SPECIAL REPORT

Reconsideration of the Uncertainty Relations and Quantum Measurements

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Discussions on uncertainty relations (UR) and quantum measurements (QMS) persisted until nowadays in publications about quantum mechanics (QM). They originate mainly from the conventional interpretation of UR (CIUR). In the most of the QM literarure, it is underestimated the fact that, over the years, a lot of deficiencies regarding CIUR were signaled. As a rule the alluded deficiencies were remarked disparately and discussed as punctual and non-essential questions. Here we approach an investigation of the mentioned deficiencies collected in a conclusive ensemble. Subsequently we expose a reconsideration of the major problems referring to UR and QMS. We reveal that all the basic presumption of CIUR are troubled by insurmountable deficiencies which require the indubitable failure of CIUR and its necessary abandonment. Therefore the UR must be deprived of their statute of crucial pieces for physics. So, the aboriginal versions of UR appear as being in postures of either (i) thought-experimental fictions or (ii) simple QM formulae and, any other versions of them, have no connection with the QMS. Then the QMS must be viewed as an additional subject comparatively with the usual questions of QM. For a theoretical description of QMS we propose an information-transmission model, in which the quantum observables are considered as random variables. Our approach directs to natural solutions and simplifications for many problems regarding UR and QMS.

1 Introduction

The uncertainty relations (UR) and quantum measurements (QMS) constitute a couple of considerable popularity, frequently regarded as a crucial pieces of quantum mechanics (QM). The respective crucial character is often glorified by assertions like:

(i) UR are expression of "*the most important principle of the twentieth century physics*" [1];

(ii) the description of QMS is "*probably the most important part of the theory (QM)*" [2].

The alluded couple constitute the basis for the so-called Conventional Interpretation of UR (CIUR). Discussions about CIUR are present in a large number of early as well as recent publications (see [1–11] and references therein). Less mentioned is the fact that CIUR ideas are troubled by a number of still unsolved deficiencies. As a rule, in the main stream of CIUR partisan publications, the alluded deficiencies are underestimated (through unnatural solutions or even by omission). Nevertheless, during the years, in scientific literature were recorded remrks such as:

(i) UR "*are probably the most controverted formulae in the whole of the theoretical physics*" [12];

(ii) "*the word ("measurement") has had such a damaging efect on the discussions that… it should be banned altogether in quantum mechanics*" [13];

(iii) "*the idea that there are defects in the foundations of orthodox quantum theory is unquestionable present in the conscience of many physicists*" [14];

(iv) "*Many scientists have considered the conceptual framework of quantum theory to be unsatisfactory. The very foundations of Quantum Mechanics is a matter that needs to be resolved in order to achieve and gain a deep physical understanding of the underlying physical procedures that constitute our world*" [15].

The above mentioned status of things require further studies and probably new views. We believe that a promising strategy to satisfy such requirements is to develop an investigation guided by the following objectives (obj.):

(obj.1) to identify the basic presumptions of CIUR;

(obj.2) to reunite together all the significant deficiencies of CIUR;

(obj.3) to examine the verity and importance of the respective deficiencies;

(obj.4) to see if such an examination defends or incriminate CIUR;

(obj.5) in the latter case to admit the failure of CIUR and its abanonment;

(obj.6) to search for a genuine reinterpretation of UR;

(obj.7) to evaluate the consequences of the UR reinterpretation for QMS;

(obj.8) to promote new views about QMS;

(obj.9) to note a number of remarks on some adjacent questions.

A such guided investigation we are approaching in the next sections of this paper. The present approach try to complete and to improve somewhat less elaborated ideas from few of our previous writings. But, due to a lot of unfortunate chances, and contrary to my desire, the respective writings were edited in modest publications [16–18] or remained as preprints registred in data bases of LANL and CERN libraries (see [19]).

2 Shortly on CIUR history and its basic presumptions

The story of CIUR began with the Heisenberg's seminal work [20] and it starts [21] from the search of general answers to the primary questions (q.):

(q.1) Are all measurements affected by measuring uncertainties?

(q.2) How can the respective uncertainties be described quantitatively?

In connection with the respective questions, in its subsequent extension, CIUR promoted the suppositions (s.):

(s.1) The measuring uncertainties are due to the perturbations of the measured microparticle (system) by its interactions with the measuring instrument;

(s.2) In the case of macroscopic systems the mentioned perturbations can be made arbitrarily small and, consequently, always the corresponding uncertainties can be considered as negligible;

(s.3) On the other hand, in the case of quantum microparticles (of atomic size) the alluded perturbations are essentially unavoidable and consequently for certain measurements (see below) the corresponding uncertainties are non-negligible.

Then CIUR limited its attention only to the quantum cases, for which restored to an amalgamation of the following motivations (m.):

(m.1) Analysis of some thought (gedanken) measuring experiments;

(m.2) Appeal to the theoretical version of UR from the existing QM.

Notification: In the present paper we will use the term "observable" (introduced by CIUR literature) for denoting a physical quantity referring to a considered microparticle (system).

Now let us return to the begining of CIUR history. Firstly [20, 22], for argumentation of the above noted motivation (m.1) were imagined some thought experiments on a quantum microparticle, destined to simultaneous measurements of two (canonically) conjugated observables A and B (such are coordinate q and momentum p or time t and energy E). The corresponding "*thought experimental*" (*te*) uncertainties were noted with $\Delta_{te}A$ and $\Delta_{te}B$. They were found as being interconnected trough the following *te*-UR

$$\Delta_{te}A \cdot \Delta_{te}B \geqslant \hbar\,, \tag{1}$$

where $\hbar$ denotes the reduced Planck constant.

As regard the usage of motivation (m.2) in order to promote CIUR few time later was introduced [23, 24] the so-called Robertson Schrödinger UR (RSUR):

$$\Delta_{\Psi}A \cdot \Delta_{\Psi}B \geqslant \frac{1}{2}\left|\left\langle [\hat{A}, \hat{B}]\right\rangle_{\Psi}\right|. \tag{2}$$

In this relation one finds usual QM notations i.e.: (i) $\hat{A}$ and $\hat{B}$ denote the quantum operators associated with the observables A and B of the same microparticle, (ii) $\Delta_{\Psi}A$ and $\Delta_{\Psi}B$ signify the standard deviation of the respective observables, (iii) $\langle(\ldots)\rangle_{\Psi}$ represents the mean value of $(\ldots)$ in the state described by the wave function Ψ, (iv) $[\hat{A}, \hat{B}]$ depict the commutator of the operators $\hat{A}$ and $\hat{B}$ (for some other details about the QM notations and validity of RSUR (2) see the next section).

CIUR was built by regarding the relations (1) and (2), as standard (reference) elements. It started through the writings (and public lectures) of the so-called Copenhagen School partisans. Later CIUR was adopted, more or less explicitely, in a large number of publications.

An attentive examination of the alluded publications show that in the main CIUR is builded onthe following five basic presumptions (***P***):

***P*₁** : Quantities $\Delta_{te}A$ and $\Delta_{\Psi}A$ from relations (1) and (2) denoted by a unique symbol ΔA, have similar significance of measuring uncertainty for the observable A refering to the same microparticle. Consequently the respective relations have the same generic interpretation as UR regarding the simultaneous measurements of observables A and B of the alluded microparticle;

$\boldsymbol{P_2}$: In case of a solitary observable A, for a microparticle, the quantity ΔA can have always an unbounded small value. Therefore such an obvservable can be measured without uncertainty in all cases of microparticles (systems) and states;

$\boldsymbol{P_3}$: When two observables A and B are commutable (i.e $[\hat{A},\, \hat{B}] = 0$) relation (2) allows for the quantities ΔA and ΔB, regarding the same microparticle, to be unlimitedly small at the same time. That is why such observables can be measured simultaneously and without uncertainties for any microparticle (system) or state. Therefore they are considered as compatible;

$\boldsymbol{P_4}$: If two observables A and B are non-commutable (i.e. $[\hat{A},\, \hat{B}] \neq 0$) relation (2) shows that, for a given microparticle, the quantities ΔA and ΔB can be never reduced concomitantly to null values. For that reason such observables can be measured simultaneously only with non-null and interconnected uncertainties, irrespective of the microparticle (system) or state. Hence such observables are considered as incompatible;

$\boldsymbol{P_5}$: Relations (1) and (2), Planck's constant $\hbar$ as well as the measuring peculiarities noted in $\boldsymbol{P_4}$ are typically QM things which have not analogies in classical (non-quantum) macroscopic physics.

Here it must recorded the fact that, in individual publications from the literature which promote CIUR, the above noted presumptions $\boldsymbol{P_1}$–$\boldsymbol{P_5}$ often appear in non-explicit forms and are mentioned separately or only few of them. Also in the same publications the deficiencies of CIUR are omited or underestimated. On the other hand in writings which tackle the deficiencies of CIUR the respective deficiencies are always discussed as separate pieces not reunited in some elucidative ensembles. So, tacitly, in our days CIUR seems to remain a largely adopted doctrine which dominates the questions regarding the foundation and interpretation of QM.

3 Examination of CIUR deficiencies regarded in an elucidative collection

In oder to evaluate the true significance of deficiences regarding CIUR we think that it must discussed together many such deficiences reunited, for a good examination, in an elucidative collection. Such a kind of discussion we try to present below in this section.

Firstly let us examine the deficiences regarding the relation (1). For such a purpose we note the following remark ($\boldsymbol{R}$):

$\boldsymbol{R_1}$: On the relation (1)
In reality the respective relation is an improper piece for a reference/standard element of a supposed solid doctrine such as CIUR. This fact is due to the to the circumstance that such a relation has a transitory/temporary character because it was founded on old resolution criteria (introduced by Abe and Rayleigh — see [22,25]). But the respective criteria were improved in the so-called super-resolution techniques worked out in modern experimental physics (see [26–31] and references). Then it is possible to imagine some super-resolution-thought-experiments (*srte*). So, for the corresponding *srte*-uncertainties $\Delta_{srte}A$ and $\Delta_{srte}B$ of two observables A and B the following relation can be promoted

$$\Delta_{srte}A \cdot \Delta_{srte}B \leqslant \hbar\,. \tag{3}$$

Such a relation is possibly to replace the CIUR basic formula (1). But the alluded possibility invalidate the presumtion $\boldsymbol{P_1}$ and incriminate CIUR in connection with one of its main points.
End of $\boldsymbol{R_1}$

For an argued examination of CIUR deficiences regarding the relation (2) it is of main importance the following remark:

$\boldsymbol{R_2}$: On the aboriginal QM elements
Let us remind briefly some significant elements, selected from the aboriginal framework of usual QM. So we consider a QM microparticle whose state (of orbital nature) is described by the wave function Ψ. Two observables A_j ($j = 1, 2$) of the respective particle will be described by the operators $\hat{A}_j$. The notation (f, g) will be used for the scalar product of the functions f and g. Correspondingly, the quantities $\langle A_j\rangle_\Psi = (\Psi\,, \hat{A}_j\Psi)$ and $\delta_\Psi \hat{A}_j = \hat{A}_j - \langle \hat{A}_j\rangle_\Psi$ will depict the mean (expected) value respectively the deviation-operator of the observable A_j regarded as a random variable. Then, by denoting the two observable with $A_1 = A$ and $A_2 = B$, we can be write the following Cauchy-Schwarz relation:

$$\begin{aligned}\left(\delta_\Psi\hat{A}\Psi,\, \delta_\Psi\hat{A}\Psi\right)\left(\delta_\Psi\hat{B}\Psi,\, \delta_\Psi\hat{B}\Psi\right) \geqslant \\ \geqslant \left|\left(\delta_\Psi\hat{A}\Psi,\, \delta_\Psi B\Psi\right)\right|^2.\end{aligned} \tag{4}$$

