A Method of Successive Approximations in the Framework of the Geometrized Lagrange Formalism

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It is shown that in the weak field approximation the new geometrical approach can lead to the linear field equations for the several independent fields. For the stronger fields and in the second order approximation the field equations become non-linear, and the fields become dependent. This breaks the superposition principle for every separate field and produces the interaction between different fields.The unification of the gravitational and electromagnetic field theories is performed in frames of the geometrical approach in the pseudo-Riemannian space and in the curved Berwald-Moor space.

1 Introduction

In paper [1] the new (geometrical) approach was suggested for the field theory. It is applicable for any Finsler space [2] for which in any point of the main space $x^1, x^2, \ldots, x^n$ the indicatrix volume $\left(V_{ind}(x^1, x^2, \ldots, x^n) \right)_{ev}$ can be defined, provided the tangent space is Euclidean. Then the action I for the fields present in the metric function of the Finsler space is defined within the accuracy of a constant factor as a volume of a certain n-dimensional region V :

$$I = \text{const} \cdot \int_V^{(n)} \frac{dx^1 dx^2 \ldots dx^n}{(V_{ind}(x^1, x^2, \ldots, x^n))_{ev}} . \tag{1}$$

Thus, the field Lagrangian is defined in the following way

$$L = \text{const} \cdot \frac{1}{(V_{ind}(x^1, x^2, \ldots, x^n))_{ev}} . \tag{2}$$

In papers [3, 4] the spaces conformally connected with the Minkowski space and with the Berwald-Moor space were regarded. These spaces have a single scalar field for which the field equation was written and the particular solutions were found for the spherical symmetry and for the rhombododecaedron symmetry of the space.

The present paper is a continuation of those papers dealing with the study and development of the geometric field theory.

2 Pseudo-Riemannian space with the signature $(+---)$

Let us consider the pseudo-Riemannian space with the signature $(+---)$ and select the Minkowski metric tensor $\overset{\circ}{g}_{ij}$ in the metric tensor $g_{ij}(x)$, of this space explicitly

$$g_{ij}(x) = \overset{\circ}{g}_{ij} + h_{ij}(x) . \tag{3}$$

Let us suppose that the field $h_{ij}(x)$ is weak, that is

$$|h_{ij}(x)| \ll 1 . \tag{4}$$

According to [1], the Lagrangian for a pseudo-Riemannian space with the signature $(+---)$ is equal to

$$L = \sqrt{-\det(g_{ij})} . \tag{5}$$

Let us calculate the value of $[-\det(g_{ij})]$ within the accuracy of $|h_{ij}(x)|^2$:

$$-\det(g_{ij}) \simeq 1 + L_1 + L_2 , \tag{6}$$

where

$$L_1 = \overset{\circ}{g}{}^{ij} h_{ij} \equiv h_{00} - h_{11} - h_{22} - h_{33} , \tag{7}$$

$$\begin{aligned} L_2 = &-h_{00}(h_{11} + h_{22} + h_{33}) + h_{11}h_{22} + h_{11}h_{33} + \\ &+ h_{22}h_{33} - h_{12}^2 - h_{13}^2 - h_{23}^2 + h_{03}^2 + h_{02}^2 + h_{01}^2 . \end{aligned} \tag{8}$$

The last formula can be rewritten in a more convenient way

$$\begin{aligned} L_2 = &-\begin{vmatrix} h_{00} & h_{01} \\ h_{01} & h_{11} \end{vmatrix} - \begin{vmatrix} h_{00} & h_{02} \\ h_{02} & h_{22} \end{vmatrix} - \\ &-\begin{vmatrix} h_{00} & h_{03} \\ h_{03} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{vmatrix} + \\ &+\begin{vmatrix} h_{11} & h_{13} \\ h_{13} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{22} & h_{23} \\ h_{23} & h_{33} \end{vmatrix} , \end{aligned} \tag{9}$$

then

$$L \simeq 1 + \frac{1}{2} L_1 + \frac{1}{2} \left[L_2 - \frac{1}{4} L_1^2 \right] . \tag{10}$$

To obtain the field equations in the first order approximation, one should use the Lagrangian L_1, and to do the same in the second order approximation — the Lagrangian $\left(L_1 + L_2 - \frac{1}{4} L_1^2 \right)$.

