(z_1-z_3)} A e^{\hat{t}\log(z_4-z_2)}. \quad (67)$$

Then if we interchange z_1 and z_3 we have

$$W(z_3,w)W(w,z_2)W(z_1,w)W(w,z_4) = \\ = e^{-\hat{t}\log[-(z_1-z_3)]}Ae^{\hat{t}\log(z_4-z_2)}. \tag{68}$$

From (67) and (68) as a choice of branch we have

$$W(z_3,w)W(w,z_2)W(z_1,w)W(w,z_4) = \\ = RW(z_1,w)W(w,z_2)W(z_3,w)W(w,z_4)\,, \tag{69}$$

where $R = e^{-i\pi\hat{t}}$ is the monodromy of the KZ equation. In (69) z_1 and z_3 denote two points on a closed curve such that along the direction of the curve the point z_1 is before the point z_3 and in this case we choose a branch such that the angle of $z_3 - z_1$ minus the angle of $z_1 - z_3$ is equal to π.

Remark We may use other representations of the product $W(z_1,z_2)W(z_3,z_4)$. For example we may use the following representation:

$$W(z_1,w)W(w,z_2)W(z_3,w)W(w,z_4) = \\ = e^{-\hat{t}\log(z_1-z_3)}e^{-2\hat{t}\log(z_1-w)}e^{-2\hat{t}\log(z_3-w)}\times \\ \times Ae^{\hat{t}\log(z_4-z_2)}e^{2\hat{t}\log(z_4-w)}e^{2\hat{t}\log(z_2-w)}. \tag{70}$$

Then the interchange of z_1 and z_3 changes only $z_1 - z_3$ to $z_3 - z_1$. Thus the formula (69) holds. Similarly all other representations of $W(z_1,z_2)W(z_3,z_4)$ will give the same result. ⋄

Now from (69) we can take a convention that the ordering (66) represents that the curve represented by $W(z_1,z_2)$ is up-crossing the curve represented by $W(z_3,z_4)$ while (65) represents zero crossing of these two curves.

Similarly from the dual KZ equation as a choice of branch which is consistent with the above formula we have

$$W(z_1,w)W(w,z_4)W(z_3,w)W(w,z_2) = \\ = W(z_1,w)W(w,z_2)W(z_3,w)W(w,z_4)R^{-1}, \tag{71}$$

where z_2 is before z_4. We take a convention that the ordering in (71) represents that the curve represented by $W(z_1,z_2)$ is under-crossing the curve represented by $W(z_3,z_4)$. Here along the orientation of a closed curve the piece of curve represented by $W(z_1,z_2)$ is before the piece of curve represented by $W(z_3,z_4)$. In this case since the angle of $z_3 - z_1$ minus the angle of $z_1 - z_3$ is equal to π we have that the angle of $z_4 - z_2$ minus the angle of $z_2 - z_4$ is also equal to π and this gives the R^{-1} in this formula (71).

From (69) and (71) we have

$$W(z_3,z_4)W(z_1,z_2) = RW(z_1,z_2)W(z_3,z_4)R^{-1}, \tag{72}$$

where z_1 and z_2 denote the end points of a curve which is before a curve with end points z_3 and z_4. From (72) we see that the algebraic structure of these quantum Wilson lines $W(z,z')$ is analogous to the quasi-triangular quantum group [66,69].

8 Computation of quantum Dirac-Wilson loop

Consider again the quantum Wilson line $W(z_1,z_4)$ given by $W(z_1,z_4) = W(z_1,z_2)W(z_2,z_4)$. Let us set $z_1 = z_4$. In this case the quantum Wilson line forms a closed loop. Now in (61) with $z_1 = z_4$ we have that the quantities $e^{-\hat{t}\log\pm(z_1-z_2)}$ and $e^{\hat{t}\log\pm(z_1-z_2)}$ which come from the two-side KZ equations cancel each other and from the multi-valued property of the log function we have:

$$W(z_1,z_1) = R^N A_{14}\,, \qquad N = 0,\pm1,\pm2,\ldots \tag{73}$$

where $R = e^{-i\pi\hat{t}}$ is the monodromy of the KZ equation [69].

Remark It is clear that if we use other representation of the quantum Wilson loop $W(z_1,z_1)$ (such as the representation $W(z_1,z_1) = W(z_1,w_1)W(w_1,w_2)W(w_2,z_1)$) then we will get the same result as (73).

Remark For simplicity we shall drop the subscript of A_{14} in (73) and simply write $A_{14} = A$.

9 Winding number of Dirac-Wilson loop as quantization

We have the equation (73) where the integer N is as a winding number. Then when the gauge group is $U(1)$ we have

$$W(z_1,z_1) = R^N_{U(1)}A\,, \tag{74}$$

where $R_{U(1)}$ denotes the monodromy of the KZ equation for $U(1)$. We have

$$R^N_{U(1)} = e^{iN\frac{\pi e_0^2}{k_0+g_0}}\,, \qquad N = 0,\pm1,\pm2,\ldots \tag{75}$$

where the constant e_0 denotes the bare electric charge (and $g_0 = 0$ for $U(1)$ group). The winding number N is as the quantization property of photon. We show in the following Section that the Dirac-Wilson loop $W(z_1,z_1)$ with the abelian $U(1)$ group is a model of the photon.

10 Magnetic monopole is a photon with a specific frequency

We see that the Dirac-Wilson loop is an exactly solvable nonlinear observable. Thus we may regard it as a quantum soliton of the above gauge theory. In particular for the abelian gauge theory with $U(1)$ as gauge group we regard the Dirac-Wilson loop as a quantum soliton of the electromagnetic field. We now want to show that this soliton has all the properties of photon and thus we may identify it with the photon.

First we see that from (75) it has discrete energy levels of light-quantum given by

$$h\nu := N\frac{\pi e_0^2}{k_0}\,, \qquad N = 0,1,2,3,\ldots \tag{76}$$

where h is the Planck's constant; ν denotes a frequency and the constant $k_0 > 0$ is determined from this formula. This formula is from the monodromy $R_{U(1)}$ for the abelian gauge theory. We see that the Planck's constant h comes out from this winding property of the Dirac-Wilson loop. Then since this Dirac-Wilson loop is a loop we have that it has the polarization property of light by the right hand rule along the loop and this polarization can also be regarded as the spin of photon. Now since this loop is a quantum soliton which behaves as a particle we have that this loop is a basic particle of the above abelian gauge theory where the abelian gauge property is considered as the fundamental property of electromagnetic field. This shows that the Dirac-Wilson loop has properties of photon. We shall later show that from this loop model of photon we can describe the absorption and emission of photon by an electron. This property of absorption and emission is considered as a basic principle of the light-quantum hypothesis of Einstein [1]. From these properties of the Dirac-Wilson loop we may identify it with the photon.

On the other hand from Dirac's analysis of the magnetic monopole we have that the property of magnetic monopole comes from a closed line integral of vector potential of the electromagnetic field which is similar to the Dirac-Wilson loop [4]. Now from this Dirac-Wilson loop we can define the magnetic charge q and the minimal magnetic charge q_{min} which are given by:

$$eq := enq_{min} := n_e e_0 n \frac{n_m e_0 \pi}{k_0}, \quad n = 0, 1, 2, 3, \dots \quad (77)$$

where $e := n_e e_0$ is as the observed electric charge for some positive integer n_e; and $q_{min} := \frac{n_m e_0 \pi}{k_0}$ for some positive integer n_m and we write $N = nn_e n_m, n = 0, 1, 2, 3, \dots$ (by absorbing the constant k_0 to e_0^2 we may let $k_0 = 1$).

This shows that the Dirac-Wilson loop gives the property of magnetic monopole for some frequencies. Since this loop is a quantum soliton which behaves as a particle we have that this Dirac-Wilson loop may be identified with the magnetic monopole for some frequencies. Thus we have that photon may be identified with the magnetic monopole for some frequencies. With this identification we have the following interesting conclusion: Both the energy quantization of electromagnetic field and the charge quantization property come from the same property of photon. Indeed we have:

$$nh\nu_1 := n \frac{n_e n_m e_0^2 \pi}{k_0} = eq, \qquad n = 0, 1, 2, 3, \dots \quad (78)$$

where ν_1 denotes a frequency. This formula shows that the energy quantization gives the charge quantization and thus these two quantizations are from the same property of the photon when photon is modelled by the Dirac-Wilson loop and identified with the magnetic monopole for some frequencies. We notice that between two energy levels neq_{min} and $(n+1)eq_{min}$ there are other energy levels which may be regarded as the excited states of a particle with charge ne.

11 Nonlinear loop model of electron

In this Section let us also give a loop model to the electron. This loop model of electron is based on the above loop model of the photon. From the loop model of photon we also construct an observable which gives mass to the electron and is thus a mass mechanism for the electron.

Let $W(z,z)$ denote a Dirac-Wilson loop which represents a photon. Let Z denotes the complex variable for electron in (3). We then consider the following observable:

$$W(z,z)Z. \quad (79)$$

Since $W(z,z)$ is solvable we have that this observable is also solvable where in solving $W(z,z)$ the variable Z is fixed. We let this observable be identified with the electron. Then we consider the following observable:

$$Z^* W(z,z)Z. \quad (80)$$

This observable is with a scalar factor Z^*Z where Z^* denotes the complex conjugate of Z and we regard it as the mass mechanism of the electron (79). For this observable we model the energy levels with specific frequencies of $W(z,z)$ as the mass levels of electron and the mass m of electron is the lowest energy level $h\nu_1$ with specific frequency ν_1 of $W(z,z)$ and is given by:

$$mc^2 = h\nu_1, \quad (81)$$

where c denotes the constant of the speed of light and the frequency ν_1 is given by (78). From this model of the mass mechanism of electron we have that electron is with mass m while photon is with zero mass because there does not have such a mass mechanism $Z^*W(z,z)Z$ for the photon. From this definition of mass we have the following formula relating the observed electric charge e of electron, the magnetic charge q_{min} of magnetic monopole and the mass m of electron:

$$mc^2 = eq_{min} = h\nu_1. \quad (82)$$

By using the nonlinear model $W(z,z)Z$ to represent an electron we can then describe the absorption and emission of a photon by an electron where photon is as a parcel of energy described by the loop $W(z,z)$, as follows. Let $W(z,z)Z$ represents an electron and let $W_1(z_1,z_1)$ represents a photon. Then the observable $W_1(z_1,z_1)W(z,z)Z$ represents an electron having absorbed the photon $W_1(z_1,z_1)$. This property of absorption and emission is as a basic principle of the hypothesis of light-quantum stated by Einstein [1]. Let us quote the following paragraph from [1]:

> ...First, the light-quantum was conceived of as a parcel of energy as far as the properties of pure radiation (no coupling to matter) are concerned. Second, Einstein made the assumption — he call it the heuristic principle — that also in its coupling to matter (that is, in emission and absorption), light is created or annihilated in similar discrete parcels of energy. That, I

> believe, was Einstein's one revolutionary contribution to physics. It upset all existing ideas about the interaction between light and matter...

12 Photon with specific frequency carries electric and magnetic charges

In this loop model of photon we have that the observed electric charge $e := n_e e_0$ and the magnetic charge q_{min} are carried by the photon with some specific frequencies. Let us here describe the physical effects from this property of photon that photon with some specific frequency carries the electric and magnetic charge. From the nonlinear model of electron we have that an electron $W(z,z)Z$ also carries the electric charge when a photon $W(z,z)$ carrying the electric and magnetic charge is absorbed to form the electron $W(z,z)Z$. This means that the electric charge of an electron is from the electric charge carried by a photon. Then an interaction (as the electric force) is formed between two electrons (with the electric charges).

On the other hand since photon carries the constant e_0^2 of the bare electric charge e_0 we have that between two photons there is an interaction which is similar to the electric force between two electrons (with the electric charges). This interaction however may not be of the same magnitude as the electric force with the magnitude e^2 since the photons may not carry the frequency for giving the electric and magnetic charge. Then for stability such interaction between two photons tends to give repulsive effect to give the diffusion phenomenon among photons.

Similarly an electron $W(z,z)Z$ also carries the magnetic charge when a photon $W(z,z)$ carrying the electric and magnetic charge is absorbed to form the electron $W(z,z)Z$. This means that the magnetic charge of an electron is from the magnetic charge carried by a photon. Then a closed-loop interaction (as the magnetic force) may be formed between two electrons (with the magnetic charges).

On the other hand since photon carries the constant e_0^2 of the bare electric charge e_0 we have that between two photons there is an interaction which is similar to the magnetic force between two electrons (with the magnetic charges). This interaction however may not be of the same magnitude as the magnetic force with the magnetic charge q_{min} since the photons may not carry the frequency for giving the electric and magnetic charge. Then for stability such interaction between two photons tends to give repulsive effect to give the diffusion phenomenon among photons.

13 Statistics of photons and electrons

The nonlinear model $W(z,z)Z$ of an electron gives a relation between photon and electron where the photon is modelled by $W(z,z)$ which is with a specific frequency for $W(z,z)Z$ to be an electron, as described in the above Sections. We want to show that from this nonlinear model we may also derive the required statistics of photons and electrons that photons obey the Bose-Einstein statistics and electrons obey the Fermi-Dirac statistics. We have that $W(z,z)$ is as an operator acting on Z. Let $W_1(z,z)$ be a photon. Then we have that the nonlinear model $W_1(z,z)W(z,z)Z$ represents that the photon $W_1(z,z)$ is absorbed by the electron $W(z,z)Z$ to form an electron $W_1(z,z)W(z,z)Z$. Let $W_2(z,z)$ be another photon. The we have that the model $W_1(z,z)W_2(z,z)W(z,z)Z$ again represents an electron where we have:

$$\begin{aligned} W_1(z,z)W_2(z,z)W(z,z)Z &= \\ &= W_2(z,z)W_1(z,z)W(z,z)Z\,. \end{aligned} \tag{83}$$

More generally the model $\prod_{n=1}^{N} W_n(z,z)W(z,z)Z$ represents that the photons $W_n(z,z), n = 1,2,\ldots,N$ are absorbed by the electron $W(z,z)Z$. This model shows that identical (but different) photons can appear identically and it shows that photons obey the Bose-Einstein statistics. From the polarization of the Dirac-Wilson loop $W(z,z)$ we may assign spin 1 to a photon represented by $W(z,z)$.

Let us then consider statistics of electrons. The observable $Z^*W(z,z)Z$ gives mass to the electron $W(z,z)Z$ and thus this observable is as a scalar and thus is assigned with spin 0. As the observable $W(z,z)Z$ is between $W(z,z)$ and $Z^*W(z,z)Z$ which are with spin 1 and 0 respectively we thus assign spin $\frac{1}{2}$ to the observable $W(z,z)Z$ and thus electron represented by this observable $W(z,z)Z$ is with spin $\frac{1}{2}$.

Then let Z_1 and Z_2 be two independent complex variables for two electrons and let $W_1(z,z)Z_1$ and $W_2(z,z)Z_2$ represent two electrons. Let $W_3(z,z)$ represents a photon. Then the model $W_3(z,z)(W_1(z,z)Z_1 + W_2(z,z)Z_2)$ means that two electrons are in the same state that the operator $W_3(z,z)$ is acted on the two electrons. However this model means that a photon $W(z,z)$ is absorbed by two distinct electrons and this is impossible. Thus the models $W_3(z,z)W_1(z,z)Z_1$ and $W_3(z,z)W_2(z,z)Z_2$ cannot both exist and this means that electrons obey Fermi-Dirac statistics.

Thus this nonlinear loop model of photon and electron gives the required statistics of photons and electrons.

14 Photon propagator and quantum photon propagator

Let us then investigate the quantum Wilson line $W(z_0,z)$ with $U(1)$ group where z_0 is fixed for the photon field. We want to show that this quantum Wilson line $W(z_0,z)$ may be regarded as the quantum photon propagator for a photon propagating from z_0 to z.

As we have shown in the above Section on computation of quantum Wilson line; to compute $W(z_0,z)$ we need to write $W(z_0,z)$ in the form of two (connected) Wilson lines: $W(z_0,z) = W(z_0,z_1)W(z_1,z)$ for some z_1 point. Then we have:

$$W(z_0,z_1)W(z_1,z) = \\ = e^{-\hat{t}\log[\pm(z_1-z_0)]}Ae^{\hat{t}\log[\pm(z-z_1)]}, \quad (84)$$

where $\hat{t} = -\frac{e_0^2}{k_0}$ for the $U(1)$ group (k_0 is a constant and we may for simplicity let $k_0 = 1$) where the term $e^{-\hat{t}\log[\pm(z-z_0)]}$ is obtained by solving the first form of the dual form of the KZ equation and the term $e^{\hat{t}\log[\pm(z_0-z)]}$ is obtained by solving the second form of the dual form of the KZ equation.

Then we may write $W(z_0,z)$ in the following form:

$$W(z_0,z) = W(z_0,z_1)W(z_1,z) = e^{-\hat{t}\log\frac{(z_1-z_0)}{(z-z_1)}}A. \quad (85)$$

Let us fix z_1 with z such that:

$$\frac{|z_1-z_0|}{|z-z_1|} = \frac{r_1}{n_e^2} \quad (86)$$

for some positive integer n_e such that $r_1 \leqslant n_e^2$; and we let z be a point on a path of connecting z_0 and z_1 and then a closed loop is formed with z as the starting and ending point. (This loop can just be the photon-loop of the electron in this electromagnetic interaction by this photon propagator (85).) Then (85) has a factor $e_0^2\log\frac{r_1}{n_e^2}$ which is the fundamental solution of the two dimensional Laplace equation and is analogous to the fundamental solution $\frac{e^2}{r}$ (where $e := e_0 n_e$ denotes the observed (renormalized) electric charge and r denotes the three dimensional distance) of the three dimensional Laplace equation for the Coulomb's law. Thus the operator $W(z_0,z) = W(z_0,z_1)W(z_1,z)$ in (85) can be regarded as the quantum photon propagator propagating from z_0 to z.