For an observable A_j considered as a random variable the quantity $\Delta_\Psi A_j = \left(\delta_\Psi\hat{A}_j\Psi,\, \delta_\Psi\hat{A}_j\Psi\right)^{\frac{1}{2}}$ signifies its standard deviation. From (4) it results directly that the standard deviations $\Delta_\Psi A$ and $\Delta_\Psi B$ of the mentioned observables satisfy the relation

$$\Delta_\Psi A \cdot \Delta_\Psi B \geqslant \left|\left(\delta_\Psi\hat{A}\Psi,\, \delta_\Psi B\Psi\right)\right|, \tag{5}$$

which can be called *Cauchy-Schwarz formula* (CSF). Note that CSF (5) (as well as the relation (4)) is always valid, i.e. for all observables, paricles and states. Here it is important to specify the fact that the CSF (5) is an aboriginal piece which implies the subsequent and restricted RSUR (1) only in the cases when the operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions

$$\left(\hat{A}_j\Psi, \hat{A}_k\Psi\right) = \left(\Psi, \hat{A}_j\hat{A}_k\Psi\right), \qquad (j, k = 1, 2)\,. \tag{6}$$

Indeed in such cases one can write the relation

$$\begin{aligned}&\left(\delta_\Psi\hat{A}\Psi,\, \delta_\Psi\hat{B}\Psi\right) = \\ &= \tfrac{1}{2}\left(\Psi,\, \left(\delta_\Psi\hat{A}\cdot\delta_\Psi\hat{B}\Psi + \delta_\Psi\hat{B}\cdot\delta_\Psi\hat{A}\right)\Psi\right) - \\ &- \tfrac{i}{2}\left(\Psi,\, i\,[\hat{A},\, \hat{B}]\Psi\right),\end{aligned} \tag{7}$$

where the two terms from the right hand side are purely real and imaginary quantities respectively. Therefore in the mentioned cases from (5) one finds

$$\Delta_\Psi A \cdot \Delta_\Psi B \geqslant \frac{1}{2}\left|\left\langle[\hat{A},\, \hat{B}]\right\rangle_\Psi\right| \tag{8}$$

i.e. the well known RSUR (2).

The above reminded aboriginal QM elements prove the following fact. In reality for a role of standard (reference) piece regarding the interpretation of QM aspects must be considered the CSF (5) but not the RSUR (2). But such a reality

incriminate in an indubitable manner all the basic presumptions $\boldsymbol{P_1}$–$\boldsymbol{P_5}$ of CIUR.
End of $\boldsymbol{R_2}$

The same QM elenments reminded in $\boldsymbol{R_2}$, motivate the next remark:

$\boldsymbol{R_3}$: On a denomination used by CIUR
The denomination "*uncertainty*" used by CIUR for quantities like $\Delta_\Psi A$ from (2) is groundless because of the following considerations. As it was noted previously in the aboriginal QM framework, $\Delta_\Psi A$ signifies the standard deviation of the observable A regarded as a random variable. The mentioned framework deals with theoretical concepts and models about the intrinsic (inner) properties of the considered particle but not with aspects of the measurements performed on the respective particle. Consequently, for a quantum microparticle, the quantity $\Delta_\Psi A$ refers to the intrinsic characteristics (reflected in fluctuations) of the observable A. Moreover it must noted the following realities:

(i) For a particle in a given state the quantity $\Delta_\Psi A$ has a well defined value connected with the corresponding wave function Ψ;

(ii) The value of $\Delta_\Psi A$ is not related with the possible modifications of the accuracy regarding the measurement of the observable A.

The alluded realities are attested by the fact that for the same state of the measured particle (i.e. for the same value of $\Delta_\Psi A$) the measuring uncertainties regarding the observable A can be changed through the improving or worsening of experimental devices/procedures. Note that the above mentioned realities imply and justify the observation [32] that, for two variables x and p of the same particle, the usual CIUR statement "*as Δx approaches zero, Δp becomes infinite and vice versa*" is a doubtful speculation. Finally we can conclude that the ensemble of the things revealed in the present remark contradict the presumptions $\boldsymbol{P_2}$–$\boldsymbol{P_4}$ of CIUR. But such a conclusion must be reported as a serious deficience of CIUR.
End of $\boldsymbol{R_3}$

A class of CIUR conceptual deficiences regards the following pairs of canonically conjugated observables: L_z-φ, N-ϕ and E-t ($L_z = z$ component of angular momentum, $\varphi =$ azimuthal angle, $N =$ number, $\phi =$ phase, $E =$ energy, $t =$ time). The respective pairs were and still are considered as being unconformable with the accepted mathematical rules of QM. Such a fact roused many debates and motivated various approaches planned to elucidate in an acceptable manner the missing conformity (for significant references see below within the remarks $\boldsymbol{R_4}$–$\boldsymbol{R_6}$). But so far such an elucidation was not ratified (or admited unanimously) in the scientific literature. In reality one can prove that, for all the three mentioned pairs of observables, the alluded unconformity refers not to conflicts with aboriginal QM rules but to serious disagreements with RSUR (2). Such proofs and their consequences for CIUR we will discuss below in the following remarks:

$\boldsymbol{R_4}$: On the pair L_z-φ
The parts of above alluded problems regarding of the pair L_z-φ were examined in all of their details in our recent paper [33]. There we have revealed the following indubitable facts:

(i) In reality the pair L_z-φ is unconformable only in respect with the secondary and limited piece which is RSUR (2);

(ii) In a deep analysis, the same pair proves to be in a natural conformity with the true QM rules presented in $\boldsymbol{R_2}$;

(iii) The mentioned conformity regards mainly the CSF (5) which can degenerate in the trivial equality $0 = 0$ in some cases rgarding the pair L_z-φ.

But such facts points out an indubitable deficience of CIUR's basic presumption $\boldsymbol{P_4}$.
End of $\boldsymbol{R_4}$

$\boldsymbol{R_5}$: On the pair N- ϕ
The involvement of pair N-ϕ in debates regarding CIUR started [35] subsequently of the Dirac's idea [36] to transcribe the ladder (lowering and raising) operators $\hat{a}$ and $\hat{a}^+$ in the forms

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{N}}\,, \qquad \hat{a}^+ = \sqrt{\hat{N}}e^{-i\hat{\phi}}. \tag{9}$$

By adopting the relation $[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$ from (9) it follows that the operators $\hat{N}$ and $\hat{\phi}$ satisfy the commutation formula

$$[\hat{N}, \hat{\phi}] = i\,. \tag{10}$$

This relation was associated directly with the RSUR (2) respectively with the presumption $\boldsymbol{P_4}$ of CIUR. The mentioned association guided to the rash impression that the N-ϕ pair satisfy the relation

$$\Delta_\Psi N \cdot \Delta_\Psi \phi \geqslant \frac{1}{2}\,. \tag{11}$$

But, lately, it was found that relation (11) is false — at least in some well-specified situations. Such a situation appears in the case of a quantum oscillator (QO). The mentioned falsity can be pointed out as follows. The Schrödinger equation for a QO stationary state has the form:

$$E\Psi = \frac{1}{2m_0}\,\hat{p}^2\Psi + \frac{1}{2}\,m_0\,\omega^2\hat{x}^2\Psi\,, \tag{12}$$

where m_0 and ω represent the mass and (angular) frequency of QO while $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $x = x\cdot$ denote the operators of the Cartesian moment p and coordinate x. Then the operators $\hat{a}$, $\hat{a}^+$ and $\hat{N}$ have [34] the expressions

$$\hat{a} = \frac{m_0\omega\hat{x} + i\hat{p}}{\sqrt{2m_0\omega\hbar}}\,, \quad \hat{a}^+ = \frac{m_0\omega\hat{x} - i\hat{p}}{\sqrt{2m_0\omega\hbar}}\,, \quad \hat{N} = \hat{a}^+\hat{a}\,. \tag{13}$$

The solution of the equation (12) is an eigenstate wave function of the form

$$\Psi_n(x) = \Psi_n(\xi) \propto \exp\left(-\frac{\xi^2}{2}\right)\mathcal{H}_n(\xi)\,, \qquad (14)$$

where $\xi = x\sqrt{\frac{m_0\omega}{\hbar}}$, while $n = 0, 1, 2, 3, \ldots$ signifies the oscillation quantum number and $\mathcal{H}_n(\xi)$ stand for Hermite polinomials of ξ. The noted solution correspond to the energy eigenvalue $E = E_n = \hbar\omega(n + \frac{1}{2})$ and satisfy the relation $\hat{N}\Psi_n(x) = n \cdot \Psi_n(x)$.

It is easy to see that in a state described by a wave function like (14) one find the results

$$\Delta_\Psi N = 0\,, \quad \Delta_\Psi \phi \leqslant 2\pi\,. \qquad (15)$$

The here noted restriction $\Delta\phi \leqslant 2\pi$ (more exactly $\Delta\phi = \pi/\sqrt{3}$ — see below in (19)) is due to the natural fact that the definition range for ϕ is the interval $[0, 2\pi)$. Through the results (15) one finds a true falsity of the presumed relation (11). Then the harmonization of N-ϕ pair with the CIUR doctrine reaches to a deadlock. For avoiding the mentioned deadlock in many publications were promoted various adjustements regarding the pair N-ϕ (see [35, 37–43] and references therein). But it is easy to observe that all the alluded adjustements are subsequent (and dependent) in respect with the RSUR (2) in the following sense. The respective adjustements consider the alluded RSUR as an absolutely valid formula and try to adjust accordingly the description of the pair N-ϕ for QO. So the operators $\hat{N}$ and $\hat{\phi}$, defined in (9) were replaced by some substitute (*sbs*) operators $\hat{N}_{sbs} = f(\hat{N})$ and $\hat{\phi}_{sbs} = g(\hat{\phi})$, where the functions f and g are introduced through various ad hoc procedures. The so introduced substitute operators $\hat{N}_{sbs}$ and $\hat{\phi}_{sbs}$ pursue to be associated with corresponding standard deviations $\Delta_\Psi N_{sbs}$ and $\Delta_\Psi \phi_{sbs}$ able to satisfy relations resembling more or less with RSUR (2) or with (11). But we appreciate as very doubtful the fact that the afferent "*substitute observables*" N_{sbs} and ϕ_{sbs} can have natural (or even useful) physical significances. Probably that this fact as well as the ad hoc character of the functions f and g constitute the reasons for which until now, in scientific publications, it does not exist a unanimous agreement able to guarantee a genuine elucidation of true status of the N-ϕ pair comparatively with CIUR concepts.

Our opinion is that an elucidation of the mentioned kind can be obtained only through a discussion founded on the aboriginal QM elements presented above in the remark $\boldsymbol{R_2}$. For approaching such a discussion here we add the following supplementary details. For the alluded QO the Schrödinger equation (12) as well as its solution (14) are depicted in a "*coordinate x-representation*". But the same equation and solution can be described in a "*phase ϕ-representation*". By taking into account the relation (10) it results directly that tn the ϕ-representation the operators $\hat{N}$ and $\hat{\phi}$ have the expressions $\hat{N} = i\left(\frac{\partial}{\partial\phi}\right)$ and $\hat{\phi} = \phi\cdot$. In the same representation the Schrödinger equation (12) takes the form

$$E\Psi_n(\phi) = \hbar\omega\left(i\frac{\partial}{\partial\phi} + \frac{1}{2}\right)\Psi_n(\phi) \qquad (16)$$

where $\phi \in [0, 2\pi)$. Then the solution of the above equation is given by the relation

$$\Psi_n(\phi) = \frac{1}{\sqrt{2\pi}}\exp(in\phi) \qquad (17)$$

with $n = \frac{E}{\hbar\omega} - \frac{1}{2}$. If, similarly with te case of a classical oscillator, for a QO the energy E is considered to have non-negative values one finds $n = 0, 1, 2, 3, \ldots$.

Now, for the case of a QO, by taking into account the wave function (17), the operators $\hat{N}$ and $\hat{\phi}$ in the ϕ-representation, as well as the aboriginal QM elements presented in $\boldsymbol{R_2}$, we can note the following things. In the respective case it is verified the relation

$$\left(\hat{N}\Psi_n, \hat{\Phi}\Psi_n\right) = \left(\Psi_n, \hat{N}\hat{\Phi}\Psi_n\right) + i\,. \qquad (18)$$

This relation shows directly the circumstance that in the mentioned case the conditions (6) are not fulfiled by the operators $\hat{N}$ and $\hat{\phi}$ in connection with the wave function (17). But such a circumstance point out the observation that in the case under discussion the RSUR (2)/(8) is not valid. On the other hand one can see that CSF (5) remains true. In fact it take the form of the trivial equality $0 = 0$ because in the due case one obtains

$$\Delta_\Psi N = 0\,, \;\; \Delta_\Psi \phi = \frac{\pi}{\sqrt{3}}\,, \;\; \left(\delta\hat{N}\Psi_n, \delta\,\hat{\phi}\,\Psi_n\right) = 0\,. \qquad (19)$$

The above revealed facts allow us to note the following conclusions. In case of QO states (described by the wave functions (14) or (17)) the N-ϕ pair is in a complete disagreement with the RSUR (2)/(8) and with the associated basic presumption $\boldsymbol{P_4}$ of CIUR. But, in the alluded case, the same pair is in a full concordance with the aboriginal QM element by the CSF (5). Then it is completely clear that the here noted concclusions reveal an authentic deficience of CIUR.

Observation: Often in CIUR literature the N-ϕ pair is discussed in connection with the situations regarding ensembles of particles (e.g. fuxes of photons). But, in our opinion, such situations are completely different comparatively with the above presented problem about the N-ϕ pair and QO wave functions (states). In the alluded situations the Dirac's notations/formulas (9) can be also used but they must be utilized strictly in connection with the wave functions describing the respective ensembles. Such utilization can offer examples in which the N-ϕ pair satisfy relations which are semblable with RSUR (2) or with the relation (11). But it is less probable that the alluded examples are able to consolidate the CIUR concepts. This because in its primary form CIUR regards on the first place the individual quantum particles but not ensembles of such particles.

End of $\boldsymbol{R_5}$

R_6: On the E-t pair

Another pair of (canonically) conjugated observables which are unconformable in relation with the CIUR ideas is given by energy E and time t. That is why the respective pair was the subject of a large number of (old as well as recent) controversial discussions (see [2, 44–48] and references therein). The alluded discussions were generated by the following observations. On one hand, in conformity with the CIUR tradition, in terms of QM, E and t regarded as conjugated observables, ought to be described by the operators

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \qquad \hat{t} = t\cdot \tag{20}$$

respectively by the commutation relation

$$\left[\hat{E}, \hat{t}\right] = i\hbar. \tag{21}$$

In accordance with the RSUR (2) such a description require the formula

$$\Delta_\Psi E \cdot \Delta_\Psi t \geqslant \frac{\hbar}{2}. \tag{22}$$

On the other hand because in usual QM the time t is a deterministic but not a random variable for any quantum situation (particle/system and state) one finds the expressions

$$\Delta_\Psi E = \text{a finite quantity}, \qquad \Delta_\Psi t \equiv 0. \tag{23}$$

But these expressions invalidate the relation (22) and consequently show an anomaly in respect with the CIUR ideas (especially with the presumption P_4). For avoiding the alluded anomaly CIUR partisans invented a lot of adjusted $\Delta_\Psi E - \Delta_\Psi t$ formulae destined to substitute the questionable relation (22) (see [2, 44–48] and references). The mentioned formulae can be written in the generic form

$$\Delta_v E \cdot \Delta_v t \geqslant \frac{\hbar}{2}. \tag{24}$$

Here $\Delta_v E$ and $\Delta_v t$ have various (v) significances such as:

(i) $\Delta_1 E$ = line-breadth of the spectrum characterizing the decay of an excited state and $\Delta_1 t$ = half-life of the respective state;

(ii) $\Delta_2 E = \hbar\Delta\omega$ = spectral width (in terms of frequency ω) of a wave packet and $\Delta_2 t$ = temporal width of the respective packet;

(iii) $\Delta_3 E = \Delta_\Psi E$ and $\Delta_3 t = \Delta_\Psi A \cdot \left(d\left\langle A\right\rangle_\Psi / dt\right)^{-1}$, with A = an arbitrary observable.

Note that in spite of the efforts and imagination implied in the disputes connected with the formulae (24) the following observations remain of topical interest.

(i) The diverse formulae from the family (24) are not mutually equivalent from a mathematical viewpoint. Moreover they have no natural justification in the framework of usual QM (that however give a huge number of good results in applications);

(ii) In the specific literature (see [2, 44–48] and references therein) none of the formulas (24) is agreed unanimously as a correct substitute for relation (22).

Here it must be added also another observation regarding the E-t pair. Even if the respective pair is considered to be described by the operators (20), in the true QM terms, one finds the relation

$$\left(\hat{E}\Psi,\, \hat{t}\Psi\right) = \left(\Psi,\, \hat{E}\,\hat{t}\Psi\right) - i\hbar. \tag{25}$$

This relation shows clearly that for the E-t pair the condition (6) is never satisfied. That is why for the respective pair the RSUR (2)/(8) is not applicable at all. Nevertheless for the same pair, described by the operators (20), the CSF (5) is always true. But because in QM the time t is a deterministic (i.e. non-random) variable in all cases the mentioned CSF degenerates into the trivial equality $0 = 0$.

Due to the above noted observations we can conclude that the applicability of the CIUR ideas to the E-t pair persists in our days as a still unsolved question. Moreover it seems to be most probabble the fact that the respective question can not be solved naturally in accordance with the authentic and aboriginal QM procedures. But such a fact must be reported as a true and serious deficience of CIUR.

End of R_6

In the above remarks R_1–R_6 we have approached few facts which through detailed examinations reveal indubitable deficiences of CIUR.The respective facts are somewhat known due to their relative presence in the published debates. But there are a number of other less known things which poit out also deficiences of CIUR. As a rule, in publications, the respective things are either ignored or mentioned with very rare occasions. Now we attempt to re-examine the mentioned things in a spirit similar with the one promoted in the remarks R_1–R_6 from the upper part of this section. The announced reexamination is given below in the next remarks.

R_7: On the commutable observables

For commutable observables CIUR adopt the presumtion P_3 because the right hand side term from RSUR (2) is a null quantity. But as we have shown in remark R_2 the respective RSUR is only a limited by-product of the general relation which is the CSF (5). However by means of the alluded CSF one can find examples where two commutable observable A and B can have simultaneously non-null values for their standard deviations ΔA and ΔB.

An example of the mentioned kind is given by the cartesian momenta p_x and p_y for a particle in a 2D potential well. The observables p_x and p_y are commutable because $[\hat{p_x}, \hat{p_y}] = 0$. The well is delimited as follows: the potential energy V is null for $0 < x_1 < a$ and $0 < y_1 < b$ respectively $V = \infty$ otherwise, where $0 < a < b$, $x_1 = \frac{(x+y)}{\sqrt{2}}$ and $y_1 = \frac{(y-x)}{\sqrt{2}}$. Then for the particle in the lowest energetic state one finds

$$\Delta_\Psi p_x = \Delta_\Psi p_y = \hbar\, \frac{\pi}{ab} \sqrt{\frac{a^2+b^2}{2}}\,, \tag{26}$$

$$\left|\left\langle\left(\delta_\Psi \hat{p}_x \Psi,\, \delta_\Psi \hat{p}_y \Psi\right)\right\rangle\right| = \left(\frac{\hbar\pi}{ab}\right)^2 \cdot \left(\frac{b^2-a^2}{2}\right). \tag{27}$$

With these expressions it results directly that for the considered example the momenta p_x and p_y satisfy the CSF (5) in a non-trivial form (i.e. as an inequality with a non-null value for the right hand side term).

The above noted observations about commutable observables constitute a fact that conflicts with the basic presumption $\boldsymbol{P_3}$ of CIUR. Consequently such a fact must be reported as an element which incriminates the CIUR doctrine.

End of $\boldsymbol{R_7}$

$\boldsymbol{R_8}$: On the eigenstates

The RSUR (2) fails in the case when the wave function Ψ describes an eigenstate of one of the operators $\hat{A}$ or $\hat{B}$. The fact was mentioned in [49] but it seems to remain unremarked in the subsequent publications. In terms of the here developed investigations the alluded failure can be discussed as follows. For two non-commutable observables A and B in an eigenstate of A one obtains the set of values: $\Delta_\Psi A = 0$, $0 < \Delta_\Psi B < \infty$ and $\langle[\hat{A},\, \hat{B}]\rangle_\Psi \neq 0$. But, evidently, the respective values infringe the RSUR(2). Such situations one finds particularly with the pairs L_z-φ in some cases detailed in [33] and N-ϕ in situations presented above in $\boldsymbol{R_5}$.

Now one can see that the question of eigenstates does not engender any problem if the quantities $\Delta_\Psi A$ and $\Delta_\Psi B$ are regarded as QM standard deviations (i.e.characteristics of quantum fluctuations) (see the next Section). Then the mentioned set of values show that in the respective eigenstate A has not fluctuations (i.e. A behaves as a deterministic variable) while B is endowed with fluctuations (i.e. B appears as a random variable). Note also that in the cases of specified eigenstates the RSUR (2) are not valid. This happens because of the fact that in such cases the conditions (6) are not satisfied. The respective fact is proved by the observation that its opposite imply the absurd result

$$a \cdot \langle B\rangle_\Psi = \left\langle [\hat{A},\, \hat{B}]\right\rangle_\Psi + a \cdot \langle B\rangle_\Psi \tag{28}$$

with $\langle[\hat{A},\, \hat{B}]\rangle_\Psi \neq 0$ and $a =$ eigenvalue of $\hat{A}$ (i.e. $\hat{A}\Psi = a\Psi$). But in the cases of the alluded eigenststes the CSF (5) remain valid. It degenerates into the trivial equality $0 = 0$ (because $\delta_\Psi \hat{A}\,\Psi = 0$).

So one finds a contradiction with the basic presumption $\boldsymbol{P_4}$ — i.e. an additional and distinct deficiency of CIUR.