We remark that when there are many photons we may introduce the space variable $\mathbf{x}$ as a statistical variable via the Lorentz metric $ds^2 = dt^2 - d\mathbf{x}^2$ to obtain the Coulomb's law $\frac{e^2}{r}$ from the fundamental solution $e_0^2\log\frac{r_1}{n_e^2}$ as a statistical law for electricity (We shall give such a space-time statistics later).

The quantum photon propagator (85) gives a repulsive effect since it is analogous to the Coulomb's law $\frac{e^2}{r}$. On the other hand we can reverse the sign of $\hat{t}$ such that this photon operator can also give an attractive effect:

$$W(z_0,z) = W(z_0,z_1)W(z_1,z) = e^{\hat{t}\log\frac{(z-z_1)}{(z_1-z_0)}}A, \quad (87)$$

where we fix z_1 with z_0 such that:

$$\frac{|z-z_1|}{|z_1-z_0|} = \frac{r_1}{n_e^2} \quad (88)$$

for some positive integer n_e such that $r_1 \geqslant n_e^2$; and we again let z be a point on a path of connecting z_0 and z_1 and then a closed loop is formed with z as the starting and ending point. (This loop again can just be the photon-loop of the electron in this electromagnetic interaction by this photon propagator (85).) Then (87) has a factor $-e_0^2\log\frac{r_1}{n_e^2}$ which is the fundamental solution of the two dimensional Laplace equation and is analogous to the attractive fundamental solution $-\frac{e^2}{r}$ of the three dimensional Laplace equation for the Coulomb's law.

Thus the quantum photon propagator in (85), and in (87), can give repulsive or attractive effect between two points z_0 and z for all z in the complex plane. These repulsive or attractive effects of the quantum photon propagator correspond to two charges of the same sign and of different sign respectively.

On the other hand when $z = z_0$ the quantum Wilson line $W(z_0,z_0)$ in (85) which is the quantum photon propagator becomes a quantum Wilson loop $W(z_0,z_0)$ which is identified as a photon, as shown in the above Sections.

Let us then derive a form of photon propagator from the quantum photon propagator $W(z_0,z)$. Let us choose a path connecting z_0 and $z = z(s)$. We consider the following path:

$$z(s) = z_1 + a_0\big[\theta(s_1-s)e^{-i\beta_1(s_1-s)} + \\ + \theta(s-s_1)e^{i\beta_1(s_1-s)}\big], \quad (89)$$

where $\beta_1 > 0$ is a parameter and $z(s_0) = z_0$ for some proper time s_0; and a_0 is some complex constant; and θ is a step function given by $\theta(s) = 0$ for $s < 0$, $\theta(s) = 1$ for $s \geqslant 0$. Then on this path we have:

$$W(z_0,z) = \\ = W(z_0,z_1)W(z_1,z) = e^{\hat{t}\log\frac{(z-z_1)}{(z_1-z_0)}}A = \\ = e^{\hat{t}\log\frac{a_0[\theta(s-s_1)e^{-i\beta_1(s_1-s)}+\theta(s_1-s)e^{i\beta_1(s_1-s)}]}{(z_1-z_0)}}A = \\ = e^{\hat{t}\log b[\theta(s-s_1)e^{-i\beta_1(s_1-s)}+\theta(s_1-s)e^{i\beta_1(s_1-s)}]}A = \\ = b_0\big[\theta(s-s_1)e^{-i\hat{t}\beta_1(s_1-s)} + \theta(s_1-s)e^{i\hat{t}\beta_1(s_1-s)}\big]A \quad (90)$$

for some complex constants b and b_0. From this chosen of the path (89) we have that the quantum photon propagator is proportional to the following expression:

$$\frac{1}{2\lambda_1}\big[\theta(s-s_1)e^{-i\lambda_1(s-s_1)} + \theta(s_1-s)e^{i\lambda_1(s-s_1)}\big] \quad (91)$$

where we define $\lambda_1 = -\hat{t}\beta_1 = e_0^2\beta_1 > 0$. We see that this is the usual propagator of a particle $x(s)$ of harmonic oscillator with mass-energy parameter $\lambda_1 > 0$ where $x(s)$ satisfies the following harmonic oscillator equation:

$$\frac{d^2x}{ds^2} = -\lambda_1^2 x(s). \quad (92)$$

We regard (91) as the propagator of a photon with mass-energy parameter λ_1. Fourier transforming (91) we have the following form of photon propagator:

$$\frac{i}{k_E^2 - \lambda_1}, \quad (93)$$

where we use the notation k_E (instead of the notation k) to denote the proper energy of photon. We shall show in the next Section that from this photon propagator by space-time statistics we can get a propagator with the k_E replaced by the energy-momemtum four-vector k which is similar to the Feynman propagator (with a mass-energy parameter $\lambda_1 > 0$). We thus see that the quantum photon propagator $W(z_0, z)$ gives a classical form of photon propagator in the conventional QED theory.

Then we notice that while $\lambda_1 > 0$ which may be think of as the mass-energy parameter of a photon the original quantum photon propagator $W(z_0, z)$ can give the Coulomb potential and thus give the effect that the photon is massless. Thus the photon mass-energy parameter $\lambda_1 > 0$ is consistent with the property that the photon is massless. Thus in the following Sections when we compute the vertex correction and the Lamb shift we shall then be able to let $\lambda_1 > 0$ without contradicting the property that the photon is massless. This then can solve the infrared-divergence problem of QED.

We remark that if we choose other form of paths for connecting z_0 and z we can get other forms of photon propagator corresponding to a choice of gauge. From the property of gauge invariance the final result should not depend on the form of propagators. We shall see that this is achieved by renormalization. This property of renormalizable is as a property related to the gauge invariance. Indeed we notice that the quantum photon propagator with a photon-loop $W(z, z)$ attached to an electron represented by Z has already given the renormalized charge e (and the renormalized mass m of the electron) for the electromagnetic interaction.

It is clear that this renormalization by the quantum photon propagator with a photon-loop $W(z, z)$ is independent of the chosen photon propagator (because it does not need to choose a photon propagator). Thus the renormalization method as that in the conventional QED theory for the chosen of a photon propagator (corresponding to a choice of gauge) should give the observable result which does not depend on the form of the photon propagators since these two forms of renormalization must give the same effect of renormalization.

In the following Section and the Sections from Section 16 to Section 23 on Quantum Electrodynamics (QED) we shall investigate the renormalization method which is analogous to that of the conventional QED theory and the computation of QED effects by using this renormalization method.

15 Renormalization

In this Section and the following Sections from Section 16 to Section 23 on Quantum Electrodynamics (QED) we shall use the density (3) and the notations from this density where $A_j, j = 1, 2$ are real components of the photon field. Following the conventional QED theory let us consider the following renormalization:

$$\begin{aligned} &A_j = z_A^{\frac{1}{2}} A_{jR}\,, \;\; j = 1, 2; \quad Z = z_Z^{\frac{1}{2}} Z_R\,; \\ &e_0 = \frac{z_e}{z_Z z_A^{\frac{1}{2}}}\, e = \frac{1}{n_e}\, e\,; \end{aligned} \tag{94}$$

where z_A, z_Z, and z_e are renormalization constants to be determined and $A_{Rj}, j = 1, 2$, Z_R are renormalized fields. From this renormalization the density D of QED in (3) can be written in the following form:

$$\begin{aligned} D &= \tfrac{1}{2}\, z_A \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right)^* \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right) + \\ &+ z_Z \left(\tfrac{dZ_R^*}{ds} + ie\,(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) Z_R^*\right) \times \\ &\times \left(\tfrac{dZ_R}{ds} - ie\,(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) Z_R\right) = \\ &= \Big\{ \tfrac{1}{2} \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right)^* \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right) + \\ &+ \tfrac{dZ_R^*}{ds}\tfrac{dZ_R}{ds} + \mu^2 Z_R^* Z_R - \mu^2 Z_R^* Z_R + \\ &+ i\,e\,(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) Z_R^* \tfrac{dZ_R}{ds} - \\ &- i\,e\,(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) \tfrac{dZ_R^*}{ds} Z_R + \\ &+ e^2 (\textstyle\sum_{j=1}^2 A_{Rj}\tfrac{dx^j}{ds})^2 Z_R^* Z_R \Big\} + \\ &+ \Big\{ (z_A - 1)\big[\tfrac{1}{2} \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right)^* \left(\tfrac{\partial A_{1R}}{\partial x^2} - \tfrac{\partial A_{2R}}{\partial x^1}\right)\big] + \\ &+ (z_Z - 1) \tfrac{dZ_R^*}{ds}\tfrac{dZ_R}{ds} + \\ &+ (z_e - 1) \big[+ie\,(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) Z_R^* \tfrac{dZ_R}{ds} - \\ &- i\,e(\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds}) \tfrac{dZ_R^*}{ds} Z_R \big] + \\ &+ (\tfrac{z_e^2}{z_Z} - 1) e^2 (\textstyle\sum_{j=1}^2 A_{jR}\tfrac{dx^j}{ds})^2 Z_R^* Z_R \Big\} := \\ &:= D_{phy} + D_{cnt}\,, \end{aligned} \tag{95}$$

where D_{phy} is as the physical term and the D_{cnt} is as the counter term; and in D_{phy} the positive parameter μ is introduced for perturbation expansion and for renormalization.

Similar to that the Ward-Takahashi identities in the conventional QED theory are derived by the gauge invariance of the conventional QED theory; by using the gauge invariance of this QED theory we shall also derive the corresponding Ward-Takahashi identities for this QED theory in the Section on electron self-energy. From these Ward-Takahashi identities we then show that there exists a renormalization procedure such that $z_e = z_Z$; as similar to that in the conventional QED theory. From this relation $z_e = z_Z$ we then have:

$$e_0 = \frac{e}{z_A^{\frac{1}{2}}} = \frac{1}{n_e}\, e \tag{96}$$

and that in (95) we have $\frac{z_e^2}{z_Z} - 1 = z_e - 1$.

16 Feynman diagrams and Feynman rules for QED

Let us then transform ds in (3) to $\frac{1}{(\beta+ih)}ds$ where $\beta, h > 0$ are parameters and h is as the Planck constant. The parameter h will give the dynamical effects of QED (as similar to the conventional QED). Here for simplicity we only consider the limiting case that $\beta \to 0$ and we let $h = 1$. From this transformation we get the Lagrangian $\mathcal{L}$ from $-\int_{s_0}^{s_1} D ds$ changing to $\int_{s_0}^{s_1} \mathcal{L} ds$. Then we write $\mathcal{L} = \mathcal{L}_{phy} + \mathcal{L}_{cnt}$ where $\mathcal{L}_{phy}$ corresponds to D_{phy} and $\mathcal{L}_{cnt}$ corresponds to D_{cnt}. Then from the following term in $\mathcal{L}_{phy}$:

$$-i\left[\left(\frac{dZ_R}{ds}\right)^* \frac{dZ_R}{ds} - \mu^2 Z_R^* Z_R\right] \tag{97}$$

and by the perturbation expansion of $e^{\int_{s_0}^{s_1} \mathcal{L} ds}$ we have the following propagator:

$$\frac{i}{p_E^2 - \mu^2} \tag{98}$$

which is as the (primitive) electron propagator where p_E denotes the proper energy variable of electron.

Then from the pure gauge part of $\mathcal{L}_{phy}$ we get the photon propagator (93), as done in the above Sections and the Section on photon propagator.

Then from $\mathcal{L}_{phy}$ we have the following seagull vertex term:

$$ie^2\left(\sum_{j=1}^{2} A_{jR}\frac{dx^j}{ds}\right)^2 Z_R^* Z_R\,. \tag{99}$$

This seagull vertex term gives the vertex factor ie^2 (We remark that the ds of the paths $\frac{dx^j}{ds}$ are not transformed to $-ids$ since these paths are given paths and thus are independent of the transformation $ds \to -ids$).

From this vertex by using the photon propagator (93) in the above Section we get the following term:

$$\frac{ie^2}{2\pi}\int \frac{i\,dk_E}{k_E^2 - \lambda_1^2} = -\frac{ie^2}{2\lambda_1} =: -i\,\omega^2. \tag{100}$$

The parameter ω is regarded as the mass-energy parameter of electron. Then from the perturbation expansion of $e^{\int_{s_0}^{s_1} \mathcal{L} ds}$ we have the following geometric series (which is similar to the Dyson series in the conventional QED):

$$\begin{aligned}&\tfrac{i}{p_E^2-\mu^2} + \tfrac{i}{p_E^2-\mu^2}(-i\omega^2 + i\mu^2)\tfrac{i}{p_E^2-\mu^2} + \ldots = \\ &= \tfrac{i}{p_E^2-\mu^2-\omega^2+\mu^2} = \tfrac{i}{p_E^2-\omega^2}\,,\end{aligned} \tag{101}$$

where the term $i\mu$ of $-i\,\omega^2 + i\mu^2$ is the $i\mu$ term in $\mathcal{L}_{phy}$. (The other term $-i\mu$ in $\mathcal{L}_{phy}$ has been used in deriving (98).) Thus we have the following electron propagator:

$$\frac{i}{p_E^2 - \omega^2}\,. \tag{102}$$

This is as the electron propagator with mass-energy parameter ω. From ω we shall get the mass m of electron. (We shall later introduce a space-time statistics to get the usual electron propagator of the Dirac equation. This usual electron propagator is as the statistical electron propagator.) As the Feynman diagrams in the conventional QED we represent this electron propagator by a straight line.

In the above Sections and the Section on the photon propagator we see that the photon-loop $W(z,z)$ gives the renormalized charge $e = n_e e_0$ and the renormalized mass m of electron from the bare charge e_0 by the winding numbers of the photon loop such that m is with the winding number factor n_e. Then we see that the above one-loop energy integral of the photon gives the mass-energy parameter ω of electron which gives the mass m of electron. Thus these two types of photon-loops are closely related (from the relation of photon propagator and quantum photon propagator) such that the mass m obtained by the winding numbers of the photon loop $W(z,z)$ reappears in the one-loop energy integral (100) of the photon.

Thus we see that even there is no mass term in the Lagrangian of this gauge theory the mass m of the electron can come out from the gauge theory. This actually resolves the mass problem of particle physics that particle can be with mass even without the mass term. Thus we do not need the Higgs mechanism for generating masses to particles.

On the other hand from the one-loop-electron form of the seagull vertex we have the following term:

$$\frac{ie^2}{2\pi}\int \frac{idp_E}{p_E^2 - \mu^2} = -\frac{ie^2}{2\mu} =: -i\lambda_2^2\,. \tag{103}$$

So for photon from the perturbation expansion of $e^{\int_{s_0}^{s_1} \mathcal{L} ds}$ we have the following geometric series:

$$\begin{aligned}&\tfrac{i}{k_E^2-\lambda_1^2} + \tfrac{i}{k_E^2-\lambda_1^2}(-i\lambda_2^2)\tfrac{i}{k_E^2-\lambda_1^2} + \ldots = \\ &= \tfrac{i}{k_E^2-\lambda_1^2-\lambda_2^2} =: \tfrac{i}{k_E^2-\lambda_0^2}\,,\end{aligned} \tag{104}$$

where we define $\lambda_0^2 = \lambda_1^2 + \lambda_2^2$. Thus we have the following photon propagator:

$$\frac{i}{k_E^2 - \lambda_0^2}\,, \tag{105}$$

which is of the same form as (93) where we replace λ_1 with λ_0. As the Feynman diagrams in the conventional QED we represent this photon propagator by a wave line.

Then the following interaction term in $\mathcal{L}_{phy}$:

$$\begin{aligned}&-ie\tfrac{dZ_R^*}{ds}\left(\textstyle\sum_{j=1}^{2} A_{jR}\tfrac{dx^j}{ds}\right)Z_R + \\ &\qquad + ie\tfrac{dZ_R}{ds}\left(\textstyle\sum_{j=1}^{2} A_{jR}\tfrac{dx^j}{ds}\right)Z_R^*\end{aligned} \tag{106}$$

gives the vertex factor $(-ie)(p_E + q_E)$ which corresponds to the usual vertex of Feynman diagram with two electron straight lines (with energies p_E and q_E) and one photon wave line in the conventional QED.

Then as the Feynman rules in the conventional QED a sign factor $(-1)^n$, where n is the number of the electron loops in a Feynman diagram, is to be included for the Feynman diagram.

17 Statistics with space-time

Let us introduce space-time as a statistical method for a large amount of basic variables Z_R and A_{1R}, A_{2R}. As an illustration let us consider the electron propagator $\frac{i}{p_E^2-\omega^2}$ and the following Green's function corresponding to it:

$$\frac{i}{2\pi}\int\frac{e^{-ip_E(s-s')}dp_E}{p_E^2-\omega^2}\,, \tag{107}$$

where s is the proper time.