End of $\boldsymbol{R_8}$

$\boldsymbol{R_9}$: On the multi-temporal relations

Now let us note the fact RSUR (2)/(8) as well as its precursor CSF (5) are one-temporal formulas. This because all the quantities implied in the respective formulas refer to the same instant of time. But the mentioned formulas can be generalized into multi-temporal versions, in which the corresponding quantities refer to different instants of time. So CSF (5) is generalizable in the form

$$\Delta_{\Psi_1} A \cdot \Delta_{\Psi_2} B \geqslant \left|\left(\delta_{\Psi_1} \hat{A}\,\Psi_1,\, \delta_{\Psi_2} \hat{B}\Psi_2\,,\right)\right| \tag{29}$$

where Ψ_1 and Ψ_2 represent the wave function for two different instants of time t_1 and t_2. If in (29) one takes $|t_2 - t_1| \to \infty$ in the CIUR vision the quantities $\Delta_{\Psi_1} A$ and $\Delta_{\Psi_2} B$ have to refer to A and B regarded as independent solitary observables. But in such a regard if $\left(\delta_{\Psi_1} \hat{A}\,\Psi_1,\, \delta_{\Psi_2} \hat{B}\,\Psi_2\right) \neq 0$ the relation (29) refute the presumption $\boldsymbol{P_2}$ and so it reveals another additional deficience of CIUR. Note here our opinion that the various attempts [50, 51], of extrapolating the CIUR vision onto the relations of type (29) are nothing but artifacts without any real (physical) justification. We think that the relation (29) does not engender any problem if it is regarded as fluctuations formula (in the sense which will be discussed in the next Section). In such a regard the cases when $\left(\delta_{\Psi_1} \hat{A}\,\Psi_1,\, \delta_{\Psi_2} \hat{B}\,\Psi_2\right) \neq 0$ refer to the situations in which, for the time moments t_1 and t_2, the corresponding fluctuations of A and B are correlated (i.e. statistically dependent).

Now we can say that, the previuosly presented discussion on the multi-temporal relations, disclose in fact a new deficiency of CIUR.

End of $\boldsymbol{R_9}$

$\boldsymbol{R_{10}}$: On the many-observable relations

Mathematically the RSUR (2)/(8) is only a restricted by-product of CSF (5) which follows directly from the two-observable true relation (4). But further one the alluded relation (4) appear to be merely a simple two-observable version of a more general many-observable formula. Such a genaral formula has the form

$$\det\left[\left(\delta_\Psi \hat{A}_j \Psi,\, \delta_\Psi \hat{A}_k \Psi\right)\right] \geqslant 0\,. \tag{30}$$

Here $\det[\alpha_{jk}]$ denotes the determinant with elements α_{jk} and $j = 1, 2, \ldots, r$; $k = 1, 2, \ldots, r$ with $r \geqslant 2$. The formula (30) results from the mathematical fact that the quantities $\left(\delta_\Psi \hat{A}_j \Psi,\, \delta_\Psi \hat{A}_k \Psi\right)$ constitute the elements of a Hermitian and non-negatively defined matrix (an abstract presentation of the mentioned fact can be found in [52]).

Then, within a consistent judgment of the things, for the many-observable relations (30), CIUR must to give an interpretation concordant with its own doctrine (summarized in its basic presumptions $\boldsymbol{P_1}$–$\boldsymbol{P_5}$). Such an interpretation was proposed in [53] but it remained as an unconvincing thing (because of the lack of real physical justifications). Other discussions about the relations of type (30) as in [38] elude any interpretation of the mentioned kind. A recent attempt [54] meant to promote an interpretation of relations like (30), for three or more observables. But the respective attempt has not a helping value for CIUR doctrine. This is because instead of consolidating the CIUR basic presumptions $\boldsymbol{P_1}$–$\boldsymbol{P_5}$) it seems rather to support the idea that the considered relations are fluctuations formulas (in the sense which will be discussed bellow in the next Section). We opine that to find a CIUR-concordant interpretation for the many-observable relations (30) is a difficult (even impossible) task on natural ways (i.e. without esoteric and/or non-physical considerations). An exemplification of the respective difficulty can be appreciated by investigating the case of observables $A_1 = p$, $A_2 = x$ and $A_3 = H$ = energy in the situations described by the wave functions (14) of a QO.

Based on the above noted appreciations we conclude that the impossibility of a natural extension of CIUR doctrine to a interpretation regarding the many-observable relations (30) rveal another deficience of the respective doctrine.

End of $\boldsymbol{R_{10}}$

$\boldsymbol{R_{11}}$: On the quantum-classical probabilistic similarity

Now let us call attention on a quantum-classical similarity which directly contradicts the presumption $\boldsymbol{P_5}$ of CIUR. The respective similarity is of probabilistic essence and regards directly the RSUR (2)/(8) as descendant from the CSF (5). Indeed the mentioned CSF is completely analogous with certain two-observable formula from classical (phenomenolgical) theory of fluctuations for thermodynamic quantities. The alluded classical formula can be written [55, 56] as follows

$$\Delta_w \mathbb{A} \cdot \Delta_w \mathbb{B} \geqslant \left| \langle \delta_w \mathbb{A} \cdot \delta_w \mathbb{B} \rangle_w \right| . \tag{31}$$

In this formula $\mathbb{A}$ and $\mathbb{B}$ signify two classical global observables which characterize a thermodynamic system in its wholeness. In the same formula w denotes the phenomenological probability distribution, $\langle (\ldots) \rangle_w$ represents the mean (expected value) of the quantity $(\ldots)$ evaluated by means of w while $\Delta_w \mathbb{A}$, $\Delta_w \mathbb{B}$ and $\langle \delta_w \mathbb{A} \cdot \delta_w \mathbb{B} \rangle_w$ stand for characteristics (standard deviations respectively correlation) regarding the fluctuations of the mentioned observables. We remind the appreciation that in classical physics the alluded characteristics and, consequently, the relations (31) describe the intrinsic (own) properties of thermodynamic systems but not the aspects of measurements performed on the respective systems. Such an appreciation is legitimated for example by the research regarding the fluctuation spectroscopy [57] where the properties of macroscopic (thermodynamic) systems are evaluated through the (spectral components of) characteristics like $\Delta_w \mathbb{A}$ and $\langle \delta_w \mathbb{A} \cdot \delta_w \mathbb{B} \rangle_w$.

The above discussions disclose the groundlessness of idea [58–60] that the relations like (31) have to be regarded as a sign of a macroscopic/classical complementarity (similar with the quantum complementarity motivated by CIUR presumption $\boldsymbol{P_4}$). According to the respective idea the quantities $\Delta_w \mathbb{A}$ and $\Delta_w \mathbb{B}$ appear as macroscopic uncertainties. Note that the mentioned idea was criticized partially in [61, 62] but without any explicit specification that the quantities $\Delta_w \mathbb{A}$ and $\Delta_w \mathbb{B}$ are quatities which characterise the macroscopic fluctuations.

The previously notified quantum-classical similarity together with the reminded significance of the quantities implied in (31) suggests and consolidates the following regard (argued also in $\boldsymbol{R_3}$). The quantities $\Delta_\Psi A$ and $\Delta_\Psi B$ from RSUR (2)/(8) as well as from CSF (5) must be regarded as describing intrinsic properties (fluctuations) of quantum observables A and B but not as uncertainties of such observables.

Now, in conclusion, one can say that the existence of classical relations (31) contravenes to both presumptions $\boldsymbol{P_1}$ and $\boldsymbol{P_5}$ of CIUR. Of course that such a conclusion must be announced as a clear deficience of CIUR.

End of $\boldsymbol{R_{11}}$

$\boldsymbol{R_{12}}$: On the higher order fluctuations moments

In classical physics the fluctuations of thermodynamic observables $\mathbb{A}$ and $\mathbb{B}$ implied in (31) are described not only by the second order probabilistic moments like $\Delta_w \mathbb{A}$, $\Delta_w \mathbb{B}$ or $\langle \delta_w \mathbb{A}\, \delta_w \mathbb{B} \rangle_w$. For a better evaluation the respective fluctuations are characterized additionally [63] by higher order moments like $\left\langle (\delta_w \mathbb{A})^r \, (\delta_w \mathbb{B})^s \right\rangle_w$ with $r + s \geqslant 3$. This fact suggests the observation that, in the context considered by CIUR, we also have to use the quantum higher order probabilistic moments like $\left(\left(\delta_\Psi \hat{A}_j\right)^r \Psi, \left(\delta_\Psi \hat{A}_k\right)^s \Psi \right)$, $r + s \geqslant 3$. Then for the respective quantum higher order moments CIUR is obliged to offer an interpretation compatible with its own doctrine. But it seems to be improbable that such an interpretation can be promoted through credible (and natural) arguments resulting from the CIUR own presumptions.

That improbability reveal one more deficience of CIUR.

End of $\boldsymbol{R_{12}}$

$\boldsymbol{R_{13}}$: On the so-called "macroscopic operators"

Another obscure aspect of CIUR was pointed out in connection with the question of the so called "macroscopic operators". The question was debated many years ago (see [64, 65] and references) and it seems to be ignored in the lsat decades, although until now it was not elucidated. The question appeared due to a forced transfer of RSUR (2) for the cases of quantum statistical systems. Through such a transfer CIUR partisans promoted the formula

$$\Delta_\rho \mathbb{A} \cdot \Delta_\rho \mathbb{B} \geqslant \frac{1}{2} \left| \left\langle [\hat{\mathbb{A}}, \hat{\mathbb{B}}] \right\rangle_\rho \right| . \tag{32}$$

This formula refers to a quantum statistical system in a state described by the statistical operator (density matrix) $\hat{\rho}$.

With $\mathbb{A}$ and $\mathbb{B}$ are denoted two macroscopic (global) observables associated with the operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$. The quantity

$$\Delta_\rho \mathbb{A} = \left\{ \mathrm{Tr}\left[\left(\hat{\mathbb{A}} - \langle \mathbb{A} \rangle_\rho \right)^2 \right] \right\}^{\frac{1}{2}}$$

denotes the standard deviation of the macroscopic observable $\mathbb{A}$ regarded as a (generalised) random variable. In its expression the respective quantity imply the notation $\langle \mathbb{A} \rangle_\rho = \mathrm{Tr}\left(\hat{\mathbb{A}} \hat{\rho}\right)$

for the mean (expected) value of the macroscopic observable $\mathbb{A}$.

Relation (32) entailed discussions because of the conflict between the following two findings:

(i) On the one hand (32) is introduced by analogy with RSUR (2) on which CIUR is founded. Then, by extrapolating CIUR, the quantities $\Delta_\rho \mathbb{A}$ and $\Delta_\rho \mathbb{B}$ from (32) should be interpreted as (global) uncertainties subjected to stipulations as the ones indicated in the basic presumption $\boldsymbol{P_1}$;

(ii) On the other hand, in the spirit of the presumption $\boldsymbol{P_5}$, CIUR agrees the posibility that macroscopic observables can be measured without any uncertainty (i.e. with unbounded accuracy). For an observable the mentioned possibility should be independent of the fact that it is measured solitarily or simultaneously with other observables. Thus, for two macroscopic (thermodynamic) observables, it is senselessly to accept CIUR basic presumptions $\boldsymbol{P_3}$ and $\boldsymbol{P_4}$.

In order to elude the mentioned conflict it was promoted the idea to abrogate the formula (32) and to replace it with an adjusted macroscopic relation concordant with CIUR vision. For such a purpose the global operators $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ from (32) were substituted [64, 65] by the so-called "macroscopic operators" $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. The respective "macroscopic operators" are considered to be representable as quasi-diagonal matrices (i.e. as matrices with non-null elements only in a "microscopic neighbourhood" of principal diagonal). Then one supposes that $\left[\hat{\mathcal{A}}, \hat{\mathcal{B}}\right] = 0$ for any pairs of "macroscopic observables" $\mathbb{A}$ and $\mathbb{B}$. Consequently instead of (32) it was introduced the formula

$$\Delta_\rho \mathcal{A} \cdot \Delta_\rho \mathcal{B} \geqslant 0\,. \tag{33}$$

In this formula CIUR partisans see the fact that the uncertainties $\Delta_\rho \mathcal{A}$ and $\Delta_\rho \mathcal{B}$ can be unboundedly small at the same time moment, for any pair of observables $\mathbb{A}$ and $\mathbb{B}$ and for any system. Such a fact constitute the CIUR vision about macroscopic observables. Today it seems to be accepted the belief that mentioned vision solves all the troubles of CIUR caused by the formula (32).

A first disapproval of the mentioned belief results from the following observations:

(i) Relation (32) cannot be abrogated if the entire mathematical apparatus of quantum statistical physics is not abrogated too. More exactly, the substitution of operators from the usual global version $\hat{\mathbb{A}}$ into a "macroscopic" variant $\hat{\mathcal{A}}$ is a senseless invention as long as in practical procedures of quantum statistical physics [66, 67] for lucrative operators one uses $\hat{\mathbb{A}}$ but not $\hat{\mathcal{A}}$;

(ii) The substitution $\hat{\mathbb{A}} \to \hat{\mathcal{A}}$ does not metamorphose automatically (32) into (33), because if two operators are quasi-diagonal, in sense required by the partisans of CIUR, it is not surely that they commute.

For an ilustration of the last observatiom we quote [68] the Cartesian components of the global magnetization $\vec{\mathbb{M}}$ of a paramagnetic system formed of N independent $\frac{1}{2}$-spins. The alluded components are described by the global operators

$$\hat{\mathbb{M}}_\alpha = \frac{\gamma\hbar}{2}\,\hat{\sigma}_\alpha^{(1)} \oplus \frac{\gamma\hbar}{2}\,\hat{\sigma}_\alpha^{(2)} \oplus \cdots \oplus \frac{\gamma\hbar}{2}\,\hat{\sigma}_\alpha^{(N)}\,, \tag{34}$$

where $\alpha = x, y, z$; γ = magneto-mechanical factor and $\hat{\sigma}_\alpha^{(i)}$ = Pauli matrices associated to the i-th spin (particle). Note that the operators (34) are quasi-diagonal in the sense required by CIUR partisans, i.e. $\hat{\mathbb{M}}_\alpha \equiv \hat{\mathcal{M}}_\alpha$. But, for all that, they do not commute because $\left[\hat{\mathcal{M}}_\alpha, \hat{\mathcal{M}}_\beta\right] = i\hbar\gamma \cdot \varepsilon_{\alpha\beta\mu} \cdot \hat{\mathcal{M}}_\mu$ ($\varepsilon_{\alpha\beta\mu}$ denote the Levi-Civita tensor).

A second disproval of the belief induced by the substitution $\hat{\mathbb{A}} \to \hat{\mathcal{A}}$ is evidenced if the relation (32) is regarded in an ab original QM approach like the one presented in $\boldsymbol{R_2}$. In such regard it is easy to see that in fact the formula (32) is only a restrictive descendant from the generally valid relation

$$\Delta_\rho \mathbb{A} \cdot \Delta_\rho \mathbb{B} \geqslant \left|\left\langle \delta_\rho \mathbb{A} \cdot \delta_\rho \mathbb{B} \right\rangle_\rho\right|, \tag{35}$$

where $\delta_\rho \hat{\mathbb{A}} = \hat{\mathbb{A}} - \langle \mathbb{A} \rangle_\rho$. In the same regard for the "macroscopic operators" $\mathcal{A}$ and $\mathcal{B}$ instead of the restricted relation (33) it must considered the more general formula

$$\Delta_\rho \mathcal{A} \cdot \Delta_\rho \mathcal{B} \geqslant \left|\left\langle \delta_\rho \mathcal{A} \cdot \delta_\rho \mathcal{B} \right\rangle_\rho\right|. \tag{36}$$

The above last two relations justify the following affirmations:

(i) Even in the situations when $\left[\hat{\mathcal{A}}, \hat{\mathcal{B}}\right] = 0$ the product $\Delta_\rho \mathcal{A} \cdot \Delta_\rho \mathcal{B}$ can be lower bounded by a non-null quantity. This happens because it is possible to find cases in which the term from the right hand side of (36) has a non-null value;

(ii) In fact the substitution $\hat{\mathbb{A}} \to \hat{\mathcal{A}}$ replace (35) with (36). But for all that the alluded replacement does not guaranttee the validity of the relation (33) and of the corresponding speculations.

The just presented facts warrant the conclusion that the relation (32) reveal a real deficiency of CIUR. The respective deficiency cannot be avoided by resorting to the so-called "macroscopic operators". But note that the same relation does not rise any problem if it is considered together with (35) as formulas which refer to the fluctuations of macroscopic (global) observables regarding thermodynamic systems.

End of $\boldsymbol{R_{13}}$

$\boldsymbol{R_{14}}$: On the similarities between calassical Boltzmann's and quantum Planck's constants k_B and $\hbar$

The quantum-classical similarity revealed in $\boldsymbol{R_{11}}$ entails also a proof against the CIUR presumption $\boldsymbol{P_5}$. According to the respective presumptions the Planck constant $\hbar$ has no analog in classical (non-quantum) physics. The announced proof can be pointed out as follows.

The here discussed similarity regards the groups of classical respectively quantum relations (31) and (5) (the last ones including their restricted descendant RSUR (2)/(8)). The respective relations imply the standard deviations $\Delta_w \mathbb{A}$ or $\Delta_\Psi A$ associated with the fluctuations of the corresponding classical and quantum observables. But mathematically the standard deviation indicate the randomness of an observable. This in the sense that the alluded deviation has a positive or null value as the corresponding observable is a random or, alternatively, a deterministic (non-random) variable. Therefore the quantities $\Delta_w \mathbb{A}$ and $\Delta_\Psi A$ can be regarded as similar indicators of randomness for the classical respectively quantum observables.

For diverse cases (of observables, systems and states) the classical standard deviations $\Delta_w \mathbb{A}$ have various expressions in which, apparently, no common element seems to be implied. Nevertheless such an element can be found out [69] as being materialized by the Boltzmann constant k_B. So, in the framework of phenomenological theory of fluctuations (in Gaussian approximation) one obtains [69]

$$(\Delta_w \mathbb{A})^2 = k_\beta \cdot \sum_\alpha \sum_\beta \frac{\partial \bar{\mathbb{A}}}{\partial \bar{\mathbb{X}}_\alpha} \cdot \frac{\partial \bar{\mathbb{A}}}{\partial \bar{\mathbb{X}}_\beta} \cdot \left(\frac{\partial^2 \bar{\mathbb{S}}}{\partial \bar{\mathbb{X}}_\alpha \partial \bar{\mathbb{X}}_\beta} \right)^{-1}. \quad (37)$$

In this relation $\bar{\mathbb{A}} = \langle \mathbb{A} \rangle_w$, $\bar{\mathbb{S}} = \bar{\mathbb{S}}(\bar{\mathbb{X}}_\alpha)$ denotes the entropy of the system written as a function of independent thermodynamic variables $\bar{\mathbb{X}}_\alpha$, $(\alpha = 1, 2, \ldots, r)$ and $(a_{\alpha\beta})^{-1}$ represent the elements for the inverse of matrix $(a_{\alpha\beta})$. Then from (37) it result that the expressions for $(\Delta_w \mathbb{A})^2$ consist of products of k_B with factors which are independent of k_B. The respective independence is evidenced by the fact that the alluded factors must coincide with deterministic (non-random) quantities from usual thermodynamics (where the fluctuations are neglected). Or it is known that such quantities do not imply k_B at all. See [69] for concrete exemplifications of the relations (37) with the above noted properties.

Then, as a first aspect, from (37) it results that the fluctuations characteristics $(\Delta_w \mathbb{A})^2$ (i.e. dispersions = squares of the standard deviations) are directly proportional to k_B and, consequently, they are non-null respectively null quantities as $k_B \neq 0$ or $k_B \to 0$. (Note that because k_B is a physical constant the limit $k_B \to 0$ means that the quantities directly proportional with k_B are negligible comparatively with other quantities of same dimensionality but independent of k_B.) On the other hand, the second aspect (mentioned also above) is the fact that $\Delta_w \mathbb{A}$ are particular indicators of classical randomness. Conjointly the two mentioned aspects show that k_B has the qualities of an authentic generic indicator of thermal randomness which is specific for classical macroscopic (thermodynamic) systems. (Add here the observation that the same quality of k_B can be revealed also [69] if the thermal randomness is studied in the framework of classical statistical mechanics).

Now let us discuss about the quantum randomness whose indicators are the standard deviations $\Delta_\Psi A$. Based on the relations (26) one can say that in many situations the expressions for $(\Delta_\Psi A)^2$ consist in products of Planck constant $\hbar$ with factors which are independent of $\hbar$. (Note that a similar situation can be discovered [33] for the standard deviations of the observables L_z and φ in the case of quantum torsion pendulum.) Then, by analogy with the above discussed classical situations, $\hbar$ places itself in the posture of generic indicator for quantum randomness.

In the mentioned roles as generic indicators k_B and $\hbar$, in direct connections with the quantities $\Delta_w \mathbb{A}$ and $\Delta_\Psi A$, regard the onefold (simple) randomness, of classical and quantum nature respectively. But in physics is also known a twofold (double) randomness, of a combined thermal and quantum nature. Such a kind of randomness one encounters in cases of quantum statistical systems and it is evaluated through the standard deviations $\Delta_\rho \mathbb{A}$ implied in relations (32) and (35). The expressions of the mentioned deviations can be obtained by means of the fluctuation-dissipation theorem [70] and have the form

$$(\Delta_\rho \mathbb{A})^2 = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} \coth\left(\frac{\hbar\omega}{2k_B T} \right) \chi''(\omega)\, d\omega\,. \quad (38)$$

Here $\chi''(\omega)$ denote the imaginary parts of the susceptibility associated with the observable $\mathbb{A}$ and T represents the temperature of the considered system. Note that $\chi''(\omega)$ is a deterministic quantity which appear also in non-stochastic framework of macroscopic physics [71]. That is why $\chi''(\omega)$ is independent of both k_B and $\hbar$. Then from (38) it results that k_B and $\hbar$ considered together appear as a couple of generic indicators for the twofold (double) randomness of thermal and quantum nature. The respective randomness is negligible when $k_B \to 0$ and $\hbar \to 0$ and significant when $k_B \neq 0$ and $\hbar \neq 0$ respectively.

The above discussions about the classical and quantum randomness respectively the limits $k_B \to 0$ and $\hbar \to 0$ must be supplemented with the following specifications.