We imagine each electron (and photon) occupies a space region (This is the creation of the concept of space which is associated to the electron. Without the electron this space region does not exist). Then we write

$$p_E(s-s') = p_E(t-t') - \mathbf{p}(\mathbf{x}-\mathbf{x}')\,, \tag{108}$$

where $\mathbf{p}(\mathbf{x}-\mathbf{x}')$ denotes the inner product of the three dimensional vectors $\mathbf{p}$ and $\mathbf{x}-\mathbf{x}'$ and $(t,\mathbf{x})$ is the time-space coordinate where $\mathbf{x}$ is in the space region occupied by $Z_R(s)$ and that

$$\omega^2 - \mathbf{p}^2 = m^2 > 0\,, \tag{109}$$

where m is the mass of electron. This mass m is greater than 0 since each Z_R occupies a space region which implies that when $t-t'$ tends to 0 we can have that $|\mathbf{x}-\mathbf{x}'|$ does not tend to 0 ($\mathbf{x}$ and $\mathbf{x}'$ denote two coordinate points in the regions occupied by $Z_R(s)$ and $Z_R(s')$ respectively) and thus (109) holds. Then by linear summing the effects of a large amount of basic variables Z_R and letting ω varies from m to ∞ from (107), (108) and (109) we get the following statistical expression:

$$\frac{i}{(2\pi)^4}\int\frac{e^{-ip(x-x')}dp}{p^2-m^2}\,, \tag{110}$$

which is the usual Green's function of a free field with mass m where p is a four vector and $x=(t,\mathbf{x})$.

The result of the above statistics is that (110) is induced from (107) with the scalar product p_E^2 of a scalar p_E changed to an indefinite inner product p^2 of a four vector p and the parameter ω is reduced to m.

Let us then introduce Fermi-Dirac statistics for electrons. As done by Dirac for deriving the Dirac equation we factorize p^2-m^2 into the following form:

$$\begin{aligned} p^2-m^2 &= (p_E-\omega)(p_E+\omega) = \\ &= (\gamma_\mu p^\mu - m)(\gamma_\mu p^\mu + m)\,, \end{aligned} \tag{111}$$

where γ_μ are the Dirac matrices. Then from (110) we get the following Green's function:

$$\begin{aligned} &\frac{i}{(2\pi)^4}\int e^{-ip(x-x')}\frac{\gamma_\mu p^\mu+m}{p^2-m^2}dp = \\ &= \frac{i}{(2\pi)^4}\int\frac{e^{-ip(x-x')}dp}{\gamma_\mu p^\mu - m}\,. \end{aligned} \tag{112}$$

Thus we have the Fermi-Dirac statistics that the statistical electron propagator is of the form $\frac{i}{\gamma_\mu p^\mu - m}$ which is the propagator of the Dirac equation and is the electron propagator of the conventional QED.

Let us then consider statistics of photons. Since the above quantum gauge theory of photons is a gauge theory which is gauge invariant we have that the space-time statistical equation for photons should be gauge invariant. Then since the Maxwell equation is the only gauge invariant equation for electromagnetism which is based on the space-time we have that the Maxwell equation must be a statistical equation for photons.

Then let us consider the vertexes. The tree vertex (106) with three lines (one for photon and two for electron) gives the factor $-ie(p_E+q_E)$ where p_E and q_E are from the factor $\frac{dZ_R}{ds}$ for electron.

We notice that this vertex is with two electron lines (or electron propagator) and one photon line (or photon propagator). In doing a statistics on this photon line when it is as an external electromagnetic field on the electron this photon line is of the statistical form $\gamma_\mu A^\mu$ where A^μ denotes the four electromagnetic potential fields of the Maxwell equation. Thus the vertex $-ie(p_E+q_E)$ after statistics is changed to the form $-ie(p_E+q_E)\frac{\gamma^\mu}{2}$ where for each γ^μ a factor $\frac{1}{2}$ is introduced for statistics.

Let us then introduce the on-mass-shell condition as in the conventional QED theory (see [6]). As similar to the on-mass-shell condition in the conventional QED theory our on-mass-shell condition is that $p_E = m$ where m is the observable mass of the electron. In this case $-ie(p_E+q_E)\frac{\gamma^\mu}{2}$ is changed to $-iem\gamma^\mu$. Then the m is absorbed to the two external spinors $\frac{1}{\sqrt{E}}u$ (where E denotes the energy of the electron satisfied the Dirac equation while the E of p_E is only as a notation) of the two electrons lines attached to this vertex such that the factor $\frac{1}{\sqrt{E}}$ of spin 0 of the Klein-Gordon equation is changed to the factor $\sqrt{\frac{m}{E}}$ of spinors of the Dirac equation. In this case we have the magnitude of p_E and q_E reappears in the two external electron lines with the factor $\sqrt{m}$. The statistical vertex then becomes $-ie\gamma^\mu$. This is exactly the usual vertex in the conventional QED. Thus after a space-time statistics on the original vertex $-ie(p_E+q_E)$ we get the statistical vertex $-ie\gamma^\mu$ of the conventional QED.

18 Basic effects of Quantum Electrodynamics

To illustrate this new theory of QED let us compute the three basic effects of QED: the one-loop photon and electron self-energies and the one-loop vertex correction.

As similar to the conventional QED we have the Feynman rules such that the one-loop photon self-energy is given by the following Feynman integral:

$$i\Pi_0(k_E) := i^2(-i)^2\frac{e^2}{2\pi}\times \times\int\frac{(2p_E+k_E)(2p_E+k_E)dp_E}{(p_E^2-\omega^2)((k_E+p_E)^2-\omega^2)}, \tag{113}$$

where e is the renormalized electric charge.

Then as the Feynman rules in the conventional QED for the space-time statistics a statistical sign factor $(-1)^j$, where j is the number of the electron loops in a Feynman diagram, will be included for the Feynman diagram. Thus for the one-loop photon self-energy (113) a statistical factor $(-1)^j$ will be introduced to this one-loop photon self-energy integral.

Then similarly we have the Feynman rules such that the one-loop electron self-energy is given by the following Feynman integral:

$$-i\Sigma_0(p_E) := i^2(-i)^2\frac{e^2}{2\pi}\times \times\int\frac{(2p_E-k_E)(2p_E-k_E)dk_E}{(k_E^2-\lambda_0^2)((p_E-k_E)^2-\omega^2)}. \tag{114}$$

Similarly we have the Feynman rules that the one-loop vertex correction is given by the following Feynman integral:

$$(-ie)\Lambda_0(p_E,q_E) := := (i)^3(-i)^3\frac{e^3}{2\pi}\int\frac{(2p_E-k_E)(2q_E-k_E)(p_E+q_E-2k_E)dk_E}{((p_E-k_E)^2-\omega^2)((q_E-k_E)^2-\omega^2)(k_E^2-\lambda_0^2)}. \tag{115}$$

Let us first compute the one-loop vertex correction and then compute the photon self-energy and the electron self-energy.

As a statistics we extend the one dimensional integral $\int dk_E$ to the n-dimensional integral $\int d^nk$ $(n\to 4)$ where $k=(k_E,\mathbf{k})$. This is similar to the dimensional regularization in the conventional quantum field theories (However here our aim is to increase the dimension for statistics which is different from the dimensional regularization which is to reduce the dimension from 4 to n to avoid the ultraviolet divergence). With this statistics the factor 2π is replaced by the statistical factor $(2\pi)^n$. From this statistics on (115) we have the following statistical one loop vertex correction:

$$\frac{e^3}{(2\pi)^n}\int_0^1 dx\int_0^1 2ydy\int d^nk\times \times\frac{4p_Eq_E(p_E+q_E)-2k_E((p_E+q_E)^2+4p_Eq_E)+5k_E^2(p_E+q_E)-2k_E^3}{[k^2-2k(pxy+q(1-x)y)-p_E^2xy-q_E^2(1-x)y+m^2y+\lambda^2(1-y)]^3}, \tag{116}$$

where $k^2=k_E^2-\mathbf{k}^2$, and $\mathbf{k}^2$ is from the free parameters ω, λ_0 where we let $\omega^2=m^2+\mathbf{k}^2$, $\lambda_0^2=\lambda^2+\mathbf{k}^2$ for the electron mass m and a mass-energy parameter λ for photon; and:

$$\left.\begin{aligned} k(pxy+q(1-x)y) &:= k_E(p_Exy+q_E(1-x)y)\\ -\mathbf{k}\cdot\mathbf{0} &= k_E(p_Exy+q_E(1-x)y)\end{aligned}\right\}. \tag{117}$$

By using the formulae for computing Feynman integrals we have that (116) is equal to (see [6, 72]):

$$\begin{aligned}&\frac{ie^3}{(2\pi)^n}\int_0^1 dx\int_0^1 2ydy\times\\ &\times\Big[\frac{4p_Eq_E(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})}{\Gamma(3)(\Delta-r^2)^{3-2}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}-\\ &-\frac{2((p_E+q_E)^2+4p_Eq_E)\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r}{\Gamma(3)(\Delta-r^2)^{3-2}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}+\\ &+\frac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-1-\frac{n}{2})\frac{n}{2}}{\Gamma(3)(\Delta-r^2)^{3-2-1}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}+\\ &+\frac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r^2}{\Gamma(3)(\Delta-r^2)^{3-2}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}-\\ &-\frac{\frac{(n+2)}{2}2\pi^{\frac{n}{2}}\Gamma(3-1-\frac{n}{2})r}{\Gamma(3)(\Delta-r^2)^{3-2-1}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}-\\ &-\frac{2\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r^3}{\Gamma(3)(\Delta-r^2)^{3-2}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}\Big]=:\\ &=: (-ie)\Lambda(p_1,p_2),\end{aligned} \tag{118}$$

where we define:

$$\left.\begin{aligned} r &:= p_Exy+q_E(1-x)y\\ \Delta &:= p_E^2xy+q_E^2(1-x)y-m^2y-\lambda^2(1-y)\end{aligned}\right\}. \tag{119}$$

We remark that in this statistics the p_E and q_E variables are remained as the proper variables which are derived from the proper time s.

Let us then introduce the Fermi-Dirac statistics on the electron and we consider the on-mass-shell case as in the conventional QED. We shall see this will lead to the theoretical results of the conventional QED on the anomalous magnetic moment and the Lamb shift.

As a Fermi-Dirac statistics we have shown in the above Section that the vertex term $-ie(p_E+q_E)$ is replaced with the vertex term $-ie(p_E+q_E)\frac{\gamma^\mu}{2}$. Then as a Fermi-Dirac statistics in the above Section we have shown that the statistical vertex is $-ie\gamma^\mu$ under the on-mass-shell condition. We notice that this vertex agrees with the vertex term in the conventional QED theory.

Let us then consider the Fermi-Dirac statistics on the one-loop vertex correction (118). Let us first consider the following term in (118):

$$\frac{ie^3}{(2\pi)^n}\int_0^1 dx\int_0^1 2ydy\times \times\frac{\pi^2(p_E+q_E)4p_Eq_E}{\Gamma(3)(\Delta-r^2)^{3-2}}\frac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}, \tag{120}$$

where we can (as an approximation) let $n=4$. From Fermi-Dirac statistics we have that this term gives the following statistics:

$$\frac{ie^3}{(2\pi)^4}\int_0^1 dx\int_0^1 2ydy\,\frac{\pi^2(p_E+q_E)\frac{1}{2}\gamma^\mu 4p_Eq_E}{\Gamma(3)(\Delta-r^2)^{3-2}}. \tag{121}$$

Then we consider the case of on-mass-shell. In this case we have $p_E=m$ and $q_E=m$. Thus from (121) we have the following term:

$$\frac{ie^3}{(2\pi)^4}\int_0^1 dx\int_0^1 2ydy\,\frac{\pi^2\gamma^\mu 4p_E q_E}{\Gamma(3)(\Delta-r^2)^{3-2}}\,, \tag{122}$$

where a mass factor $m=\frac{1}{2}(p_E+q_E)$ has been omitted and put to the external spinor of the external electron as explained in the above Section on space-time statistics. In (122) we still keep the expression $p_E q_E$ even though in this case of on-mass-shell because this factor will be important for giving the observable Lamb shift, as we shall see. In (122) because of on-mass-shell we have (as an approximation we let $n=4$):

$$(\Delta-r^2)^{3-2}=-\lambda^2(1-y)-r^2=\\ -\lambda^2(1-y)-m^2y^2\,. \tag{123}$$

Thus in the on-mass-shell case (122) is of the following form:

$$ie\gamma^\mu\frac{\alpha}{\pi}\int_0^1 dx\int_0^1 ydy\,\frac{\pi^2 p_E q_E}{-\lambda^2(1-y)-m^2y^2}\,, \tag{124}$$

where $\alpha=\frac{e^2}{4\pi}$ is the fine structure constant. Carrying out the integrations on y and on x we have that as $\lambda\to 0$ (124) is equal to:

$$(-ie)\gamma^\mu\frac{\alpha}{\pi}\frac{p_E q_E}{m^2}\log\frac{m}{\lambda}\,, \tag{125}$$

where the proper factor $p_E q_E$ will be for a linear space-time statistics of summation. We remark that (125) corresponds to a term in the vertex correction in the conventional QED theory with the infra-divergence when $\lambda=0$ (see [6]). Here since the parameter λ has not been determined we shall later find other way to determine the effect of (125) and to solve the infrared-divergence problem.

Let us first rewrite the form of the proper value $p_E q_E$. We write $p_E q_E$ in the following space-time statistical form:

$$p_E q_E=-2p'\cdot p\,, \tag{126}$$

where p and p' denote two space-time four-vectors of electron such that $p^2=m^2$ and $p'^2=m^2$. Then we have

$$\begin{aligned} p_E q_E &= \\ &=\tfrac{1}{3}(p_E q_E+p_E q_E+p_E q_E)=\tfrac{1}{3}(m^2-2p'\cdot p+m^2)\\ &=\tfrac{1}{3}(m^2-2p'\cdot p+m^2)=\tfrac{1}{3}(p'^2-2p'\cdot p+p^2)\\ &=\tfrac{1}{3}(p'-p)^2\\ &=:\tfrac{1}{3}q^2, \end{aligned} \tag{127}$$

where following the convention of QED we define $q=p'-p$. Thus from (125) we have the following term:

$$(-ie)\gamma^\mu\frac{\alpha}{3\pi}\frac{q^2}{m^2}\log\frac{m}{\lambda}\,, \tag{128}$$

where the parameter λ are to be determined. Again this term (128) corresponds to a term in the vertex correction in the conventional QED theory with the infrared-divergence when $\lambda=0$ (see [6]).

Let us then consider the following term in (118):

$$\frac{-ie^3}{(2\pi)^n}\int_0^1 dx\,\times\\ \times\int_0^1\frac{2((p_E+q_E)^2+4p_E q_E)\pi^{\frac{n}{2}}r2ydy}{\Gamma(3)(\Delta-r^2)^{3-2}(-\Delta+r^2)^{2-\frac{n}{2}}}\,. \tag{129}$$

For this term we can (as an approximation) also let $n=4$ and we have let $\Gamma(3-\frac{n}{2})=1$. As similar to the conventional QED theory we want to show that this term gives the anomalous magnetic moment and thus corresponds to a similar term in the vertex correction of the conventional QED theory (see [6]).

By Fermi-Dirac statistics the factor (p_E+q_E) in (129) of $(p_E+q_E)^2$ gives the statistical term $(p_E+q_E)\frac{1}{2}\gamma^\mu$. Thus with the on-mass-shell condition the factor (p_E+q_E) gives the statistical term $m\gamma^\mu$. Thus with the on-mass-shell condition the term $(p_E+q_E)^2$ gives the term $m\gamma^\mu(p_E+q_E)$. Then the factor (p_E+q_E) in this statistical term also give $2m$ by the on-mass-shell condition. Thus by Fermi-Dirac statistics and the on-mass-shell condition the factor $(p_E+q_E)^2$ in (129) gives the statistical term $\gamma^\mu 2m^2$. Then since this is a (finite) constant term it can be cancelled by the corresponding counter term of the vertex giving the factor $-ie\gamma^\mu$ and having the factor z_e-1 in (95). From this cancellation the renormalization constant z_e is determined. Since the constant term is depended on the $\delta>0$ which is introduced for space-time statistics we have that the renormalization constant z_e is also depended on the $\delta>0$. Thus the renormalization constant z_e (and the concept of renormalization) is related to the space-time statistics.

At this point let us give a summary of this renormalization method, as follows.

Renormalization

1. The renormalization method of the conventional QED theory is used to obtain the renormalized physical results. Here unlike the conventional QED theory the renormalization method is not for the removing of ultraviolet divergences since the QED theory in this paper is free of ultraviolet divergences.

2. We have mentioned in the above Section on photon propagator that the property of renormalizable is a property of gauge invariance that it gives the physical results independent of the chosen photon propagator.

3. The procedure of renormalization is as a part of the space-time statistics to get the statistical results which is independent of the chosen photon propagator. ⋄

Let us then consider again the above computation of the one-loop vertex correction. We now have that the (finite) constant term of the one-loop vertex correction is cancelled by the corresponding counter term with the factor z_e-1 in (95).

Thus the nonconstant term (128) is renormalized to be the following renormalized form:

$$(-ie)\,\gamma^{\mu}\,\frac{\alpha}{3\pi}\,\frac{q^2}{m^2}\log\frac{m}{\lambda}\,. \tag{130}$$

Let us then consider the following term in (129):

$$\frac{-ie^3}{(2\pi)^n}\int_0^1 dx\int_0^1 2y\,dy\,\frac{8p_E\,q_E\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r}{\Gamma(3)(\Delta-r^2)^{3-\frac{n}{2}}}\,, \tag{131}$$

where we can (as an approximation) let $n=4$. With the on-mass-shell condition we have that $\Delta-r^2$ is again given by (123). Then letting $\lambda=0$ we have that (131) is given by:

$$\frac{-ie\alpha}{4\pi}\int_0^1 dx\int_0^1 y\,dy\,.\,\frac{8p_E\,q_E}{-r} \tag{132}$$

With the on-mass-shell condition we have $r=my$. Thus this term (132) is equal to:

$$(-ie)\frac{-\alpha}{4\pi m}\,8p_E\,q_E\,. \tag{133}$$

Again the factor $p_E\,q_E$ is for the exchange of energies for two electrons with proper energies p_E and q_E respectively and thus it is the vital factor. This factor is then for the space-time statistics and later it will be for a linear statistics of summation for the on-mass-shell condition. Let us introduce a space-time statistics on the factor $p_E\,q_E$, as follows. With the on-mass-shell condition we write $p_E\,q_E$ in the following form:

$$p_E\,q_E=\frac{1}{2}\,(mp_E+q_Em)=\frac{1}{2}\,m\,(p_E+q_E)\,. \tag{134}$$

Then we introduce a space-time statistics on the proper energies p_E and q_E respectively that p_E gives a statistics βp and q_E gives a statistics $\beta p'$ where p and p' are space-time four vectors such that $p^2=m^2$; $p'^2=m^2$; and β is a statistical factor to be determined.

Then we have the following Gordan relation on the space-time four vectors p and p' respectively (see [6] [72]):

$$\left.\begin{aligned} p^{\mu}&=\gamma^{\mu}(p\cdot\gamma)+i\sigma^{\mu\nu}p_{\nu}\\ p^{\mu\prime}&=(p'\cdot\gamma)\gamma^{\mu}-i\sigma^{\mu\nu}p'_{\nu}\end{aligned}\right\}, \tag{135}$$

where p^{μ} and $p^{\mu\prime}$ denote the four components of p and p' respectively. Thus from (134) and the Gordan relation (135) we have the following space-time statistics:

$$\begin{aligned}&\tfrac{1}{2}\,(mp_E+q_Em)=\\ &=\tfrac{1}{2}\,m\,\beta(\gamma^{\mu}(p\cdot\gamma)+(p'\cdot\gamma)\gamma^{\mu}-i\sigma^{\mu\nu}q_{\nu})\,,\end{aligned} \tag{136}$$

where following the convention of QED we define $q=p'-p$.

From (136) we see that the space-time statistics on p_E for giving the four vector p needs the product of two Dirac γ-matrices. Then since the introducing of a Dirac γ-matrix for space-time statistics requires a statistical factor $\frac{1}{2}$ we have that the statistical factor $\beta=\frac{1}{4}$.

Then as in the literature on QED when evaluated between polarization spinors, the $p'\cdot\gamma$ and $\gamma\cdot p$ terms are deduced to the mass m respectively. Thus the term $\frac{1}{2}m\beta(\gamma^{\mu}p\cdot\gamma+p'\cdot\gamma\gamma^{\mu})$ as a constant term can be cancelled by the corresponding counter term with the factor z_e-1 in (95).

Thus by space-time statistics on $p_E\,q_E$ from (133) we get the following vertex correction:

$$(-ie)\,\frac{i\alpha}{4\pi m}\,\sigma^{\mu\nu}q_{\nu} \tag{137}$$

where $q=p-p'$ and the factor 8 in (133) is cancelled by the statistical factor $\frac{1}{2}\beta=\frac{1}{8}$. We remark that in the way of getting (137) a factor m has been absorbed by the two polarization spinors u to get the form $\sqrt{\frac{m}{E}}u$ of the spinors of external electrons.

Then from (137) we get the following exact second order magnetic moment:

$$\frac{\alpha}{2\pi}\,\mu_0\,, \tag{138}$$

where $\mu_0=\frac{1}{2m}$ is the Dirac magnetic moment as in the literature on QED (see [6]).

We see that this result is just the second order anomalous magnetic moment obtained from the conventional QED (see [6] [72]- [78]). Here we can obtain this anomalous magnetic moment exactly while in the conventional QED this anomalous magnetic moment is obtained only by approximation under the condition that $|q^2|\ll m^2$. The point is that we do not need to carry out a complicate integration as in the literature in QED when the on-mass-shell condition is applied to the proper energies p_E and q_E, and with the on-mass-shell condition applied to the proper energies p_E and q_E the computation is simple and the computed result is the exact result of the anomalous magnetic moment.

Let us then consider the following terms in the one-loop vertex correction (118):

$$\begin{aligned}&\tfrac{ie^3}{(2\pi)^n}\textstyle\int_0^1 dx\int_0^1 2y\,dy\,\times\\ &\times\Big[\tfrac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-1-\frac{n}{2})\frac{n}{2}}{\Gamma(3)(\Delta-r^2)^{3-2-1}}\,\tfrac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}+\\ &+\tfrac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r^2}{\Gamma(3)(\Delta-r^2)^{3-2}}\,\tfrac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}-\\ &-\tfrac{\frac{(n+2)}{2}2\pi^{\frac{n}{2}}\Gamma(3-1-\frac{n}{2})r}{\Gamma(3)(\Delta-r^2)^{3-2-1}}\,\tfrac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}-\\ &-\tfrac{2\pi^{\frac{n}{2}}\Gamma(3-\frac{n}{2})r^3}{\Gamma(3)(\Delta-r^2)^{3-2}}\,\tfrac{1}{(-\Delta+r^2)^{2-\frac{n}{2}}}\Big].\end{aligned} \tag{139}$$

From the on-mass-shell condition we have $\Delta-r^2=-r^2$ where we have set $\lambda=0$. The first and the second term are with the factor (p_E+q_E) which by Fermi-Dirac statistics gives the statistics $(p_E + q_E)\frac{1}{2}\gamma^\mu$. Then from the following integration:

$$\int_0^1 dx \int_0^1 2yr dy = \\ = \int_0^1 dx \int_0^1 2y^2(p_E x + (1-x)q_E)dy \tag{140}$$

we get a factor $(p_E + q_E)$ for the third and fourth terms. Thus all these four terms by Fermi-Dirac statistics are with the statistics $(p_E + q_E)\frac{1}{2}\gamma^\mu$. Then by the on-mass-shell condition we have that the statistics $(p_E + q_E)\frac{1}{2}\gamma^\mu$ gives the statistics $m\gamma^\mu$. Thus (139) gives a statistics which is of the form $(\gamma^\mu \cdot \text{constant})$. Thus this constant term can be cancelled by the corresponding counter term with the factor $z_e - 1$ in (95).

Thus under the on-mass-shell condition the renormalized vertex correction $(-ie)\Lambda_R(p', p)$ from the one-loop vertex correction is given by the sum of (128) and (137):

$$(-ie)\Lambda_R(p', p) = \\ = (-ie)\left[\gamma^\mu \frac{\alpha}{3\pi}\frac{q^2}{m^2}\log\frac{m}{\lambda} + \frac{i\alpha}{4\pi m}\sigma^{\mu\nu}q_\nu\right]. \tag{141}$$

19 Computation of the Lamb shift: Part I

The above computation of the vertex correction has not been completed since the parameter λ has not been determined. This appearance of the nonzero λ is due to the on-mass-shell condition. Let us in this Section complete the above computation of the vertex correction by finding another way to get the on-mass-shell condition. By this completion of the above computation of the vertex correction we are then able to compute the Lamb shift.

As in the literature of QED we let $\omega_{\min}$ denote the minimum of the (virtual) photon energy in the scatting of electron. Then as in the literature of QED we have the following relation between $\omega_{\min}$ and λ when $\frac{v}{c} \ll 1$ where v denotes the velocity of electron and c denotes the speed of light (see [6, 68–74]):

$$\log 2\omega_{\min} = \log\lambda + \frac{5}{6}. \tag{142}$$

Thus from (141) we have the following form of the vertex correction:

$$(-ie)\gamma^\mu \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left[\log\frac{m}{2\omega_{\min}} + \frac{5}{6}\right] + \\ + (-ie)\gamma^\mu \frac{ie\alpha i\sigma^{\mu\nu}q_\nu}{4\pi m}. \tag{143}$$

Let us then find a way to compute the following term in the vertex correction (143):

$$(-ie)\gamma^\mu \frac{\alpha}{3\pi}\frac{q^2}{m^2}\log\frac{m}{2\omega_{\min}}. \tag{144}$$

The parameter $2\omega_{\min}$ is for the exchanging (or shifting) of the proper energies p_E and q_E of electrons. Thus the magnitudes of p_E and q_E correspond to the magnitude of $\omega_{\min}$. When the $\omega_{\min}$ is chosen the corresponding p_E and q_E are also chosen and vise versa.

Since $\omega_{\min}$ is chosen to be very small we have that the corresponding proper energies p_E and q_E are very small that they are no longer equal to the mass m for the on-mass-shell condition and they are for the virtual electrons. Then to get the on-mass-shell condition we use a linear statistics of summation on the vital factor $p_E q_E$. This means that the large amount of the effects $p_E q_E$ of the exchange of the virtual electrons are to be summed up to statistically getting the on-mass-shell condition.

Thus let us consider again the one-loop vertex correction (118) where we choose p_E and q_E such that $p_E \ll m$ and $q_E \ll m$. This chosen corresponds to the chosen of $\omega_{\min}$. We can choose p_E and q_E as small as we want such that $p_E \ll m$ and $q_E \ll m$. Thus we can let $\lambda = 0$ and set $p_E = q_E = 0$ for the p_E and q_E in the denominators $(\Delta - r^2)^{3-2}$ in (118). Thus (118) is approximately equal to:

$$\frac{ie^3}{(2\pi)^n}\int_0^1 dx \int_0^1 dy \Big[\frac{4p_E q_E(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-2)}{-m^2} - \\ - \frac{2((p_E+q_E)^2+4p_E q_E)\pi^{\frac{n}{2}}\Gamma(3-2)r}{-m^2} + \\ + \frac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})\frac{n}{2}}{(-\Delta+r^2)^{2-\frac{n}{2}}} + \frac{5(p_E+q_E)\pi^{\frac{n}{2}}\Gamma(3-2)r^2}{-m^2} - \\ - \frac{\frac{(n+2)}{2}2\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})r}{(\Delta-r^2)^{2-\frac{n}{2}}} + \frac{2\pi^{\frac{n}{2}}\Gamma(3-2)r^3}{-m^2}\Big]. \tag{145}$$

Let us then first consider the four terms in (145) without the factor $\Gamma(2-\frac{n}{2})$. For these four terms we can (as an approximation) let $n = 4$. Carry out the integrations $\int_0^1 dx \int_0^1 y dy$ of these four terms we have that the sum of these four terms is given by:

$$(ie)\frac{\alpha\pi^2}{4\pi^3 m^2}\big[4p_E q_E(p_E + q_E) - \\ - \tfrac{1}{2}((p_E + q_E)^2 + 4p_E q_E)(p_E + q_E) + \\ + \tfrac{5}{9}(p_E + q_E)(p_E^2 + q_E^2 + p_E q_E) - \\ - \tfrac{1}{8}(p_E^3 + q_E^3 + p_E^2 q_E + p_E q_E^2)\big] = \\ = (ie)\frac{\alpha\pi^2}{4\pi^3 m^2}(p_E + q_E)\left[\tfrac{5}{72}p_E^2 + \tfrac{5}{72}q_E^2 - \tfrac{14}{9}p_E q_E\right], \tag{146}$$

where the four terms of the sum are from the corresponding four terms of (145) respectively.

Then we consider the two terms in (145) with the factor $\Gamma(2-\frac{n}{2})$. Let $\delta := 2 - \frac{n}{2} > 0$. We have:

$$\Gamma(\delta)\cdot(-\Delta + r^2)^{-\delta} = \tag{147}$$
$$= \left(\tfrac{1}{\delta} + \text{a finite limit term as } \delta \to 0\right)\cdot e^{-\delta\log(-\Delta+r^2)}.$$

We have:

$$\tfrac{1}{\delta}\cdot e^{-\delta\log(-\Delta+r^2)} = \\ = \tfrac{1}{\delta}\cdot\left[1 - \delta\log(-\Delta + r^2) + 0(\delta^2)\right]. \tag{148}$$

Then we have:

$$-\tfrac{1}{\delta}\cdot\delta\log(-\Delta+r^2) =$$
$$= -\log m^2 y - \log\tfrac{1}{m^2}\times$$
$$\times\big[m^2 - p_E^2 x - q_E^2(1-x) + (p_E x + q_E(1-x))^2 y\big] =$$
$$= -\log m^2 y - \log\Big[1 - \tfrac{p_E^2 x(1-xy)+q_E^2(1-x)(1-(1-x)y)}{m^2} +$$
$$+ \tfrac{2p_E q_E x(1-x)y}{m^2} + 0\Big(\tfrac{p_E^2+q_E^2}{m^2}\Big)\Big]. \qquad (149)$$

Then the constant term $-\log m^2 y$ in (149) can be cancelled by the corresponding counter term with the factor $z_e - 1$ in (95) and thus can be ignored. When $p_E^2 \ll m^2$ and $q_E^2 \ll m^2$ the second term in (149) is approximately equal to:

$$f(x,y) := \tfrac{p_E^2 x(1-xy)+q_E^2(1-x)(1-(1-x)y)}{m^2} - \tfrac{2p_E q_E x(1-x)y}{m^2}. \qquad (150)$$

Thus by (150) the sum of the two terms in (145) having the factor $\Gamma(2-\frac{n}{2})$ is approximately equal to:

$$\tfrac{ie^3}{(2\pi)^n}\int_0^1 dx\int_0^1 y\,dy\, f(x,y)\times$$
$$\times\big[5(p_E+q_E)\pi^{\frac{n}{2}}\tfrac{n}{2} - 2\pi^{\frac{n}{2}}\tfrac{(n+2)}{2}r\big], \qquad (151)$$

where we can (as an approximation) let $n=4$. Carrying out the integration $\int_0^1 dx\int_0^1 y\,dy$ of the two terms in (151) we have that (151) is equal to the following result:

$$(ie)\tfrac{\alpha\pi^2}{4\pi^3 m^2}(p_E+q_E)\times$$
$$\times\big[(-5\cdot\tfrac{1}{9}\cdot 2p_E q_E) + (-\tfrac{7}{24}p_E^2 - \tfrac{7}{24}q_E^2 + \tfrac{3}{9}p_E q_E)\big], \qquad (152)$$

where the first term and the second term in the $[\cdot]$ are from the first term and the second term in (151) respectively.

Combining (146) and (151) we have the following result which approximately equal to (145) when $p_E^2 \ll m^2$ and $q_E^2 \ll m^2$:

$$(-ie)\frac{\alpha\pi^2}{4\pi^3 m^2}(p_E+q_E)\left[\frac{2}{9}p_E^2 + \frac{2}{9}q_E^2 + \frac{7}{3}p_E q_E\right], \qquad (153)$$

where the exchanging term $\frac{7}{3}p_E q_E$ is of vital importance.

Now to have the on-mass-shell condition let us consider a linear statistics of summation on (153). Let there be a large amount of virtual electrons $z_j, j\in J$ indexed by a set J with the proper energies $p_{Ej}^2 \ll m^2$ and $q_{Ej}^2 \ll m^2$, $j\in J$. Then from (153) we have the following linear statistics of summation on (153):

$$\tfrac{(-ie)\alpha\pi^2(p_{Ej_0}+q_{Ej_0})}{4\pi^3 m^2}\times$$
$$\times\Big[\tfrac{2}{9}\textstyle\sum_j(p_{Ej}^2+q_{Ej}^2) + \tfrac{7}{3}\sum_j p_{Ej}q_{Ej}\Big], \qquad (154)$$

where for simplicity we let:

$$p_{Ej}+q_{Ej} = p_{Ej'}+q_{Ej'} = p_{Ej_0}+q_{Ej_0} = 2m_0 \qquad (155)$$

for all $j, j'\in J$ and for some (bare) mass $m_0 \ll m$ and for some $j_0\in J$. Then by applying Fermi-Dirac statistics on the factor $p_{Ej_0}+q_{Ej_0}$ in (154) we have the following Fermi-Dirac statistics for (154):

$$(-ie)\tfrac{\alpha\pi^2}{4\pi^3 m^2}\tfrac{1}{2}\gamma^\mu(p_{Ej_0}+q_{Ej_0})\times$$
$$\times\big[\tfrac{2}{9}\textstyle\sum_j(p_{Ej}^2+q_{Ej}^2) + \tfrac{7}{3}\sum_j p_{Ej}q_{Ej}\big] = \qquad (156)$$
$$= (-ie)\tfrac{\alpha\pi^2\gamma^\mu m_0}{4\pi^3 m^2}\big[\tfrac{2}{9}\textstyle\sum_j(p_{Ej}^2+q_{Ej}^2) + \tfrac{7}{3}\sum_j p_{Ej}q_{Ej}\big].$$

Then for the on-mass-shell condition we require that the linear statistical sum $m_0\frac{7}{3}\sum_j p_{Ej}q_{Ej}$ in (156) is of the following form:

$$m_0\frac{7}{3}\sum_j p_{Ej}q_{Ej} = \beta_0 m\,\frac{7}{3}\,q^2, \qquad (157)$$

where $q^2 = (p' - p^2)$ and the form $mq^2 = m(p' - p^2)$ is the on-mass-shell condition which gives the electron mass m; and that β_0 is a statistical factor (to be determined) for this linear statistics of summation and is similar to the statistical factor $(2\pi)^n$ for the space-time statistics.