(i) In the case of the classical randomness it must considered the following fact. In the respective case one associates the limits $k_B \to 0$ respectively "*(classical) microscopic approach*" $\to$ "*(classical) macroscopic approach*". But in this context $k_B \to 0$ is concomitant with the condition $N \to 0$ (N = number of microscopic constituents (molecules) of the considered system). The respective concomitance assures the transformation $k_B N \to \nu R$, i.e. transition of physical quantities from "*microscopic version*" into a "*macroscopic version*" (because R sidnify the macoscopic gass constant and ν denotes the macroscopic amount of substance;

(ii) On the other hand in connection with the quantum case it must taken into account the following aspect. The corresopnding randomness regards the cases of observables of orbital and spin types respectively;

(iii) In the orbital cases the limit $\hbar \to 0$ is usually associated with the quantum $\to$ classical limit. The respective limit implies an unbounded growth of the values of some quantum numbers so as to ensure a correct limit for the associated observables regarding orbital movements. Then one finds [72, 73] that, when $\hbar \to 0$, the orbital-type randomness is in one of the following two situations:

(a) it converts oneself in a classical-type randomness of the corresponding observables (e.g. in the cases of φ and L_z of a torsional pendulum or of x and p of a rectilinear oscillator), or

(b) it disappears, the corresponding observables becoming deterministic classical variables (e.g. in the case of the distance r of the electron in respect with the nucleus in a hydrogen atom);

(iv) The quantum randomness of spin-type regards the spin observables. In the limit $\hbar \to 0$ such observables disappear completely (i.e. they lose both their mean values and the affined fluctuations).

In the alluded posture the Planck constant $\hbar$ has an authentic classical analog represented by the Boltzmann constant k_B. But such an analogy contradicts strongly the presumption $\boldsymbol{P_5}$ and so it reveals a new deficience of CIUR.

End of $\boldsymbol{R_{14}}$

Within this section, throgh the remarks $\boldsymbol{R_1}$–$\boldsymbol{R_{14}}$, we examined a collection of things whose ensemble point out deficiencies which incriminate all the basic presumptions $\boldsymbol{P_1}$–$\boldsymbol{R_5}$ of CIUR, considered as single or grouped pieces. In regard to the truth qualities of the respective deficiences here is the place to note the following completion remark:

$\boldsymbol{R_{15}}$: On the validity of the above signallized CIUR deficiences

The mentioned deficiencies are indubitable and valid facts which can not be surmounted (avoided or rejected) by solid and verisimilar arguments taken from the inner framework of CIUR doctrine.

End of $\boldsymbol{R_{15}}$

4 Consequences of the previous examination

The discussions belonging to the examination from the previous section impose as direct consequences the following remarks:

$\boldsymbol{R_{16}}$: On the indubitable failure of CIUR

In the mentioned circumstances CIUR proves oneself to be indubitably in a failure situation which deprives it of necessary qualities of a valid scientific construction. That is why CIUR must be abandoned as a wrong doctrine which, in fact, has no real value.

End of $\boldsymbol{R_{16}}$

$\boldsymbol{R_{17}}$: On the true significance of the relations (1) **and** (2)

The alluded abandonment has to be completed by a natural re-interpretation of the basic CIUR's relations (1) and (2). We opine that the respective re-interpretation have to be done and argued by taking into account the discussions from the previous Section, mainly those from the remarks $\boldsymbol{R_1}$, $\boldsymbol{R_2}$ and $\boldsymbol{R_3}$. We appreciate that in the alluded re-interpretation must be included the following viewpoints:

(i) On the one hand the relations (1) remain as provisional fictions destitute of durable physical significance;

(ii) On the other hand the relations (2) are simple fluctuations formulae, from the same family with the microscopic and macroscopic relations from the groups (4), (5), (29), (30) respectively (31), (32), (35);

(iii) None of the relations (1) and (2) or their adjustments have not any connection with the description of QMS.

Consequently in fact the relations (1) and (2) must be regarded as pieces of fiction respectively of mathematics without special or extraordinary status/significance for physics.

End of $\boldsymbol{R_{17}}$

$\boldsymbol{R_{18}}$: On the non-influences towards the usual QM

The above noted reconsideration of CIUR does not disturb in some way the framework of usual QM as it is applied concretely in the investigations of quantum microparticles. (Few elements from the respective framework are reminded above in the remark $\boldsymbol{R_2}$).

End of $\boldsymbol{R_{18}}$

5 Some considerations on the quantum and classical measurements

The question regarding the QMS description is one of the most debated subject associated with the CIUR history. It generated a large diversity of viewpoints relatively to its importance and/or approach (see [1–9] and references). The respective diversity inserts even some extreme opinions such are the ones noted in the Section 1 of the present paper. As a notable aspect many of the existing approaches regarding the alluded question are grounded on some views which presume and even try to extend the CIUR doctrine. Such views (v.) are:

(v.1) The descriptions of QMS must be developed as confirmations and extensions of CIUR concepts;

(v.2) The peculiarities of QMS incorporated in CIUR presumptions $\boldsymbol{P_2}$–$\boldsymbol{P_4}$ are connected with the corresponding features of the measuring perturbations. So in the cases of observables refered in $\boldsymbol{P_2}$–$\boldsymbol{P_3}$ respectively in $\boldsymbol{P_4}$ the alluded perturbations are supposed to have an avoidable respectively an unavoidable character;

(v.3) In the case of QMS the mentioned perturbations cause specific jumps in states of the measured quantum microparticles (systems). In many modern texts the respective jumps are suggested to be described as follows. For a quantum observable A of a microparticle in the state Ψ a QMS is assumed to give as result a single value say a_n which is one of the eigenvalues of the associated operator $\hat{A}$. Therefore the description of the respective QMS must include as essential piece a "collapse" (sudden reduction) of the wave function i.e a relation of the form:

$$\Psi\begin{pmatrix}\text{before}\\ \text{measurement}\end{pmatrix} \rightarrow \Psi_n\begin{pmatrix}\text{after}\\ \text{measurement}\end{pmatrix}, \qquad (39)$$

where $\Psi_n\,(\mathtt{after\ measurement})$ denotes the eigenfunction of the operator $\hat{A}$ corresponding to the eigenvalue a_n;

(v.4) With regard to the observables of quantum and classical type respectively the measuring inconveniences (perturbations and uncertainties) show an essential difference. Namely they are unavoidable respectively avoidable characteristics of measurements. The mentioned difference must be taken into account as a main point in the descriptions of the measurements regarding the two types of observables;

(v.5) The description of QMS ought to be incorporated as an inseparable part in the framework of QM. Adequately QM must be considered as a unitary theory both of intrinsic properties of quantum microparticles and of measurements regarding the respective properties.

Here is the place to insert piece-by-piece the next remark:

R_{19}: Counter-arguments to the above views

The above mentioned views about QMS must be appreciated in conformity with the discussions detailed in the previous sections. For such an appreciation we think that it must taken into account the following counter-arguments (c-a):

(c-a.1) According to the remark R_{16}, in fact CIUR is nothing but a wrong doctrine which must be abandoned. Consequently CIUR has to be omitted but not extended in any lucrative scientific question, particularly in the description of QMS. That is why the above view (v.1) is totally groundless;

(c-a.2) The view (v.2) is inspired and argued by the ideas of CIUR about the relations (1) and (2). But, according to the discussions from the previous sections, the respective ideas are completely unfounded. Therefore the alluded view (v.2) is deprived of any necessary and well-grounded justification;

(c-a.3) The view (v.3) is inferred mainly from the belief that the mentioned jumps have an essential importance for QMS.

But the respective belief appears as entirely unjustified if one takes into account the following natural and indubitable observation [74]: "*it seems essential to the notion of measurement that it answers a question about the given situation existing before the measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question*".

Also the same belief apperars as a fictitious thing if we take into account the quantum-classical probabilistic similarity presented in the remark R_{11}. According to the respective similarity, a quantum observable must be regarded mathematically as a random variable.Then a measurement of such a observable must consist not in a single trial (which give a unique value) but in a statistical selection/sampling (which yields a spectrum of values). For more details regarding the measurements of random observables see below in this and in the next sections.

So we can conclude that the view (v.3) is completely unjustified;

(c-a.4) The essence of the difference between classical and quantum observables supposed in view (v.4) is questionable at least because of the following two reasons:

(a) In the classical case the mentioned avoidance of the measuring inconveniences have not a significance of principle but only a relative and limited value (depending on the performances of measuring devices and procedures). Such a fact seems to be well known by experimenters.

(b) In the quantum case until now the alluded unavoidableness cannot be justified by valid arguments of experimental nature (see the above remark R_{16} and the comments regarding the relation (3));

(c-a.5) The viev (v.5) proves to be totally unjustified if the usual conventions of physics are considered. According to the respective conventions, in all the basic chapters of physics, each observable of a system is regarded as a concept "*per se*" (in its essence) which is denuded of measuring aspects. Or QM is nothing but such a basic chapter, like classical mechanics, thermodynamics, electrodynamics or statistical physics. On the other hand in physics the measurements appear as main pourposes for experiments. But note that the study of the experiments has its own problems [75] and is done in frameworks which are additional and distinct in respect with the basic chapters of physics. The above note is consolidated by the observation that [76]: "*the procedures of measurement (comparison with standards) has a part which cannot be described inside the branch of physics where it is used*".

Then, in contrast with the view (v.5), it is natural to accept the idea that QM and the description of QMS have to remain distinct scientific branches. However the two branches have to use some common concepts and symbols. This happens because, in fact, both of them also imply elements regarding the same quantum microparticles (systems).

The here presented counter-arguments contradict all the above oresented views (v.1)–(v.5) promoted in many of the existing approaches regarding the QMS description.

End of $\boldsymbol{R_{19}}$

On the basis of discussions presented in $\boldsymbol{R_{11}}$ and reminded in (c-a.3) from $\boldsymbol{R_{19}}$ a quantum observables must be considered as random variables having similar characteristics which corespond to the classical random observables. Then it results that, on principle, the description of QMS can be approached in a manner similar with the one regarding the corresponding classical measurements. That is why below we try to resume a model promoted by us in [77, 78] and destined to describe the measurement of classical random observables.

For the announced resume we consider a classical random observable from the family discussed in $\boldsymbol{R_{11}}$. Such an observable and its associated probability distribution will be depicted with the symbols $\widetilde{\mathbb{A}}$ respectively $w = w(a)$. The individual values a of $\widetilde{\mathbb{A}}$ belong to the spectrum $a \in (-\infty, \infty)$). For the considered situation a measurement preserve the spectrum of $\widetilde{\mathbb{A}}$ but change the dustribution $w(a)$ from a "*in*" (input) version $w_{in}(a)$ into an "*out*" (output) reading $w_{out}(a)$. Note that $w_{in}(a)$ describes the intrinsic properties of the measured system while $w_{out}(a)$ incorporates the information about the same system, but obtained on the recorder of measuring device. Add here the fact that, from a general perspective, the distributions $w_{in}(a)$ and $w_{out}(a)$ incorporate informations referring to the measured system. That is why a measurement appears as an "*informational input $\to$ output transmision process*". Such a process is symbolized by a transformation of the form $w_{in}(a) \to w_{out}(a)$. When the measurement is done by means of a device with stationary and linear characteristics, the the mentioned transformation can described as follows:

$$w_{out}(a) = \int_{-\infty}^{\infty} G(a, a')\, w_{in}(a')\, da' . \tag{40}$$

Here the kernel$G(a, a')$ represents a transfer probability with the significances:

(i) $G(a, a')\, da$ enotes the (infnitesimal) probability that by measurement the *in*-value a' of $\widetilde{\mathbb{A}}$ to be recorded in the *out*-interval $(a; a + da)$;

(ii) $G(a, a')\, da'$ stands for the probability that the *out*-value a to result from the *in*-values which belong to the interval $(a'; a' + da')$.

Due to the mentioned significances the kernel $G(a, a')$ satisfies the conditions

$$\int_{-\infty}^{\infty} G(a,\, a')\, da = \int_{-\infty}^{\infty} G(a,\, a')\, da' = 1\, . \tag{41}$$

Add here the fact that, from a physical perspective, the kernel $G(a, a')$ incorporates the theoretical description of all the characteristics of the measuring device. For an ideal device which ensure $w_{out}(a) = w_{in}(a)$ it must be of the form $G(a, a') = \delta(a - a')$ (with $\delta(a - a')$ denoting the Dirac's function of argument $a - a'$).

By means of $w_\eta(a)$ ($\eta = in, out$) the corresponding global (or numerical) characteristics of $\widetilde{\mathbb{A}}$ regarded as random variable can be introduced. In the spirit of usual practice of physics we refer here only to the two lowest order such characteristics. They are the η — mean (expected) value $\langle \mathbb{A} \rangle_\eta$ and η — standard deviations $\Delta_\eta \mathbb{A}$ defined as follows

$$\left.\begin{aligned} \langle \mathbb{A} \rangle_\eta &= \int_{-\infty}^{\infty} a\, w_\eta(a)\, da \\ (\Delta_\eta \mathbb{A})^2 &= \left\langle \left(\mathbb{A} - \langle \mathbb{A} \rangle_\eta \right)^2 \right\rangle_\eta \end{aligned}\right\} . \tag{42}$$

Now, from the general perspective of the present paper, it is of interest to note some observations about the measuring uncertainties (errors). Firstly it is important to remark that for the discussed observable $\mathbb{A}$, the standard deviations $\Delta_{in}\mathbb{A}$ and $\Delta_{out}\mathbb{A}$ are not estimators of the mentioned uncertainties. Of course that the above remark contradicts some loyalities induced by CIUR doctrine. Here it must be pointed out that:

(i) On the one hand $\Delta_{in}\mathbb{A}$ together with $\langle \mathbb{A} \rangle_{in}$ describe only the intrinsic properties of the measured system;

(ii) On the other hand $\Delta_{out}\mathbb{A}$ and $\langle \mathbb{A} \rangle_{out}$ incorporate composite information about the respective system and the measuring device.

Then, in terms of the above considerations, the measuring uncertainties of $\mathbb{A}$ are described by the following error indicators (characteristics)

$$\left.\begin{aligned} \varepsilon\{\langle \mathbb{A} \rangle\} &= |\langle \mathbb{A} \rangle_{out} - \langle \mathbb{A} \rangle_{in}| \\ \varepsilon\{\Delta\, \mathbb{A}\} &= |\Delta_{out}\mathbb{A} - \Delta_{in}\mathbb{A}| \end{aligned}\right\} . \tag{43}$$

Note that because $\mathbb{A}$ is a random variable for an acceptable evaluation of its measuring uncertainties it is completely insufficient the single indicator $\varepsilon\{\langle \mathbb{A} \rangle\}$. Such an evaluation requires at least the couple $\varepsilon\{\langle \mathbb{A} \rangle\}$and $\varepsilon\{\Delta\, \mathbb{A}\}$ or even the differences of the higher order moments like

$$\varepsilon\left\{\left\langle (\delta \mathbb{A})^n \right\rangle\right\} = \left| \left\langle (\delta_{out}\mathbb{A})^n \right\rangle_{out} - \left\langle (\delta_{in}\mathbb{A})^n \right\rangle_{in} \right| , \tag{44}$$

where $\delta_\eta \mathbb{A} = \widetilde{\mathbb{A}} - \langle \mathbb{A} \rangle_\eta$; $\eta = in,\ out$; $n \geqslant 3$).

Now we wish to specify the fact that the errors (uncertainties) indicators (43) and (44) are *theoretical (predicted) quantities*. This because all the above considerations consist in a theoretical (mathematical) modelling of the discussed measuring process. Or within such a modelling we operate only with theoretical (mathematical) elements presumed to reflect in a plausible manner all the main characteristics of the respective process. On the other hand, comparatively, in experimental physics, the indicators regarding the measuring errors (uncertainties) are *factual entities* because they are estimated on the basis of factual experimental data. But such entities are discussed in the framework of observational error studies.

6 An informational model for theoretical description of QMS

In the above, (c-a.5) from $\boldsymbol{R}_{19}$, we argued forthe idea that QM and the description of QMS have to remain distinct scientific branches which nevertheless have to use some common concepts and symbols. Here we wish to put in a concrete form the respective idea by recommending a reconsidered model for description of QMS. The announced model will assimilate some elements discussed in the previous section in connecton with the measuremens of classical random observables.

We restrict our considerations only to the measurements of quantum observables of orbital nature (i.e. coordinates, momenta, angles, angular momenta and energy). The respective observables are described by the following operators $\hat{A}_j$ $(j = 1, 2, \ldots, n)$ regarded as generalized random variables. As a measured system we consider a spinless microparticle whose state is described by the wave function $\Psi = \Psi(\vec{r})$, taken in the coordinate representation ($\vec{r}$ stand for microparticle position). Add here the fact that, because we consider only a non-relativistic context, the explicit mention of time as an explicit argument in the expression of Ψ is unimportant.

Now note the observation that the wave function $\Psi(\vec{r})$ incorporate information (of probabilistic nature) about the measured system. That is why a QMS can be regarded as a *process of information transmission*: from the measured microparticle (system) to the recorder of the measuring device. Then, on the one hand, the input (in) information described by $\Psi_{in}(\vec{r})$ refers to the intrinsic (own) properties of the respective micropraticle (regarded as information source). The expression of $\Psi_{in}(\vec{r})$ is deducible within the framework of usual QM (e.g. by solving the adequate Schrödinger equation). On the other hand, the output (out) information, described by the wave function $\Psi_{out}(\vec{r})$, refers to the data obtained on the device recorder (regarded as information receiver). So the measuring device plays the role of the transmission channel for the alluded information. Accordingly the measurement appears as a processing information operation. By regarding the things as above the description of the QMS must be associated with the transformation

$$\Psi_{in}(\vec{r}) \rightarrow \Psi_{out}(\vec{r}) . \tag{45}$$

As in the classical model (see the previous section), without any loss of generality, here we suppose that the quantum observables have identical spectra of values in both in- and out-situations. In terms of QM the mentioned supposition means that the operators $\hat{A}_j$ have the same mathematical expressions in both in- and out-readings. The respective expressions are the known ones from the usual QM.

In the framework delimited by the above notifications the description of QMS requires to put the transformation (45) in concrete forms by using some of the known QM rules. Additionally the same description have to assume suggestions from the discussions given in the previous section about measurements of classical random obsevables. That is why, in our opinion, the transformation (45) must be detailed in terms of quantum probabilities carriers. Such carriers are the probabilistic densities ρ_η and currents $\vec{J}_\eta$ defined by

$$\rho_\eta = \left|\Psi_\eta\right|^2 , \qquad \vec{J}_\eta = \frac{\hbar}{m_0} \left|\Psi_\eta\right|^2 \cdot \nabla\Phi_\eta . \tag{46}$$

Here $|\Psi_\eta|$ and Φ_η represents the modulus and the argument of Ψ_η respectively (i.e. $\Psi_\eta = |\Psi_\eta|\, exp(i\Phi_\eta)$) and m_0 denotes the mass of microparticle.

The alluded formulation is connected with the observations [79] that the couple ρ–$\vec{J}$ "*encodes the probability distributions of quantum mechanics*" and it "*is in principle measurable by virtue of its effects on other systems*". To be added here the possibility [80] of taking in QM as primary entity the couple ρ_{in}–$\vec{J}_{in}$ but not the wave function Ψ_{in} (i.e. to start the construction of QM from the continuity equation for the mentioned couple and subsequently to derive the Schrödinger equation for Ψ_{in}).

According to the above observations the transformations (45) have to be formulated in terms of ρ_η and $\vec{J}_\eta$. But ρ_η and $\vec{J}_\eta$ refer to the position and the motion kinds of probabilities respectively. Experimentally the two kinds can be regarded as measurable by distinct devices and procedures. Consequently the mentioned formulation has to combine the following two distinct transformations

$$\rho_{in} \rightarrow \rho_{out} , \qquad \vec{J}_{in} \rightarrow \vec{J}_{out} . \tag{47}$$

The considerations about the classical relation (40) suggest that, by completely similar arguments, the transformations (47) admit the following formulations

$$\rho_{out}(\vec{r}) = \iiint \Gamma(\vec{r}, \vec{r}\,') \; \rho_{in}\left(\vec{r}\,'\right) d^3\vec{r}\,' \tag{48}$$

$$J_{out;\,\alpha} = \sum_{\beta=1}^{3} \iiint \Lambda_{\alpha\beta}(\vec{r}, \vec{r}\,') \; J_{in;\,\beta}(\vec{r}\,') \; d^3\vec{r}\,' . \tag{49}$$

In (49) $J_{\eta;\alpha}$ with $\eta = in, out$ and $\alpha = 1, 2, 3 = x, y, z$ denote Cartesian components of $\vec{J}_\eta$.

Note the fact that the kernels Γ and $\Lambda_{\alpha\beta}$ from (48) and (49) have significance of transfer probabilities, completely analogous with the meaning of the classical kernel $G(a, a')$ from (40). This fact entails the following relations

$$\iiint \Gamma(\vec{r}, \vec{r}\,')\, d^3\vec{r} = \iiint \Gamma(\vec{r}, \vec{r}\,')\, d^3\vec{r}\,' = 1 , \tag{50}$$

$$\sum_{\alpha=1}^{3} \iiint \Lambda_{\alpha\beta}\left(\vec{r}, \vec{r}'\right) \, d^3\vec{r} = \\ = \sum_{\beta=1}^{3} \iiint \Lambda_{\alpha\beta}\left(\vec{r}, \vec{r}'\right) \, d^3\vec{r}' = 1 \, . \tag{51}$$

The kernels Γ and $\Lambda_{\alpha\beta}$ describe the transformations induced by QMS in the data (information) about the measured microparticle. Therefore they incorporate some extra-QM elements regarding the characteristics of measuring devices and procedures. The respective elements do not belong to the usual QM framework which refers to the intrinsic (own) characteristics of the measured microparticle (system).

The above considerations facilitate an evaluation of the effects induced by QMS on the probabilistic estimators of here considered orbital observables A_j. Such observables are described by the operators $\hat{A}_j$ whose expressions depend on $\vec{r}$ and ∇. According to the previous discussions the mentioned operators are supposed to remain invariant under the transformations which describe QMS. So one can say that in the situations associated with the wave functions Ψ_η ($\eta = in, out$) the mentioned observables are described by the following probabilistic estimators/characteristics (of lower order): mean values $\langle A_j \rangle_\eta$, correlations $C_\eta\left(A_j,\, A_k\right)$ and standard deviations $\Delta_\eta A_j$. With the usual notation $(f, g) = \int f^* g \, d^3\vec{r}$ for the scalar product of functions f and g, the mentioned estimators are defined by the relations

$$\left.\begin{aligned} &\langle A_j \rangle_\eta = \left(\Psi_\eta,\, \hat{A}_j \Psi_\eta\right) \\ &\delta_\eta \hat{A}_j = \hat{A}_j - \langle A_j \rangle_\eta \\ &C_\eta\left(A_j,\, A_k\right) = \left(\delta_\eta \hat{A}_j \, \Psi_\eta,\, \delta_\eta \hat{A}_k \, \Psi_\eta\right) \\ &\Delta_\eta A_j = \sqrt{C_\eta\left(A_j,\, A_j\right)} \end{aligned}\right\} . \tag{52}$$

Add here the fact that the in-version of the estimators (52) are calculated by means of the wave function Ψ_{in}, known from the considerations about the inner properties of the investigated system (e.g. by solving the corresponding Schrödinger equation).