Then we notice that (156) is for computing (144) and thus its exchanging term corresponding to $\sum_j p_{Ej}q_{Ej}$ must be equal to (144). From (156) we see that there is a statistical factor 4 which does not appear in (144). Since this exchanging term in (156) must be equal to (144) we conclude that the statistical factor β_0 must be equal to 4 so as to cancel the statistical factor 4 in (156). (We also notice that there is a statistical factor π^2 in the numerator of (156) and thus it requires a statistical factor 4 to form the statistical factor $(2\pi)^2$ and thus $\beta_0 = 4$.) Thus we have that for the on-mass-condition we have that (156) is of the following statistical form:

$$(-ie)\frac{\alpha\pi^2}{\pi^3 m^2}\,m\gamma^\mu\left[\beta_2\,\frac{2}{9}\,m^2 + \beta_2'\,\frac{2}{9}\,m^2 + \frac{7}{3}\,q^2\right]. \qquad (158)$$

Then from (158) we have the following statistical form:

$$(-ie)\frac{\alpha\pi^2}{\pi^3 m^2}\,\gamma^\mu\left[\beta_2\,\frac{2}{9}\,m^2 + \beta_2'\,\frac{2}{9}\,m^2 + \frac{7}{3}\,q^2\right], \qquad (159)$$

where the factor m of $m\gamma^\mu$ has been absorbed to the two external spinors of electron. Then we notice that the term corresponding to $\beta_2\frac{2}{9}m^2 + \beta_2'\frac{2}{9}m^2$ in (159) is as a constant term and thus can be cancelled by the corresponding counter term with the factor $z_e - 1$ in (95). Thus from (159) we have the following statistical form of effect which corresponds to (144):

$$(-ie)\gamma^\mu\,\frac{\alpha}{\pi m^2}\,\frac{7}{3}\,q^2. \qquad (160)$$

This effect (160) is as the total effect of q^2 computed from the one-loop vertex with the minimal energy $\omega_{\min}$ and thus includes the effect of q^2 from the anomalous magnetic moment. Thus we have that (144) is computed and is given by the following statistical form:

$$(-ie)\gamma^\mu \tfrac{\alpha}{3\pi}\tfrac{q^2}{m^2}\log\tfrac{m}{2\omega_{\min}} = \\ = (-ie)\gamma^\mu \tfrac{\alpha}{3\pi}\tfrac{q^2}{m^2}\left[7-\tfrac{3}{8}\right], \tag{161}$$

where the term corresponding to the factor $\frac{3}{8}$ is from the anomalous magnetic moment (137) as computed in the literature of QED (see [6]). This completes our computation of (144). Thus under the on-mass-shell condition the renormalized one-loop vertex $(-ie)\Lambda_R(p',p)$ is given by:

$$(-ie)\Lambda_R(p',p) = \\ = (-ie)\left[\gamma^\mu \tfrac{\alpha q^2}{3\pi m^2}\left(7+\tfrac{5}{6}-\tfrac{3}{8}\right)+\tfrac{i\alpha}{4\pi m}\,\sigma^{\mu\nu}q_\nu\right]. \tag{162}$$

This completes our computation of the one-loop vertex correction.

20 Computation of photon self-energy

To compute the Lamb shift let us then consider the one-loop photon self energy (113). As a statistics we extend the one dimensional integral $\int dp_E$ to the n-dimensional integral $\int d^np$ $(n \to 4)$ where $p = (p_E, \mathbf{p})$. This is similar to the dimensional regularization in the existing quantum field theories (However here our aim is to increase the dimension for statistics which is different from the dimensional regularization which is to reduce the dimension from 4 to n to avoid the ultraviolet divergence). With this statistics the factor 2π is replaced by the statistical factor $(2\pi)^n$. From this statistics on (113) we have that the following statistical one-loop photon self-energy:

$$(-1)i^2(-i)^2\tfrac{e^2}{(2\pi)^n}\times \\ \times\int_0^1 dx\int\tfrac{(4p_E^2+4p_Ek_E+k_E^2)d^np}{(p^2+2pkx+k_E^2x-m^2)^2}, \tag{163}$$

where $p^2 = p_E^2 - \mathbf{p}^2$, and $\mathbf{p}^2$ is from $\omega^2 = m^2 + \mathbf{p}^2$; and:

$$pk := p_Ek_E - \mathbf{p}\cdot\mathbf{0} = p_Ek_E\,. \tag{164}$$

As a Feynman rule for space-time statistics a statistical factor (-1) has been introduced for this photon self-energy since it has a loop of electron particles.

By using the formulae for computing Feynman integrals we have that (163) is equal to:

$$\tfrac{(-1)ie^2}{(2\pi)^n}\int_0^1 dx\times \\ \times\left[\tfrac{k_E^2(4x^2-4x+1)\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})}{\Gamma(2)(m^2-k_E^2x(1-x))^{2-\frac{n}{2}}}+\tfrac{\pi^{\frac{n}{2}}\Gamma(2-1-\frac{n}{2})\frac{n}{2}}{\Gamma(2)(m^2-k_E^2x(1-x))^{2-1-\frac{n}{2}}}\right]. \tag{165}$$

Let us first consider the first term in the $[\cdot]$ in (165). Let $\delta := 2-\frac{n}{2} > 0$. As for the one-loop vertex we have

$$\Gamma(\delta)\cdot(m^2-k_E^2\,x(1-x))^{-\delta} = \\ = \left(\tfrac{1}{\delta}+\text{a finite term as }\delta\to 0\right)\cdot e^{-\delta\log(m^2-k_E^2x(1-x))}. \tag{166}$$

We have

$$\tfrac{1}{\delta}\cdot e^{-\delta\log(m^2-k_E^2x(1-x))} = \\ = \tfrac{1}{\delta}\cdot\left[1-\delta\log(m^2-k_E^2x(1-x))+0(\delta^2)\right]. \tag{167}$$

Then we have

$$-\tfrac{1}{\delta}\cdot\delta\log(m^2-k_E^2x(1-x)) = \\ = -\log m^2 - \log\left[1-\tfrac{k_E^2x(1-x)}{m^2}\right]. \tag{168}$$

Then the constant term $-\log m^2$ in (168) can be cancelled by the corresponding counter term with the factor $z_A - 1$ in (95) and thus can be ignored. When $k_E^2 \ll m^2$ the second term in (168) is approximately equal to:

$$\frac{k_E^2x(1-x)}{m^2}\,. \tag{169}$$

Carrying out the integration $\int_0^1 dx$ in (163) with $-\log\left[1-\frac{k_E^2x(1-x)}{m^2}\right]$ replaced by (169), we have the following result:

$$\int_0^1 dx(4x^2-4x+1)\frac{k_E^2x(1-x)}{m^2} = \frac{k_E^2}{30m^2}\,. \tag{170}$$

Thus as in the literature in QED from the photon self-energy we have the following term which gives contribution to the Lamb shift:

$$\frac{k_E^2}{30m^2} = \frac{(p_E-q_E)^2}{30m^2}\,, \tag{171}$$

where $k_E = p_E - q_E$ and p_E, q_E denote the proper energies of virtual electrons. Let us then consider statistics of a large amount of photon self-energy (168). When there is a large amount of photon self-energies we have the following linear statistics of summation:

$$\frac{\sum_i k_{Ei}^2}{30m^2}\,, \tag{172}$$

where each i represent a photon. Let us write:

$$k_{Ei}^2 = (p_{Ei}-q_{Ei})^2 = p_{Ei}^2 - 2p_{Ei}q_{Ei} + q_{Ei}^2\,. \tag{173}$$

Thus we have:

$$\textstyle\sum_i k_{Ei}^2 = \sum_i(p_{Ei}-q_{Ei})^2 = \\ = \sum_i(p_{Ei}^2+q_{Ei}^2) - 2\sum_i p_{Ei}q_{Ei}\,. \tag{174}$$

Now as the statistics of the vertex correction we have the following statistics:

$$\sum_i p_{Ei}q_{Ei} = 4(p'-p)^2 = 4q^2, \tag{175}$$

where 4 is a statistical factor which is the same statistical factor of case of the vertex correction and p, p' are on-mass-shell four vectors of electrons. As the the statistics of the vertex correction this statistical factor cancels another statistical factor 4. On the other hand as the statistics of the vertex correction we have the following statistics:

$$\sum_i p_{Ei}^2 = \beta_3 m^2, \qquad \sum_i q_{Ei}^2 = \beta_4 m^2, \tag{176}$$

where β_3 and β_4 are two statistical factors. As the case of the vertex correction these two sums give constant terms and thus can be cancelled by the corresponding counter term with the factor $z_A - 1$ in (95). Thus from (174) we have that the linear statistics of summation $\sum_i k_{Ei}^2$ gives the following statistical renormalized photon self-energies Π_R and Π_M (where we follow the notations in the literature of QED for photon self-energies Π_M):

$$\begin{aligned} i\Pi_R(k_E) = ik_E^2\Pi_M(k_E) = \\ = ik_E^2 \frac{\alpha}{4\pi}\frac{8q^2}{30m^2} = ik_E^2\frac{\alpha}{3\pi}\frac{q^2}{5m^2}\,, \end{aligned} \tag{177}$$

where we let $k_{Ei}^2 = k_E^2$ for all i.

Let us then consider the second term in the $[\cdot]$ in (165). This term can be written in the following form:

$$\begin{aligned} &\frac{\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})\frac{n}{2}}{(1-\frac{n}{2})\Gamma(2)(m^2-k_E^2x(1-x))^{2-1-\frac{n}{2}}} = \\ &= \frac{\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})\frac{n}{2}}{(1-\frac{n}{2})\Gamma(2)}\left[(m^2 - k_E^2x(1-x)) + 0(\delta)\right] = \\ &= k_E^2\left[\frac{1}{\delta}\cdot\frac{(-1)\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})\frac{n}{2}}{(1-\frac{n}{2})\Gamma(2)}\,x(1-x)\right] + \\ &+ \left[\frac{1}{\delta}\cdot\frac{\pi^{\frac{n}{2}}\Gamma(2-\frac{n}{2})\frac{n}{2}}{(1-\frac{n}{2})\Gamma(2)}\,m^2 + 0(\delta))\right] \end{aligned} \tag{178}$$

Then the first term in (178) under the integration $\int_0^1 dx$ is of the form $(k_E^2 \cdot \text{constant})$. Thus this term can also be cancelled by the counter-term with the factor $z_A - 1$ in (95). In summary the renormalization constant z_A is given by the following equation:

$$\begin{aligned} (-1)^3 i(z_A - 1) = (-i)\Big\{\frac{1}{\delta}\cdot\frac{e^2\pi^{\frac{n}{2}}}{(2\pi)^n}\int_0^1 dx\,\times \\ \times\left[(4x^2 - 4x + 1) - \frac{nx(1-x)}{2-n}\right] + c_A\Big\}, \end{aligned} \tag{179}$$

where c_A is a finite constant when $\delta \to 0$. From this equation we have that z_A is a very large number when $\delta > 0$ is very small. Thus $e_0 = z_e\left(z_Z z_A^{1/2}\right)^{-1} e = \frac{1}{n_e}e$ is a very small constant when $\delta > 0$ is very small (and since $\frac{e^2}{4\pi} = \alpha = \frac{1}{137}$ is small) where shall show that we can let $z_e = z_Z$.

Then the second term in (178) under the integration $\int_0^1 dx$ gives a parameter $\lambda_3 > 0$ for the photon self-energy since $\delta > 0$ is as a parameter.

Combing the effects of the two terms in the $[\cdot]$ in (165) we have the following renormalized one-loop photon self-energy:

$$i\left(\Pi_R(k_E) + \lambda_3\right). \tag{180}$$

Then we have the following Dyson series for photon propagator:

$$\begin{aligned} &\frac{i}{k_E^2-\lambda_0} + \frac{i}{k_E^2-\lambda_0}(i\Pi_R(k_E) + i\lambda_3)\frac{i}{k_E^2-\lambda_0} + \ldots = \\ &= \frac{i}{k_E^2(1+\Pi_M)-(\lambda_0-\lambda_3)} =: \\ &=: \frac{i}{k_E^2(1+\Pi_M)-\lambda_R}\,, \end{aligned} \tag{181}$$

where λ_R is as a renormalized mass-energy parameter. This is as the renormalized photon propagator. We have the following approximation of this renormalized photon propagator:

$$\frac{i}{k_E^2(1+\Pi_M) - \lambda_R} \approx \frac{i}{k_E^2 - \lambda_R}(1 - \Pi_M)\,. \tag{182}$$

21 Computation of the Lamb shift: Part II

Combining the effect of vertex correction and photon self-energy we can now compute the Lamb shift. Combining the effect of photon self-energy $(-ie\gamma^\mu)[-\Pi_M]$ and vertex correction we have:

$$\begin{aligned} &(-ie)\Lambda_R(p', p) + (-ie\gamma^\mu)[-\Pi_M] = \\ &= (-ie)\left[\gamma^\mu\frac{\alpha q^2}{3\pi m^2}\left(7 + \frac{5}{6} - \frac{3}{8} - \frac{1}{5}\right) + \frac{i\alpha}{4\pi m}\sigma^{\mu\nu}q_\nu\right]. \end{aligned} \tag{183}$$

As in the literature of QED let us consider the states $2S_{\frac{1}{2}}$ and the $2P_{\frac{1}{2}}$ in the hydrogen atom [6, 72–78]. Following the literature of QED for the state $2S_{\frac{1}{2}}$ an effect of $\frac{\alpha q^2}{3\pi m^2}\left(\frac{3}{8}\right)$ comes from the anomalous magnetic moment which cancels the same term with negative sign in (183). Thus by using the method in the computation of the Lamb shift in the literature of QED we have the following second order shift for the state $2S_{\frac{1}{2}}$:

$$\Delta E_{2S_{\frac{1}{2}}} = \frac{m\alpha^5}{6\pi}\left(7 + \frac{5}{6} - \frac{1}{5}\right). \tag{184}$$

Similarly by the method of computing the Lamb shift in the literature of QED from the anomalous magnetic moment we have the following second order shift for the state $2P_{\frac{1}{2}}$:

$$\Delta E_{2P_{\frac{1}{2}}} = \frac{m\alpha^5}{6\pi}\left(-\frac{1}{8}\right). \tag{185}$$

Thus the second order Lamb shift for the states $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ is given by:

$$\Delta E = \Delta E_{2S_{\frac{1}{2}}} - \Delta E_{2P_{\frac{1}{2}}} = \frac{m\alpha^5}{6\pi}\left(7 + \frac{5}{6} - \frac{1}{5} + \frac{1}{8}\right) \tag{186}$$

or in terms of frequencies for each of the terms in (186) we have:

$$\begin{aligned} \Delta\nu = 952 + 113.03 - 27.13 + 16.96 = \\ = 1054.86 \text{ Mc/sec.} \end{aligned} \tag{187}$$

This agrees with the experimental results [6, 72–78]:

$$\begin{aligned} \Delta\nu^{\text{exp}} &= 1057.86 \pm 0.06 \text{ Mc/sec} \\ \text{and } &= 1057.90 \pm 0.06 \text{ Mc/sec.} \end{aligned} \tag{188}$$

22 Computation of the electron self-energy

Let us then consider the one-loop electron self-energy (113). As a statistics we extend the one dimensional integral $\int dk_E$ to the n-dimensional integral $\int d^n k$ $(n \to 4)$ where $k = (k_E, \mathbf{k})$. This is similar to the dimensional regularization in the existing quantum field theories (However here our aim is to increase the dimension for statistics which is different from the dimensional regularization which is to reduce the dimension from 4 to n to avoid the ultraviolet divergence). With this statistics the factor 2π is replaced by the statistical factor $(2\pi)^n$. From this statistics on (114) we have that the following statistical one-loop electron self-energy $-i\Sigma(p_E)$:

$$-i\Sigma(p_E) := i^2(-i)^2 \frac{e^2}{(2\pi)^n} \int_0^1 dx \int d^n k \times \\ \times \frac{(k_E^2 - 4p_E k_E + 4p_E^2) d^n k}{(k^2 - 2kpx + p_E^2 x - xm^2 - (1-x)\lambda^2)^2}, \tag{189}$$

where $k^2 = k_E^2 - \mathbf{k}^2$, and $\mathbf{k}^2$ is from $\omega^2 = m^2 + \mathbf{k}^2$ and $\lambda_0^2 = \lambda^2 + \mathbf{k}^2$; and $kp := k_E p_E - \mathbf{k} \cdot \mathbf{0} = k_E p_E$. By using the formulae for computing Feynman integrals we have that (189) is equal to:

$$\frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big[\frac{p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \Gamma(2 - \frac{n}{2})}{\Gamma(2)(xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x))^{2-\frac{n}{2}}} +$$
$$+ \frac{\pi^{\frac{n}{2}} \Gamma(2 - 1 - \frac{n}{2}) \frac{n}{2}}{\Gamma(2)(xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x))^{2-1-\frac{n}{2}}} \Big] =$$
$$= \frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big\{ p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \times$$
$$\times \left[\left(\tfrac{1}{\delta} + O(\delta) \right) \cdot e^{-\delta \log(xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x))} \right] -$$
$$- \pi^{\frac{n}{2}} \tfrac{n}{2} \tfrac{1}{\delta} \left[xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x) + 0(\delta) \right] \Big\} =$$
$$= \frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big\{ p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \left[\tfrac{1}{\delta} - \tfrac{1}{\delta} \times \right.$$
$$\left. \times \delta \log(xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x)) + 0(\delta) \right] -$$
$$- \pi^{\frac{n}{2}} \tfrac{n}{2} \tfrac{1}{\delta} \left[xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x) + 0(\delta) \right] \Big\} =$$
$$= \frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big\{ p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \times$$
$$\times \left[\tfrac{1}{\delta} - \log(xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x)) + 0(\delta) \right] -$$
$$- \pi^{\frac{n}{2}} \tfrac{n}{2} \tfrac{1}{\delta} \left[xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x) + 0(\delta) \right] \} =$$
$$= \frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big\{ p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \times$$
$$\times \left[\tfrac{1}{\delta} - \log(xm^2 + (1-x)\lambda^2) - \right.$$
$$\left. - \log\left(1 - \frac{p_E^2 x(1-x)}{xm^2 + (1-x)\lambda^2}\right) + 0(\delta) \right] -$$
$$- \pi^{\frac{n}{2}} \tfrac{n}{2} \tfrac{1}{\delta} \left[xm^2 + (1-x)\lambda^2 - p_E^2 x(1-x) + 0(\delta) \right] \Big\} =:$$

$$=: \frac{ie^2}{(2\pi)^n} \int_0^1 dx \Big\{ p_E^2(x^2 - 4x + 4)\pi^{\frac{n}{2}} \left[\tfrac{1}{\delta} - \right.$$
$$- \log(xm^2 + (1-x)\lambda^2) - \log\left(1 - \frac{p_E^2 x(1-x)}{xm^2 + (1-x)\lambda^2}\right) +$$
$$\left. + 0(\delta) \right] + p_E^2 \cdot \tfrac{1}{\delta} \pi^{\frac{n}{2}} \tfrac{n}{2} x(1-x) \Big\} + i\omega_3, \tag{190}$$

where $\omega_3 > 0$ is as a mass-energy parameter.

Then we notice that from the expressions for $\Sigma_0(p_E)$ and $\Lambda_0(p_E, q_E)$ in (114) and (115) we have the following identity:

$$\frac{\partial}{\partial p_E} \Sigma_0(p_E) = -\Lambda_0(p_E, p_E) + \\ + \frac{i4e^2}{2\pi} \int dk \frac{-k_E + 2p_E}{(k_E^2 - \lambda_0^2)((p_E - k_E)^2 - \omega^2)}. \tag{191}$$

This is as a Ward-Takahashi identity which is analogous to the corresponding Ward-Takahashi identity in the conventional QED theory [6].

From (114) and (115) we get their statistical forms by changing $\int dk$ to $\int d^n k$. From this summation form of statistics and the identity (191) we then get the following statistical Ward-Takahashi identity:

$$\frac{\partial}{\partial p_E} \Sigma(p_E) = -\Lambda(p_E, p_E) + \\ + \frac{i4e^2}{(2\pi)^n} \int_0^1 dx \int d^n k \frac{-k_E + 2p_E}{(k^2 - 2kpx + p_E^2 x - xm^2 - (1-x)\lambda^2)^2}, \tag{192}$$

where $\Sigma(p_E)$ denotes the statistical form of $\Sigma_0(p_E)$ and is given by (189) and $\Lambda(p_E, q_E)$ denotes the statistical form of $\Lambda_0(p_E, q_E)$ as in the above Sections.

After the differentiation of (190) with respect to p_E the remaining factor p_E of the factor p_E^2 of (190) is absorbed to the external spinors as the mass m and a factor $\frac{\gamma^\mu}{2}$ is introduced by space-time statistics, as the case of the statistics of the vertex correction $\Lambda_0(p_E, q_E)$ in the above Sections. From the absorbing of a factor p_E to the external spinors for both sides of this statistical Ward-Takahashi identity we then get a statistical Ward-Takahashi identity where the Taylor expansion (of the variable p_E) of both sides of this statistical Ward-Takahashi identity are with constant term as the beginning term. From this Ward-Takahashi identity we have that these two constant terms must be the same constant. Then the constant term, denoted by $C(\delta)$, of the vertex correction of this Ward-Takahashi identity is cancelled by the counterterm with the factor $z_e - 1$ in (95), as done in the above computation of the renormalized vertex correction $A_R(p', p)$. (At this point we notice that in computing the constant term of the vertex correction some terms with the factor p_E has been changed to constant terms under the on-mass-shell condition $p_E = m$. This then modifies the definition of $C(\delta)$).

On the other hand let us denote the constant term for the electron self-energy by $B(\delta)$. Then from the above statistical Ward-Takahashi identity we have the following equality:

$$B(\delta) + a_1 \cdot \frac{1}{\delta} + b_1 = C(\delta), \tag{193}$$

where a_1, b_1 are finite constants when $\delta \to 0$ and the term $a_1 \cdot \frac{1}{\delta}$ is from the second term in the right hand side of (192).

Let us then compute the constant term $B(\delta)$ for the electron self-energy, as follows. As explained in the above the constant term for the electron self-energy can be obtained by differentiation of (190) with respect to p_E and the removing of the remaining factor p_E of p_E^2. We have:

$$\begin{aligned} &\frac{\partial}{\partial p_E}\Big\{ \frac{ie^2}{(2\pi)^n} \int_0^1 dx p_E^2 (x^2-4x+4)\pi^{\frac{n}{2}} \times \\ &\times \left[\tfrac{1}{\delta} - \log(xm^2+(1-x)\lambda^2 - p_E^2 x(1-x))\right] + \\ &+ p_E^2 \cdot \tfrac{1}{\delta}\pi^{\frac{n}{2}} \tfrac{n}{2} \frac{ie^2}{(2\pi)^n} \int_0^1 x(1-x)dx + i\omega_3 \Big\} = \\ &= \frac{ie^2}{(2\pi)^n} \int_0^1 dx 2p_E (x^2-4x+4)\pi^{\frac{n}{2}} \times \\ &\times \left[\tfrac{1}{\delta} - \log(xm^2+(1-x)\lambda^2 - p_E^2 x(1-x))\right] + \\ &+ \frac{ie^2}{(2\pi)^n} \int_0^1 dx \frac{p_E^2(x^2-4x+4)\pi^{\frac{n}{2}} \cdot 2p_E x(1-x)}{xm^2+(1-x)\lambda^2 - p_E^2 x(1-x)} + \\ &+ 2p_E \cdot \tfrac{1}{\delta}\pi^{\frac{n}{2}} \tfrac{n}{2} \frac{ie^2}{(2\pi)^n} \int_0^1 x(1-x)dx\,. \end{aligned} \tag{194}$$

Then by Taylor expansion of (194) and by removing a factor $2p_E$ from (194) the constant term for the electron self-energy is given by:

$$\begin{aligned} B(\delta) &:= \frac{-e^2}{(2\pi)^n}\int_0^1 dx (x^2-4x+4)\pi^{\frac{n}{2}} \times \\ &\times \left[\tfrac{1}{\delta} - - \log(xm^2+(1-x)\lambda^2)\right] - \\ &- \tfrac{1}{\delta}\pi^{\frac{n}{2}} \tfrac{n}{2} \frac{e^2}{(2\pi)^n}\int_0^1 x(1-x)dx\,. \end{aligned} \tag{195}$$

Then as a renormalization procedure for the electron self-energy we choose a $\delta_1 > 0$ which is related to the δ for the renormalization of the vertex correction such that:

$$B(\delta_1) = B(\delta) + a_1 \cdot \frac{1}{\delta} + b_1\,. \tag{196}$$

This is possible since $B(\delta)$ has a term proportional to $\frac{1}{\delta}$. From this renormalization procedure for the electron self-energy we have:

$$B(\delta_1) = C(\delta)\,. \tag{197}$$

This constant term $B(\delta_1)$ for the electron self-energy is to be cancelled by the counter-term with the factor $z_Z - 1$ in (95). We have the following equation to determine the renormalization constant z_Z for this cancellation:

$$(-1)^3 i(z_Z - 1) = (-i)B(\delta_1)\,. \tag{198}$$

Then from the equality (197) we have $z_e = z_Z$ where z_e is determined by the following equation:

$$(-1)^3 i(z_e - 1) = (-i)C(\delta)\,. \tag{199}$$

Cancelling $B(\delta_1)$ from the electron self-energy (190) we get the following renormalized one-loop electron self-energy:

$$\begin{aligned} &-ip_E^2\Sigma_R(p_E) + i\omega_3^2 := -ip_E^2\tfrac{\alpha}{4\pi} \times \\ &\times \int_0^1 dx(x^2-4x+4)\log\left[1 - \frac{p_E^2 x(1-x)}{xm^2+(1-x)\lambda^2}\right] + i\omega_3^2\,. \end{aligned} \tag{200}$$

We notice that in (200) we can let $\lambda = 0$ since there is no infrared divergence when $\lambda = 0$. This is better than the computed electron self-energy in the conventional QED theory where the computed one-loop electron self-energy is with infrared divergence when $\lambda = 0$ [6].

From this renormalized electron self-energy we then have the renormalized electron propagator obtained by the following Dyson series:

$$\begin{aligned} &\frac{i}{p_E^2-\omega^2} + \frac{i}{p_E^2-\omega^2}(-ip_E^2\Sigma_R(p_E) + i\omega_3^2)\frac{i}{p_E^2-\omega^2} + \ldots = \\ &= \frac{i}{p_E^2(1-\Sigma_R(p_E)) - (\omega^2-\omega_3^2)} =: \\ &=: \frac{i}{p_E^2(1-\Sigma_R(p_E)) - \omega_R^2}\,, \end{aligned} \tag{201}$$

where $\omega_R^2 := \omega^2 - \omega_3^2$ is as a renormalized electron mass-energy parameter. Then by space-time statistics from the renormalized electron propagator (201) we can get the renormalized electron propagator in the spin-$\frac{1}{2}$ form, as that the electron propagator $\frac{i}{\gamma_\mu p^\mu - m}$ in the spin-$\frac{1}{2}$ form can be obtained from the electron propagator $\frac{i}{p_E^2-\omega^2}$.

23 New effect of QED

Let us consider a new effect for electron scattering which is formed by two seagull vertexes with one photon loop and four electron lines. This is a new effect of QED because the conventional spin $\frac{1}{2}$ theory of QED does not have this seagull vertex. The Feynman integral corresponding to the photon loop is given by

$$\begin{aligned} &\frac{i^2(i)^2e^4}{2\pi}\int \frac{dk_E}{(k_E^2-\lambda_0^2)((p_E-q_E-k_E)^2-\lambda_0^2)} = \\ &= \frac{e^4}{2\pi}\int_0^1\int \frac{dk_E}{(k_E^2-2k_E(p_E-q_E)x+(p_E-q_E)^2x-\lambda_0^2)^2} = \\ &= \frac{e^4}{2\pi}\int_0^1\int \frac{dk_E}{(k_E^2-2k_E(p_E-q_E)x+(p_E-q_E)^2x-\lambda_0^2)^2}\,. \end{aligned} \tag{202}$$

Let us then introduce a space-time statistics. Since the photon propagator of the (two joined) seagull vertex interactions is of the form of a circle on a plane we have that the appropriate space-time statistics of the photons is with the two dimensional space for the circle of the photon propagator. From this two dimensional space statistics we then get a three dimensional space statistics by multiplying the statistical factor $\frac{1}{(2\pi)^3}$ of the three dimensional space statistics and by concentrating in a two dimensional subspace of the three dimensional space statistics.

Thus as similar to the four dimensional space-time statistics with the three dimensional space statistics in the above

Sections from (202) we have the following space-time statistics with the two dimensional subspace:

$$\frac{e^4}{(2\pi)^4}\int_0^1\int\frac{d^3k}{(k_E^2-2k_E(p_E-q_E)x+(p_E-q_E)^2x-\mathbf{k}^2-\lambda_4^2)^2}= \\ =\frac{e^4}{(2\pi)^4}\int_0^1 dx\int\frac{d^3k}{(k^2-2k\cdot(p_E-q_E,0)x+(p_E-q_E)^2x-\lambda_4^2)^2}\,, \tag{203}$$

where the statistical factor $\frac{1}{(2\pi)^3}$ of three dimensional space has been introduced to give the factor $\frac{1}{(2\pi)^4}$ of the four dimensional space-time statistics; and we let $k=(k_E,\mathbf{k})$, $k^2=$ $=k_E^2-\mathbf{k}^2$ and since the photon energy parameter λ_0 is a free parameter we can write $\lambda_0^2=\mathbf{k}^2+\lambda_4^2$ for some λ_4.

Then a delta function concentrating at 0 of a one dimensional momentum variable is multiplied to the integrand in (203) and the three dimensional energy-momentum integral in (203) is changed to a four dimensional energy-momentum integral by taking the corresponding one more momentum integral.

From this we get a four dimensional space-time statistics with the usual four dimensional momentum integral and with the statistical factor $\frac{1}{(2\pi)^4}$. After this additional momentum integral we then get (203) as a four dimensional space-time statistics with the two dimensional momentum variable.

Then to get a four dimensional space-time statistics with the three dimensional momentum variable a delta function concentrating at 0 of another one dimensional momentum variable is multiplied to (203) and the two dimensional momentum variable of (203) is extended to the corresponding three dimensional momentum variable. From this we then get a four dimensional space-time statistics with the three dimensional momentum variable.

Then we have that (203) is equal to:

$$\frac{e^4}{(2\pi)^4}\frac{i\pi^{\frac{3}{2}}\Gamma(2-\frac{3}{2})}{\Gamma(2)}\int_0^1\frac{dx}{((p_E-q_E)^2x(1-x)-\lambda_4^2)^{\frac{1}{2}}}\,. \tag{204}$$

Then since the photon mass-energy parameter λ_4 is a free parameter for space-time statistics we can write λ_4 in the following form:

$$\lambda_4^2=(\mathbf{p}-\mathbf{q})^2x(1-x)\,, \tag{205}$$

where $\mathbf{p}-\mathbf{q}$ denotes a two dimensional momentum vector.

Then we let $p-q=(p_E-q_E,\mathbf{p}-\mathbf{q})$. Then we have:

$$(p_E-q_E)^2\,x(1-x)-\lambda_4^2= \\ =(p_E-q_E)^2\,x(1-x)-(\mathbf{p}-\mathbf{q})^2\,x(1-x)= \\ =(p-q)^2\,x(1-x)\,. \tag{206}$$

Then we have that (204) is equal to:

$$\frac{e^4}{(2\pi)^4}\frac{i\pi^{\frac{3}{2}}\Gamma(2-\frac{3}{2})}{\Gamma(2)}\int_0^1\frac{dx}{((p-q)^2x(1-x))^{\frac{1}{2}}} \\ =\frac{e^4}{(2\pi)^4}\frac{i\pi\pi^{\frac{3}{2}}\Gamma(2-\frac{3}{2})}{\Gamma(2)}\frac{1}{((p-q)^2)^{\frac{1}{2}}}= \\ =\frac{e^4i}{16\pi((p-q)^2)^{\frac{1}{2}}}=\frac{e^2\alpha i}{4((p-q)^2)^{\frac{1}{2}}}\,. \tag{207}$$

Thus we have the following potential:

$$V_{seagull}(p-q)=\frac{e^2\alpha i}{4((p-q)^2)^{\frac{1}{2}}}\,. \tag{208}$$

This potential (208) is as the seagull vertex potential.

We notice that (208) is a new effect for electron-electron or electron-positron scattering. Recent experiments on the decay of positronium show that the experimental orthopositronium decay rate is significantly larger than that computed from the conventional QED theory [33–52]. In the following Section 24 to Section 26 we show that this discrepancy can be remedied with this new effect (208).

24 Reformulating the Bethe-Salpeter equation

To compute the orthopositronium decay rate let us first find out the ground state wave function of the positronium. To this end we shall use the Bethe-Salpeter equation. It is well known that the conventional Bethe-Salpeter equation is with difficulties such as the relative time and relative energy problem which leads to the existence of nonphysical solutions in the conventional Bethe-Salpeter equation [7–32]. From the above QED theory let us reformulate the Bethe-Salpeter equation to get a new form of the Bethe-Salpeter equation. We shall see that this new form of the Bethe-Salpeter equation resolves the basic difficulties of the Bethe-Salpeter equation such as the relative time and relative energy problem.

Let us first consider the propagator of electron. Since electron is a spin-$\frac{1}{2}$ particle its statistical propagator is of the form $\frac{i}{\gamma_\mu p^\mu-m}$. Thus before the space-time statistics the spin-$\frac{1}{2}$ form of electron propagator is of the form $\frac{i}{p_E-\omega}$ which can be obtained from the electron propagator $\frac{i}{p_E^2-\omega^2}$ by the factorization: $p_E^2-\omega^2=(p_E-\omega)(p_E+\omega)$. Then we consider the following product which is from two propagators of two spin-$\frac{1}{2}$ particles:

$$[p_{E1}-\omega_1][p_{E2}-\omega_2]= \\ =p_{E1}p_{E2}-\omega_1p_{E2}-\omega_2p_{E1}+\omega_1\omega_2=: \\ =:p_E^2-\omega_b^2, \tag{209}$$

where we define $p_E^2=p_{E1}p_{E2}$ and $\omega_b^2:=\omega_1p_{E2}+\omega_2p_{E1}-$ $-\omega_1\omega_2$. Then since ω_1 and ω_2 are free mass-energy parameters we have that ω_b is also a free mass-energy parameter with the requirement that it is to be a positive parameter.