On the other hand the out-version of the respective estimators can be evaluated by using the probability density and current ρ_{out} and $\vec{J}_{out}$. So if $\hat{A}_j$ does not depend on ∇ (i.e. $\hat{A}_j = A_j(\vec{r})$) in evaluating the scalar products from (52) one can use the evident equality $\Psi_\eta \hat{A}_j \, \Psi_\eta = \hat{A}_j \, \rho_\eta$. When $\hat{A}_j$ depends on ∇ (i.e. $\hat{A}_j = A_j(\nabla)$) in the same products can be appealed to the substitution

$$\Psi_\eta^* \, \nabla \Psi_\eta = \frac{1}{2} \, \nabla \rho_\eta + \frac{im}{\hbar} \, \vec{J}_\eta \, , \tag{53}$$

$$\Psi_\eta^* \, \nabla^2 \Psi_\eta = \rho_\eta^{\frac{1}{2}} \, \nabla^2 \, \rho_\eta^{\frac{1}{2}} + \frac{im}{\hbar} \, \nabla \vec{J}_\eta - \frac{m^2}{\hbar^2} \, \frac{\vec{J}_\eta^2}{\rho_\eta} \, . \tag{54}$$

The mentioned usage seems to allow the avoidance of the implications regarding [79] "*a possible nonuniqueness of current*" (i.e. of the couple ρ_η–$\vec{J}_\eta$).

Within the above presented model of QMS the errors (uncertainties) associated with the measurements of observables A_j can be evaluated through the following indicators

$$\left.\begin{aligned} &\varepsilon\left\{\langle A_j \rangle\right\} = \left|\langle A_j \rangle_{out} - \langle A_j \rangle_{in}\right| \\ &\varepsilon\left\{C\left(A_j, A_k\right)\right\} = \left|C_{out}\left(A_j, A_k\right) - C_{in}\left(A_j, A_k\right)\right| \\ &\varepsilon\left\{\Delta \, A_j\right\} = \left|\Delta_{out} \, A_j - \Delta_{in} \, A_j\right| \end{aligned}\right\} . \tag{55}$$

These quantum error indicators are entirely similar with the classical ones (43). Of course that, mathematically, they can be completed with error indicators like $\varepsilon\left\{\left(\left(\delta_\Psi \hat{A}_j\right)^r \Psi,\, \left(\delta_\Psi \hat{A}_k\right)^s \Psi\right)\right\}$, $r + s \geqslant 3$, which regard the higher order probabilistic moments mentioned in $\boldsymbol{R_{12}}$.

The above presented model regarding the description of QMS is exemplified in the end of this paper in Annex.

Now is the place to note that the out-version of the estimators (52), as well as the error indicators (55), have a theoretical significance.

In practice the verisimilitude of such estimators and indicators must be tested by comparing them with their *experimental (factual) correspondents* (obtained by sampling and processing of the data collected from the recorder of the measuring device). If the test is confirmative both theoretical descriptions, of QM intrinsic properties of system (microparticle) and of QMS, can be considered as adequate. But if the test gives an invalidation of the results, at least one of the mentioned descriptions must be regarded as inadequate.

In the end of this section we wish to add the following two observations:

(i) The here proposed description of QMS does not imply some interconnection of principle between the measuring uncertainties of two distinct observables. This means that from the perspective of the respective description there are no reasons to discuss about a measuring compatibility or incompatibility of two observables;

(ii) The above considerations from the present section refer to the QMS of orbital observables. Similar considerations can be also done in the case of QMS regarding the spin observables. In such a case besides the probabilities of spin-states (well known in QM publications) it is important to take into account the spin current density (e.g. in the version proposed recently [81]).

7 Some conclusions

We starred the present paper from the ascertained fact that in reality CIUR is troubled by a number of still unsolved deficiencies. For a primary purpose of our text, we resumed the CIUR history and identified its basic presumptions.Then, we attempt to examine in details the main aspects as well as the validity of CIUR deficiencies regarded in an elucidative collection.

The mentioned examination, performed in Section 3 reveal the following aspects:

(i) A group of the CIUR deficiencies appear from the application of usual RSUR (2) in situations where, mathematically, they are incorrect;

(ii) The rest of the deficiencies result from unnatural associations with things of other nature (e.g. with the thought experimental relations or with the presence/ absence of $\hbar$ in some formulas);

(iii) Moreover one finds that, if the mentioned applications and associations are handled correctly, the alluded deficiencies prove themselves as being veridic and unavoidable facts. The ensemble of the respective facts invalidate all the basic presumptions of CIUR.

In consensus with the above noted findings, in Section 4, we promoted the opinion that CIUR must be abandoned as an incorrect and useless (or even misleading) doctrine. Conjointly with the respective opinion we think that the primitive UR (the so called Heisenberg's relations) must be regarded as:

(i) fluctuation formulas — in their theoretical RSUR version (2);

(ii) fictitious things, without any physical significance — in their thought-experimental version (1).

Abandonment of CIUR requires a re-examination of the question regarding QMS theoretical description. To such a requirement we tried to answer in Sections 5 and 6. So, by a detailed investigation, we have shown that the CIUR-connected approaches of QMS are grounded on dubitable (or even incorrect) views.

That is why we consider that the alluded question must be reconsidered by promoting new and more natural models for theoretical description of QMS. Such a model, of somewhat informational concept, is developed in Section 6 and it is exemplified in Annex.

Of course that, as regards the QMS theoretical description, our proposal from Section 6, can be appreciated as only one among other possible models. For example, similarly with the discussions regarding classical errors [77, 78], the QMS errors can be evaluated through the informational (Shannon) entopies.

It is to be expected that, in connection with QMS, other models will be also promoted in the next moths/years. But as a general rule all such models have to take into account the indubitable fact that the usual QM and QMS theoretical description must be refered to distinct scientific questions (objectives).

Annex: A simple exemplification for the model presented in Section 6

For the announced exemplification let us refer to a microparticle in a one-dimensional motion along the x-axis. We take $\Psi_{in}(x) = |\Psi_{in}(x)| \cdot \exp\{i\Phi_{in}(x)\}$ with

$$|\Psi_{in}(x)| \propto \exp\left\{-\frac{(x-x_0)^2}{4\sigma^2}\right\}, \quad \Phi_{in}(x) = kx\,. \tag{56}$$

Note that here as well as in other relations from this Annex we omit an explicit notation of the normalisation constants which can be added easy by the interesed readers.

Correspondingly to the Ψ and Φ from (56) we have

$$\rho_{in}(x) = |\Psi_{in}(x)|^2\,, \quad J_{in}(x) = \frac{\hbar k}{m_0}\,|\Psi_{in}(x)|^2\,. \tag{57}$$

So the intrinsic properties of the microparticle are described by the parameters x_0, σ and k.

If the errors induced by QMS are small the kernels Γ and Λ in (48)–(49) can be considered of Gaussian forms like

$$\Gamma(x,x') \propto \exp\left\{-\frac{(x-x')^2}{2\gamma^2}\right\}, \tag{58}$$

$$\Lambda(x,x') \propto \exp\left\{-\frac{(x-x')^2}{2\lambda^2}\right\}, \tag{59}$$

where γ and λ describe the characteristics of the measuring devices. Then for ρ_{out} and J_{out} one finds

$$\rho_{out}(x) \propto \exp\left\{-\frac{(x-x')^2}{2(\sigma^2+\gamma^2)}\right\}, \tag{60}$$

$$J_{out}(x) \propto \hbar k \cdot \exp\left\{-\frac{(x-x')^2}{2(\sigma^2+\lambda^2)}\right\}. \tag{61}$$

It can been seen that in the case when both $\gamma \to 0$ and $\lambda \to 0$ the kernels $\Gamma(x,x')$ and $\Lambda(x,x')$ degenerate into the Dirac's function $\delta(x-x')$. Then $\rho_{out} \to \rho_{in}$ and $J_{out} \to J_{in}$. Such a case corresponds to an ideal measurement. Alternatively the cases when $\gamma \neq 0$ and/or $\lambda \neq 0$ are associated with non-ideal measurements.

As observables of interest we consider coordinate x and momentum p described by the operators $\hat{x} = x\cdot$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. Then, in the measurement modeled by the expressions (56),(58) and (59), for the errors (uncertainties) of the considered observables one finds

$$\varepsilon\{\langle x\rangle\} = 0\,, \quad \varepsilon\{\langle p\rangle\} = 0\,, \quad \varepsilon\{C(x,p)\} = 0\,, \tag{62}$$

$$\varepsilon\{\Delta x\} = \sqrt{\sigma^2+\gamma^2} - \sigma\,, \tag{63}$$

$$\varepsilon\{\Delta p\} = \hbar \left| \left[\frac{k^2(\sigma^2+\gamma^2)}{\sqrt{(\sigma^2+\lambda^2)(\sigma^2+2\gamma^2-\lambda^2)}} - \right. \right. \\ \left. \left. - k^2 + \frac{1}{4(\sigma^2+\gamma^2)} \right]^{\frac{1}{2}} - k \right| . \tag{64}$$

If in (56) we restrict to the values $x_0 = 0$, $k = 0$ and $\sigma = \sqrt{\frac{\hbar}{2m_0\omega}}$ our system is just a linear oscillator in its ground state (m_0 = mass and ω = angular frequency). This means that the "*in*"-wave function (56) has the same expression with the one from (14) for $n = 0$. As observable of interest we consider the energy described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_0}\frac{d^2}{dx^2} + \frac{m_0\,\omega^2}{2}\,x^2. \tag{65}$$

Then for the respective observable one finds

$$\langle H\rangle_{in} = \frac{\hbar\,\omega}{2}\,, \qquad \Delta_{in}H = 0\,, \tag{66}$$

$$\langle H\rangle_{out} = \frac{\omega\left[\hbar^2 + \left(\hbar + 2m\,\omega\,\gamma^2\right)^2\right]}{4\left(\hbar + 2m_0\,\omega\,\gamma^2\right)}\,, \tag{67}$$

$$\Delta_{out}\,H = \frac{\sqrt{2}\,m\,\omega^2\,\gamma^2\left(\hbar + m\,\omega\,\gamma^2\right)}{\left(\hbar + 2m\,\omega\,\gamma^2\right)}\,. \tag{68}$$

The corresponding errors of mean value resoectively of standard deviation of oscillator energy have the expressions

$$\varepsilon\{\langle H\rangle\} = |\langle H\rangle_{out} - \langle H\rangle_{in}| \neq 0\,, \tag{69}$$

$$\varepsilon\{\Delta H\} = |\Delta_{out}\,H - \Delta_{in}H| \neq 0\,. \tag{70}$$

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