Then we introduce the following reformulated relativistic equation of Bethe-Salpeter type for two particles with spin-$\frac{1}{2}$:

$$\phi_0(p_E,\omega_b)=\frac{i^2\lambda'}{[p_{E1}-\omega_1][p_{E2}-\omega_2]}\times \\ \times\int\frac{ie^2\phi_0(q_E,\omega_b)dq_E}{((p_E-q_E)^2-\lambda_0^2)}\,, \tag{210}$$

where we use the photon propagator $\frac{i}{k_E^2-\lambda_0^2}$ (which is of the effect of Coulomb potential) for the interaction of these two particles and we write the proper energy k_E^2 of this potential in the form $k_E^2 = (p_E - q_E)^2$; and λ' is as the coupling parameter. We shall later also introduce the seagull vertex term for the potential of binding.

Let us then introduce the space-time statistics. Since we have the seagull vertex term for the potential of binding which is of the form of a circle in a two dimensional space from the above Section on the seagull vertex potential we see that the appropriate space-time statistics is with the two dimensional space. Thus with this space-time statistics from (210) we have the following reformulated relativistic Bethe-Salpeter equation:

$$\phi_0(p) = \frac{-\lambda'}{p^2 - \gamma_0^2} \int \frac{i d^3 q}{(p-q)^2} \phi_0(q) , \qquad (211)$$

where we let the free parameters ω_b and λ_0 be such that $p^2 = p_E^2 - \mathbf{p}^2$ with $\omega_b^2 = \mathbf{p}^2 + \gamma_0^2$ for some constant $\gamma_0^2 = \frac{1}{a^2} > 0$ where a is as the radius of the binding system; and $(p-q)^2 = (p_E - q_E)^2 - (\mathbf{p} - \mathbf{q})^2$ with $\lambda_0^2 = (\mathbf{p} - \mathbf{q})^2$. We notice that the potential $\frac{i\alpha}{(p-q)^2}$ of binding is now of the usual (relativistic) Coulomb potential type. In (211) the constant e^2 in (210) has been absorbed into the parameter λ' in (211).

We see that in this reformulated Bethe-Salpeter equation the relative time and relative energy problem of the conventional Bethe-Salpeter equations is resolved [7–32]. Thus this reformulated Bethe-Salpeter equation will be free of abnormal solutions.

Let us then solve (211) for the relativistic bound states of particles. We show that the ground state solution $\phi_0(p)$ can be exactly solved and is of the following form:

$$\phi_0(p) = \frac{1}{(p^2 - \gamma_0^2)^2} . \qquad (212)$$

We have:

$$\begin{aligned} & \tfrac{1}{((p-q)^2)} \tfrac{1}{(q^2-\gamma_0^2)^2} = \\ & = \tfrac{(2+1-1)!}{(2-1)!(1-1)!} \int_0^1 \tfrac{(1-x)dx}{[x(p-q)^2 + (1-x)(q^2-\gamma_0^2)^2]^3} = \\ & = \tfrac{(2+1-1)!}{(2-1)!(1-1)!} \int_0^1 \tfrac{(1-x)dx}{[q^2+2xpq+xp^2-(1-x)\gamma_0^2]^3} = \\ & = 2\int_0^1 \tfrac{(1-x)dx}{[q^2+2xpq+xp^2-(1-x)\gamma_0^2]^3} . \end{aligned} \qquad (213)$$

Thus we have:

$$\begin{aligned} & i\int \tfrac{d^3q}{((p-q)^2)(q^2-\gamma_0^2)^2} = \\ & = i2\int_0^1 (1-x)dx \int \tfrac{d^3q}{[q^2+2xpq+xp^2-(1-x)\gamma_0^2]^3} = \\ & = i^2 \tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \int_0^1 \tfrac{(1-x)dx}{[+x(1-x)p^2-(1-x)\gamma_0^2]^{\frac{3}{2}}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \int_0^1 \tfrac{dx}{[+xp^2-\gamma_0^2][(1-x)(xp^2-\gamma_0^2)]^{\frac{3}{2}}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \tfrac{\partial^2}{\partial(\gamma_0^2)^2} \int_0^1 dx \left[\tfrac{xp^2-\gamma_0^2}{1-x}\right]^{\frac{3}{2}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \tfrac{\partial^2}{\partial(\gamma_0^2)^2} \int_0^1 dx \left[\tfrac{p^2-\gamma_0^2}{1-x} - p^2\right]^{\frac{3}{2}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \tfrac{\partial^2}{\partial(\gamma_0^2)^2} \int_1^\infty \tfrac{dt}{t^2} \left[(p^2-\gamma_0^2)t - p^2\right]^{\frac{3}{2}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \int_1^\infty \tfrac{dt}{t^2} \left[(p^2-\gamma_0^2)t - p^2\right]^{\frac{-3}{2}} = \\ & = -\tfrac{2\pi^{\frac{3}{2}}\Gamma(3-\frac{3}{2})}{\Gamma(3)} \tfrac{1}{(p^2-\gamma_0^2)} \int_{\gamma_0^2}^\infty x^{\frac{-3}{2}} dx = \\ & = -\tfrac{\pi^2}{2} \tfrac{1}{\gamma_0(p^2-\gamma_0^2)} . \end{aligned} \qquad (214)$$

Then let us choose λ' such that $\lambda' = \frac{2\gamma_0}{\pi^2}$. From this value of λ' we see that the BS equation (211) holds. Thus the ground state solution is of the form (212). We see that when $p_E = 0$ and $\omega_b^2 = \mathbf{p}^2 + \gamma_0^2$ then this ground state gives the well known nonrelativistic ground state of the form $\frac{1}{(\mathbf{p}^2+\gamma_0^2)^2}$ of binding system such as the hydrogen atom.

25 Bethe-Salpeter equation with seagull vertex potential

Let us then introduce the following reformulated relativistic Bethe-Salpeter equation which is also with the seagull vertex potential of binding:

$$\begin{aligned} \phi(p) = & \tfrac{-\lambda'}{p^2-\gamma_0^2} \times \\ & \times \int \left[\tfrac{i}{(p-q)^2} + \tfrac{i\alpha}{4((p-q)^2)^{\frac{1}{2}}}\right] \phi(q) d^3q , \end{aligned} \qquad (215)$$

where a factor e^2 of both the Coulomb-type potential and the seagull vertex potential is absorbed to the coupling constant λ'.

Let us solve (215) for the relativistic bound states of particles. We write the ground state solution in the following form:

$$\phi(p) = \phi_0(p) + \alpha\phi_1(p) , \qquad (216)$$

where $\phi_0(p)$ is the ground state of the BS equation when the interaction potential only consists of the Coulomb-type potential. Let us then determine the $\phi_1(p)$.

From (215) by comparing the coefficients of the $\alpha^j, j = 0, 1$ on both sides of BS equation we have the following equation for $\phi_1(p)$:

$$\begin{aligned} \phi_1(p) = & \tfrac{-\lambda'}{p^2-\gamma_0^2} \int \left[\tfrac{i}{4((p-q)^2)^{\frac{1}{2}}}\right] \phi_0(q) d^3q + \\ & + \tfrac{-\lambda'}{p^2-\gamma_0^2} \int \left[\tfrac{i}{((p-q)^2)} + \tfrac{i\alpha}{4((p-q)^2)^{\frac{1}{2}}}\right] \phi_1(q) d^3q . \end{aligned} \qquad (217)$$

This is a nonhomogeneous linear Fredholm integral equation. We can find its solution by perturbation. As a first order approximation we have the following approximation of $\phi_1(p)$:

$$\begin{aligned}
\phi_1(p) &\approx \frac{-\lambda'}{p^2-\gamma_0^2}\int \frac{i}{4((p-q)^2)^{\frac12}}\,\phi_0(q)d^3q = \\
&= \frac{-\lambda'}{p^2-\gamma_0^2}\int \frac{i}{4((p-q)^2)^{\frac12}}\,\frac{1}{(q^2-\gamma_0^2)^2}\,d^3q = \\
&= \frac{-\lambda'}{p^2-\gamma_0^2}\,\frac{i\Gamma(1+\frac12+2-1)}{4\Gamma(1+\frac12-1)\Gamma(1+2-1)}\int_0^1 y^{\frac12}(1-y)dy\,\times \\
&\times \int \frac{d^3q}{[q^2-2qpy+p^2y-(1-y)\gamma_0^2]^{2+\frac12}} = \\
&= \frac{-\lambda'}{p^2-\gamma_0^2}\,\frac{i\Gamma(\frac12+2)}{4\Gamma(\frac12)\Gamma(2)}\int_0^1 \frac{i\pi^{\frac32}\Gamma(\frac52-\frac32)y^{\frac12}(1-y)dy}{\Gamma(\frac52)(p^2y(1-y)-(1-y)\gamma_0^2)} = \\
&= \frac{\lambda'}{p^2-\gamma_0^2}\,\frac{\pi^{\frac32}}{4\Gamma(\frac12)}\int_0^1 y^{\frac12}dy\,\frac{1}{(p^2y-\gamma_0^2)} = \\
&= \frac{\lambda'\pi}{p^2-\gamma_0^2}\,\frac{1}{4|p|\gamma_0}\log\left|\frac{|p|-\gamma_0}{|p|+\gamma_0}\right| = \\
&= \frac{\pi}{p^2-\gamma_0^2}\,\frac{2\gamma_0}{\pi^2}\,\frac{1}{4|p|\gamma_0}\log\left|\frac{|p|-\gamma_0}{|p|+\gamma_0}\right| = \\
&= \frac{1}{2\pi(p^2-\gamma_0^2)|p|}\log\left|\frac{|p|-\gamma_0}{|p|+\gamma_0}\right|,
\end{aligned} \tag{218}$$

where $|p| = \sqrt{p^2}$.

Thus we have the ground state $\phi(p) = \phi_0(p) + \alpha\phi_1(p)$ where p denotes an energy-momentum vector with a two dimensional momentum. Thus this ground state is for a two dimensional (momentum) subspace. We may extend it to the ground state of the form $\phi(p) = \phi_0(p) + \alpha\bar\phi_1(p)$ where p denotes a four dimensional energy-momentum vector with a three dimensional momentum; and due to the special nature that $\phi_1(p)$ is obtained by a two dimensional space statistics the extension $\bar\phi_1(p)$ of $\phi_1(p)$ to with a three dimensional momentum is a wave function obtained by multiplying $\phi_1(p)$ with a delta function concentrating at 0 of a one dimensional momentum variable and the variable p of $\phi_1(p)$ is extended to be a four dimensional energy-momentum vector with a three dimensional momentum.

Let us use this form of the ground state $\phi(p) = \phi_0(p) + \alpha\bar\phi_1(p)$ to compute new QED effects in the orthopositronium decay rate where there is a discrepancy between theoretical result and the experimental result [33–52].

26 New QED effect of orthopositronium decay rate

From the seagull vertex let us find new QED effect to the orthopositronium decay rate where there is a discrepancy between theory and experimental result [33–52]. Let us compute the new one-loop effect of orthopositronium decay rate which is from the seagull vertex potential.

From the seagull vertex potential the positronium ground state is modified from $\phi(p)=\phi_0(p)$ to $\phi(p)=\phi_0(p)+\alpha\bar\phi_1(p)$. Let us apply this form of the ground state of positronium to the computation of the orthopositronium decay rate.

Let us consider the nonrelativistic case. In this case we have $\phi_0(\mathbf{p}) = \frac{1}{(\mathbf{p}^2+\gamma_0^2)^2}$ and:

$$\phi_1(\mathbf{p}) = \frac{-1}{2\pi(\mathbf{p}^2+\gamma_0^2)|\mathbf{p}|}\log\left|\frac{|\mathbf{p}|-\gamma_0}{|\mathbf{p}|+\gamma_0}\right|. \tag{219}$$

Let M denotes the decay amplitude. Let M_0 denotes the zero-loop decay amplitude. Then following the approach in the computation of the positronium decay rate [33–52] the first order decay rate Γ is given by:

$$\begin{aligned}
\int 8\pi^{\frac12}\gamma_0^{\frac52}\left[\phi_0(\mathbf{p}) + \alpha\bar\phi_1(\mathbf{p})\right]M_0(\mathbf{p})d^3\mathbf{p} &=: \\
&=: \Gamma_0 + \alpha\Gamma_{seagull}\,,
\end{aligned} \tag{220}$$

where $8\pi^{\frac12}\gamma_0^{\frac52}$ is the normalized constant for the usual unnormalized ground state wave function ϕ_0 [33–52].

We have that the first order decay rate Γ_0 is given by [33–52]:

$$\begin{aligned}
\Gamma_0 &:= \frac{1}{(2\pi)^3}\int 8\pi^{\frac12}\gamma_0^{\frac52}\phi_0(\mathbf{p})M_0(\mathbf{p})d^3\mathbf{p} = \\
&= \frac{1}{(2\pi)^3}\int \frac{8\pi^{\frac12}\gamma_0^{\frac52}}{(\mathbf{p}^2+\gamma_0^2)^2}M_0(\mathbf{p})d^3\mathbf{p} \approx \\
&\approx \psi_0(\mathbf{r}=0)M_0(0) = \\
&= \frac{8\pi^{\frac12}\gamma_0^{\frac52}}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{(\mathbf{p}^2+\gamma_0^2)^2}M_0(0) = \\
&= \frac{8\pi^{\frac12}\gamma_0^{\frac52}}{(2\pi)^3}\,\frac{\pi^2}{\gamma_0}M_0(0) = \\
&= \frac{1}{(\pi a^3)^{\frac12}}\,M_0(0)\,,
\end{aligned} \tag{221}$$

where $\psi_0(\mathbf{r})$ denotes the usual nonrelativistic ground state wave function of positronium; and $a = \frac{1}{\gamma_0}$ is as the radius of the positronium. In the above equation the step $\approx$ holds since $\phi_0(\mathbf{p}) \to 0$ rapidly as $\mathbf{p} \to \infty$ such that the effect of $M_0(\mathbf{p})$ is small for $\mathbf{p} \neq 0$; as explained in [33]- [52].

Then let us consider the new QED effect of decay rate from $\bar\phi_1(\mathbf{p})$. As the three dimensional space statistics in the Section on the seagull vertex potential we have the following statistics of the decay rate from $\bar\phi_1(\mathbf{p})$:

$$\begin{aligned}
\Gamma_{seagull} &= \frac{1}{(2\pi)^3}\int 8\pi^{\frac12}\gamma_0^{\frac52}\bar\phi_1(\mathbf{p})M_0(\mathbf{p})d^3\mathbf{p} = \\
&= \frac{1}{(2\pi)^3}\int 8\pi^{\frac12}\gamma_0^{\frac52}\phi_1(\mathbf{p})M_0(\mathbf{p})d^2\mathbf{p} \approx \\
&\approx \frac{8\pi^{\frac12}\gamma_0^{\frac52}}{(2\pi)^3}\int \phi_1(\mathbf{p})M_0(0)d^2\mathbf{p} = \\
&= \frac{-8\pi^{\frac12}\gamma_0^{\frac52}}{(2\pi)^3}\int \frac{\log\left|\frac{|\mathbf{p}|-\gamma_0}{|\mathbf{p}|+\gamma_0}\right|}{2\pi(\mathbf{p}^2+\gamma_0^2)|\mathbf{p}|}d^2\mathbf{p}M_0(0) = \\
&= \frac{8\pi^{\frac12}\gamma_0^{\frac52}}{(2\pi)^3 2\pi}\,\frac{\pi^3}{2\gamma_0}M_0(0) = \\
&= \frac{1}{4(\pi a^3)^{\frac12}}M_0(0)\,,
\end{aligned} \tag{222}$$

where the step $\approx$ holds as similar the equation (221) since in the two dimensional integral of $\phi_1(\mathbf{p})$ we have that $\phi_1(\mathbf{p}) \to 0$ as $\mathbf{p} \to \infty$ such that it tends to zero as rapidly as the three dimensional case of $\phi_0(\mathbf{p}) \to 0$.

Thus we have:

$$\alpha\Gamma_{seagull} = \frac{\alpha}{4}\,\Gamma_0\,. \tag{223}$$

From the literature of computation of the orthopositronium decay rate we have that the computed orthopositronium decay rate (up to the order α^2) is given by [33–52]:

$$\Gamma_{\text{o-Ps}} = \Gamma_0\big[1 + A\tfrac{\alpha}{\pi} + \tfrac{\alpha^2}{3}\log\alpha + B(\tfrac{\alpha}{\pi})^2 - \tfrac{\alpha^3}{2\pi}\log^2\alpha\big] = \\ = 7.039934(10)\,\mu\text{s}^{-1}, \tag{224}$$

where $A = -10.286\,606(10)$, $B = 44.52(26)$ and $\Gamma_0 = \frac{9}{2}(\pi^2 - 9)m\alpha^6 = 7.211\,169\,\mu\text{s}^{-1}$.

Then with the additional decay rate from the seagull vertex potential (or from the modified ground state of positronium) we have the following computed orthopositronium decay rate (up to the order α^2):

$$\Gamma_{\text{o-Ps}} + \alpha\Gamma_{seagull} = \\ = \Gamma_0\big[1 + (A + \tfrac{\pi}{4})\tfrac{\alpha}{\pi} + \tfrac{\alpha^2}{3}\log\alpha + B(\tfrac{\alpha}{\pi})^2 - \tfrac{\alpha^3}{2\pi}\log^2\alpha\big] = \\ = 7.039934(10) + 0.01315874\ \mu\text{s}^{-1} = \\ = 7.052092(84)\,\mu\text{s}^{-1}. \tag{225}$$

This agrees with the two Ann Arbor experimental values where the two Ann Arbor experimental values are given by: $\Gamma_{\text{o-Ps}}(\text{Gas}) = 7.0514(14)\,\mu\text{s}^{-1}$ and $\Gamma_{\text{o-Ps}}(\text{Vacuum}) = 7.0482(16)\,\mu\text{s}^{-1}$ [33, 34].

We remark that for the decay rate $\alpha\Gamma_{seagull}$ we have only computed it up to the order α. If we consider the decay rate $\alpha\Gamma_{seagull}$ up to the order α^2 then the decay rate (225) will be reduced since the order α of $\Gamma_{seagull}$ is of negative value.

If we consider only the computed orthopositronium decay rate up to the order α with the term $B(\frac{\alpha}{\pi})^2$ omitted, then $\Gamma_{\text{o-Ps}} = 7.038202\ \mu\text{s}^{-1}$ (see [33–52]) and we have the following computed orthopositronium decay rate:

$$\Gamma_{\text{o-Ps}} + \alpha\Gamma_{seagull} = 7.05136074\,\mu\text{s}^{-1}. \tag{226}$$

This also agrees with the above two Ann Arbor experimental values and is closer to these two experimental values.

On the other hand the Tokyo experimental value given by $\Gamma_{\text{o-Ps}}(\text{Powder}) = 7.0398(29)\,\mu\text{s}^{-1}$ [35] may be interpreted by that in this experiment the QED effect $\Gamma_{seagull}$ of the seagull vertex potential is suppressed due to the special two dimensional statistical form of $\Gamma_{seagull}$ (Thus the additional effect of the modified ground state ϕ of the positronium is suppressed). Thus the value of this experiment agrees with the computational result $\Gamma_{\text{o-Ps}}$. Similarly the experimental result of another Ann Arbor experiment given by $7.0404(8)\,\mu\text{s}^{-1}$ [36] may also be interpreted by that in this experiment the QED effect $\Gamma_{seagull}$ of the seagull vertex potential is suppressed due to the special two dimensional statistical form of $\Gamma_{seagull}$.

27 Graviton constructed from photon

It is well known that Einstein tried to find a theory to unify gravitation and electromagnetism [1, 79, 80]. The search for such a theory has been one of the major research topics in physics [80–88]. Another major research topic in physics is the search for a theory of quantum gravity [89–120]. In fact, these two topics are closely related. In this Section, we propose a theory of quantum gravity that unifies gravitation and electromagnetism.

In the above Sections the photon is as the quantum Wilson loop with the $U(1)$ gauge group for electrodynamics. In the above Sections we have also shown that the corresponding quantum Wilson line can be regarded as the photon propagator in analogy to the usual concept of propagator. In this section from this quantum photon propagator, the quantum graviton propagator and the graviton are constructed. This construction forms the foundation of a theory of quantum gravity that unifies gravitation and electromagnetism.

It is well known that Weyl introduced the gauge concept to unify gravitation and electromagnetism [80]. However this gauge concept of unifying gravitation and electromagnetism was abandoned because of the criticism of the path dependence of the gauge (it is well known that this gauge concept later is important for quantum physics as phase invariance) [1]. In this paper we shall use again Weyl's gauge concept to develop a theory of quantum gravity which unifies gravitation and electromagnetism. We shall show that the difficulty of path dependence of the gauge can be solved in this quantum theory of unifying gravitation and electromagnetism.

Let us consider a differential of the form $g(s)ds$ where $g(s)$ is a field variable to be determined. Let us consider a symmetry of the following form:

$$g(s)ds = g'(s')ds', \tag{227}$$

where s is transformed to s' and $g'(s)$ is a field variable such that (227) holds. From (227) we have a symmetry of the following form:

$$g(s)^* g(s)ds^2 = g'^*(s')g'(s')ds'^2, \tag{228}$$

where $g^*(s)$ and $g'^{*(s)}$ denote the complex conjugate of $g(s)$ and $g'(s)$ respectively. This symmetry can be considered as the symmetry for deriving the gravity since we can write $g(s)^*g(s)ds^2$ into the following metric form for the four dimensional space-time in General Relativity:

$$g(s)^* g(s)ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{229}$$

where we write $ds^2 = a_{\mu\nu} dx^\mu dx^\nu$ for some functions $a_{\mu\nu}$ by introducing the space-time variable x^μ, $\mu = 0, 1, 2, 3$ with x^0 as the time variable; and $g_{\mu\nu} = g(s)^* g(s) a_{\mu\nu}$. Thus from the symmetry (227) we can derive General Relativity.

Let us now determine the variable $g(s)$. Let us consider $g(s) = W(z_0, z(s))$, a quantum Wilson line with $U(1)$ group where z_0 is fixed. When $W(z_0, z(s))$ is the classical Wilson line then it is of path dependence and thus there is a difficulty to use it to define $g(s) = W(z_0, z(s))$. This is also the difficulty of Weyl's gauge theory of unifying gravitation and electromagnetism. Then when $W(z_0, z(s))$ is the quantum Wilson line because of the quantum nature of unspecification of paths we have that $g(s) = W(z_0, z(s))$ is well defined where the whole path of connecting z_0 and $z(s)$ is unspecified (except the two end points z_0 and $z(s)$).

Thus for a given transformation $s' \to s$ and for any (continuous and piecewise smooth) path connecting z_0 and $z(s)$ the resulting quantum Wilson line $W'(z_0, z(s(s')))$ is again of the form $W(z_0, z(s)) = W(z_0, z(s(s')))$. Let $g'(s') = W'(z_0, z(s(s')))\frac{ds}{ds'}$. Then we have:

$$\begin{aligned} & g'^*(s') g'(s') ds'^2 = \\ & = W'^*(z_0, z(s(s'))) W'(z_0, z(s(s'))) (\tfrac{ds}{ds'})^2 ds'^2 = \\ & = W^*(z_0, z(s)) W(z_0, z(s)) (\tfrac{ds}{ds'})^2 ds'^2 = \\ & = g(s)^* g(s) ds^2. \end{aligned} \tag{230}$$

This shows that the quantum Wilson line $W(z_0, z(s))$ can be the field variable for the gravity and thus can be the field variable for quantum gravity since $W(z_0, z(s))$ is a quantum field variable.

Then we consider the operator $W(z_0, z)W(z_0, z)$. From this operator $W(z_0, z)W(z_0, z)$ we can compute the operator $W^*(z_0, z)W(z_0, z)$ which is as the absolute value of this operator. Thus this operator $W(z_0, z)W(z_0, z)$ can be regarded as the quantum graviton propagator while the quantum Wilson line $W(z_0, z)$ is regarded as the quantum photon propagator for the photon field propagating from z_0 to z. Let us then compute this quantum graviton propagator $W(z_0, z)W(z_0, z)$. We have the following formula:

$$\begin{aligned} & W(z, z_0) W(z_0, z) = \\ & \quad = e^{-\hat{t} \log[\pm(z - z_0)]} A e^{\hat{t} \log[\pm(z_0 - z)]}, \end{aligned} \tag{231}$$

where $\hat{t} = -\frac{e_0^2}{k_0}$ for the $U(1)$ group ($k_0 > 0$ is a constant and we may let $k_0 = 1$) where the term $e^{-\hat{t}\log[\pm(z - z_0)]}$ is obtained by solving the first form of the dual form of the KZ equation and the term $e^{\hat{t}\log[\pm(z_0 - z)]}$ is obtained by solving the second form of the dual form of the KZ equation.

Then we change the $W(z, z_0)$ of $W(z, z_0)W(z_0, z)$ in (231) to the second factor $W(z_0, z)$ of $W(z, z_0)W(z_0, z)$ by reversing the proper time direction of the path of connecting z and z_0 for $W(z, z_0)$. This gives the graviton propagator $W(z_0, z)W(z_0, z)$. Then the reversing of the proper time direction of the path of connecting z and z_0 for $W(z, z_0)$ also gives the reversing of the first form of the dual form of the KZ equation to the second form of the dual form of the KZ equation. Thus by solving the second form of dual form of the KZ equation we have that $W(z_0, z)W(z_0, z)$ is given by:

$$\begin{aligned} W(z_0, z) W(z_0, z) &= e^{\hat{t} \log[\pm(z - z_0)]} A e^{\hat{t} \log[\pm(z - z_0)]} = \\ &= e^{2\hat{t} \log[\pm(z - z_0)]} A \,. \end{aligned} \tag{232}$$

In (232) let us define the following constant G:

$$G := -2\hat{t} = 2\,\frac{e_0^2}{k_0}\,. \tag{233}$$

We regard this constant G as the gravitational constant of the law of Newton's gravitation and General Relativity. We notice that from the relation $e_0 = \left(z_A^{\frac{1}{2}}\right)^{-1} e = \frac{1}{n_e} e$ where the renormalization number $n_e = z_A^{\frac{1}{2}}$ is a very large number we have that the bare electric charge e_0 is a very small number. Thus the gravitational constant G given by (233) agrees with the fact that the gravitational constant is a very small constant. This then gives a closed relationship between electromagnetism and gravitation.

We remark that since in (232) the factor $-G \log r_1 = G \log \frac{1}{r_1} < 0$ (where we define $r_1 = |z - z_0|$ and r_1 is restricted such that $r_1 > 1$) is the fundamental solution of the two dimensional Laplace equation we have that this factor (together with the factor $e^{-G\log r_1} = e^{G \log \frac{1}{r_1}}$) is analogous to the fundamental solution $-G\frac{1}{r}$ of the three dimensional Laplace equation for the law of Newton's gravitation. Thus the operator $W(z_0, z)W(z_0, z)$ in (232) can be regarded as the graviton propagator which gives attractive effect when $r_1 > 1$. Thus the graviton propagator (232) gives the same attractive effect of $-G\frac{1}{r}$ for the law of Newton's gravitation.

On the other hand when $r_1 \leqslant 1$ we have that the factor $-G\log r_1 = G\log\frac{1}{r_1} \geqslant 0$. In this case we may consider that this graviton propagator gives repulsive effect. This means that when two particles are very close to each other then the gravitational force can be from attractive to become repulsive. This repulsive effect is a modification of $-G\frac{1}{r}$ for the law of Newton's gravitation for which the attractive force between two particles tends to ∞ when the distance between the two particles tends to 0.

Then by multiplying two masses m_1 and m_2 (obtained from the winding numbers of Wilson loops in (73) of two particles to the graviton propagator (232) we have the following formula:

$$G m_1 m_2 \log \frac{1}{r_1}\,. \tag{234}$$

From this formula (234) by introducing the space variable $\mathbf{x}$ as a statistical variable via the Lorentz metric: $ds^2 =$

$= dt^2 - d\mathbf{x}^2$ we have the following statistical formula which is the potential law of Newton's gravitation:

$$-GM_1M_2\,\frac{1}{r}\,, \tag{235}$$

where M_1 and M_2 denotes the masses of two objects.

We remark that the graviton propagator (232) is for matters. We may by symmetry find a propagator $f(z_0, z)$ of the following form:

$$f(z_0, z) := e^{-2\hat{t}\log[\pm(z-z_0)]}A\,. \tag{236}$$

When $|z - z_0| > 1$ this propagator $f(z_0, z)$ gives repulsive effect between two particles and thus is for anti-matter particles where by the term anti-matter we mean particles with the repulsive effect (236). Then since $|f(z_0, z)| \to \infty$ as $|z - z_0| \to \infty$ we have that two such anti-matter particles can not physically exist. However in the following Section on dark energy and dark matter we shall show the possibility of another repulsive effect among gravitons.

As similar to that the quantum Wilson loop $W(z_0, z_0)$ is as the photon we have that the following double quantum Wilson loop can be regarded as the graviton:

$$W(z_0, z)W(z_0, z)W(z, z_0)W(z, z_0)\,. \tag{237}$$

28 Dark energy and dark matter

By the method of computation of solutions of KZ equations and the computation of the graviton propagator (232) we have that (237) is given by:

$$\begin{aligned} &W(z_0, z)W(z_0, z)W(z, z_0)W(z, z_0) = \\ &= e^{2\hat{t}\log[\pm(z-z_0)]}A_g e^{-2\hat{t}\log[\pm(z-z_0)]} = \\ &= R^{2n}A_g,\; n = 0, \pm1, \pm2, \pm3, \ldots \end{aligned} \tag{238}$$

where A_g denotes the initial operator for the graviton. Thus as similar to the quantization of energy of photons we have the following quantization of energy of gravitons:

$$h\nu = 2\pi e_0^2 n, \quad n = 0, \pm1, \pm2, \pm3, \ldots \tag{239}$$

As similar to that a photon with a specific frequency can be as a magnetic monopole because of its loop nature we have that the graviton (237) with a specific frequency can also be regarded as a magnetic monopole (which is similar to but different from the magnetic monopole of the photon kind) because of its loop nature. (This means that the loop nature gives magnetic property.)

Since we still can not directly observe the graviton in experiments the quantized energies (239) of gravitons can be identified as dark energy. Then as similar to the construction of electrons from photons we construct matter from gravitons by the following formula:

$$W(z_0, z)W(z_0, z)W(z, z_0)W(z, z_0)Z\,, \tag{240}$$

where Z is a complex number as a state acted by the graviton.

Similar to the mechanism of generating mass of electron we have that the mechanism of generating the mass m_d of these particles is given by the following formula:

$$m_d c^2 = 2\pi e_0^2 n_d = \pi G n_d = h\nu_d \tag{241}$$

for some integer n_d and some frequency ν_d.

Since the graviton is not directly observable it is consistent to identify the quantized energies of gravitons as dark energy and to identify the matters (240) constructed by gravitons as dark matter.

It is interesting to consider the quantum gravity effect between two gravitons. When a graviton propagator is connected to a graviton we have that this graviton propagator is extended to contain a closed loop since the graviton is a closed loop. In this case as similar to the quantum photon propagator this extended quantum graviton propagator can give attractive or repulsive effect. Then for stability the extended quantum graviton propagator tends to give the repulsive effect between the two gravitons. Thus the quantum gravity effect among gravitons can be repulsive which gives the diffusion of gravitons and thus gives a diffusion phenomenon of dark energy. Furthermore for stability more and more open-loop graviton propagators in the space form closed loops. Thus more and more gravitons are forming and the repulsive effect of gravitons gives the accelerating expansion of the universe [53–57].

Let us then consider the quantum gravity effect between two particles of dark matter. When a graviton propagator is connected to two particles of dark matter not by connecting to the gravitons acting on the two particles of dark matter we have that the graviton propagator gives only attractive effect between the two particles of dark matter. Thus as similar to the gravitational force among the usual non-dark matters the gravitational force among dark matters are mainly attractive. Then when the graviton propagator is connected to two particles of dark matter by connecting to the gravitons acting on the two particles of dark matter then as the above case of two gravitons we have that the graviton propagator can give attractive or repulsive effect between the two particles of dark matter.

29 Conclusion

In this paper a quantum loop model of photon is established. We show that this loop model is exactly solvable and thus may be considered as a quantum soliton. We show that this nonlinear model of photon has properties of photon and magnetic monopole and thus photon with some specific frequency may be identified with the magnetic monopole. From the discrete winding numbers of this loop model we can derive the quantization property of energy for the Planck's formula of radiation and the quantization property of electric charge. We show that the charge quantization is derived from the energy quantization. On the other hand from the nonlinear model of photon a nonlinear loop model of electron is established. This model of electron has a mass mechanism which generates mass to the electron where the mass of the electron is from the photon-loop. With this mass mechanism for generating mass the Higgs mechanism of the conventional QED theory for generating mass is not necessary.

We derive a QED theory which is not based on the four dimensional space-time but is based on the one dimensional proper time. This QED theory is free of ultraviolet divergences. From this QED theory the quantum loop model of photon is established. In this QED theory the four dimensional space-time is derived for statistics. Using the space-time statistics, we employ Feynman diagrams and Feynman rules to compute the basic QED effects such as the vertex correction, the photon self-energy and the electron self-energy. From these QED effects we compute the anomalous magnetic moment and the Lamb shift. The computation is of simplicity and accuracy and the computational result is better than that of the conventional QED theory in that the computation is simpler and it does not involve numerical approximation as that in the conventional QED theory where the Lamb shift is approximated by numerical means.

From the QED theory in this paper we can also derive a new QED effect which is from the seagull vertex of this QED theory. By this new QED effect and by a reformulated Bethe-Salpeter (BS) equation which resolves the difficulties of the BS equation (such as the existence of abnormal solutions) and gives a modified ground state wave function of the positronium. Then from this modified ground state wave function of the positronium a new QED effect of the orthopositronium decay rate is derived such that the computed orthopositronium decay rate agrees with the experimental decay rate. Thus the *orthopositronium lifetime puzzle* is completely resolved where we also show that the recent resolution of this orthopositronium lifetime puzzle only partially resolves this puzzle due to the special nature of two dimensional space statistics of this new QED effect.

By this quantum loop model of photon a theory of quantum gravity is also established where the graviton is constructed from the photon. Thus this theory of quantum gravity unifies gravitation and electromagnetism. In this unification of gravitation and electromagnetism we show that the universal gravitation constant G is proportional to e_0^2 where e_0 is the bare electric charge which is a very small constant and is related to the renormalized charge e by the formula $e_0 = \frac{1}{n_e} e$ where the renormalized number n_e is a very large winding number of the photon-loop. This relation of G with e_0 (and thus with e) gives a closed relationship between gravitation and electromagnetism. Then since gravitons are not directly observable the quantized energies of gravitons are as dark energy and the particles constructed by gravitons are as dark matter. We show that the quantum gravity effect among particles of dark matter is mainly attractive (and it is possible to be repulsive when a graviton loop is formed in the graviton propagator) while the quantum gravity effect among gravitons can be repulsive which gives the diffusion of gravitons and thus gives the diffusion phenomenon of dark energy and the accelerating expansion of the universe.

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