3 Scalar field

For the single scalar field $\varphi(x)$ the simplest representation of tensor $h_{ij}(x)$ has the form:

$$h_{ij}(x) \equiv h_{ij}^{(\varphi)}(x) = \pm \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} . \tag{11}$$

That is why

$$L_\varphi = \sqrt{-\det(g_{ij})} = \sqrt{1 \pm L_1} \simeq 1 \pm \frac{1}{2} L_1 - \frac{1}{8} L_1^2 , \tag{12}$$

where

$$L_1 = \left(\frac{\partial \varphi}{\partial x^0}\right)^2 - \left(\frac{\partial \varphi}{\partial x^1}\right)^2 - \left(\frac{\partial \varphi}{\partial x^2}\right)^2 - \left(\frac{\partial \varphi}{\partial x^3}\right)^2 . \tag{13}$$

In the first order approximation, we can use the Lagrangian L_1 to obtain the following field equation

$$\frac{\partial^2 \varphi}{\partial x^0 \partial x^0} - \frac{\partial^2 \varphi}{\partial x^1 \partial x^1} - \frac{\partial^2 \varphi}{\partial x^2 \partial x^2} - \frac{\partial^2 \varphi}{\partial x^3 \partial x^3} = 0 , \tag{14}$$

which presents the wave equation. The stationary field that depends only on the radius

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} , \tag{15}$$

will satisfy the equation

$$\frac{d}{dr}\left(r^2 \frac{d\varphi}{dr}\right) = 0 , \tag{16}$$

the integration of which gives

$$\frac{d\varphi}{dr} = -C_1 \frac{1}{r^2} \quad \Rightarrow \quad \varphi(r) = C_0 + C_1 \frac{1}{r} . \tag{17}$$

In the second order approximation one should use the Lagrangian $\left(L_1 - \frac{1}{4} L_1^2\right)$ to obtain the field equation in the second order approximation

$$\overset{\circ}{g}{}^{ij} \frac{\partial}{\partial x^i} \left[\left(\pm 1 - \frac{1}{2} L_1\right) \frac{\partial \varphi}{\partial x^j}\right] = 0 ; \tag{18}$$

this equation is already non-linear.

The strict field equation for the tensor $h_{ij}(x)$ (11) is

$$\overset{\circ}{g}{}^{ij} \frac{\partial}{\partial x^i} \left(\frac{\frac{\partial \varphi}{\partial x^j}}{\sqrt{1 \pm L_1}} \right) = 0 , \tag{19}$$

then the stationary field depending only on the radius must satisfy the equation

$$\frac{d}{dr} \left(r^2 \frac{\frac{d\varphi}{dr}}{\sqrt{1 \mp \left(\frac{d\varphi}{dr}\right)^2}} \right) = 0 . \tag{20}$$

Integrating it, we get

$$\frac{d\varphi}{dr} = - \frac{C_1}{\sqrt{r^4 \pm C_1^2}} \Rightarrow$$

$$\Rightarrow \varphi(r) = C_0 + \int_r^\infty \frac{C_1}{\sqrt{r^4 \pm C_1^2}} \, dr . \tag{21}$$

The field with the upper sign and the field with the lower sign differ qualitatively: the upper sign "+" in Eq. (11) gives a finite field with no singularity in the whole space, the lower sign "−" in Eq. (11) gives a field defined everywhere but for the spherical region

$$0 \leqslant r \leqslant \sqrt{|C_1|} , \tag{22}$$

in which there is no field, while

$$r > \sqrt{|C_1|} , r \to \sqrt{|C_1|} \quad \Rightarrow \quad \frac{d\varphi}{dr} \to -C_1 \cdot \infty . \tag{23}$$

At the same time in the infinity $(r \to \infty)$ both solutions $\varphi_\pm(r)$ behave as the solution of the wave equation Eq. (17).

If we know the Lagrangian, we can write the energy-momentum tensor T^i_j for the these solutions and calculate the energy of the system derived by the light speed c:

$$P_0 = \text{const} \int^{(3)} T^0_0 dV . \tag{24}$$

To obtain the stationary spherically symmetric solutions, we get

$$T^0_0 = - \frac{r^2}{\sqrt{r^4 \pm C_1^2}} , \tag{25}$$

that is why for both upper and lower signs $P_0 \to \infty$.

The metric tensor of Eq. (3,11) is the simplest way to "insert" the gravity field into the Minkowski space — the initial flat space containing no fields. Adding several such terms as in Eq. (11) to the metric tensor, we can describe more and more complicated fields by tensor $h_{ij} = h_{ij}^{(grav)}$.

4 Covariant vector field

To construct the twice covariant symmetric tensor $h_{ij}(x)$ with the help of a covariant field $A_i(x)$ not using the connection objects, pay attention to the fact that the alternated partial derivative of a tensor is a tensor too

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} , \tag{26}$$

but a skew-symmetric one. Let us construct the symmetric tensor on the base of tensor F_{ij}. To do this, first, form a scalar

$$\begin{aligned} L_A &= \overset{\circ}{g}{}^{ij} \, \overset{\circ}{g}{}^{km} F_{ik} F_{jm} = \\ &= 2 \, \overset{\circ}{g}{}^{ij} \, \overset{\circ}{g}{}^{km} \left(\frac{\partial A_k}{\partial x^i} \frac{\partial A_m}{\partial x^j} - \frac{\partial A_k}{\partial x^i} \frac{\partial A_j}{\partial x^m} \right) , \end{aligned} \tag{27}$$

which gives the following expressions for two symmetric tensors

$$h^{(1)}_{ij} = \overset{\circ}{g}{}^{km}\left(2\frac{\partial A_k}{\partial x^i}\frac{\partial A_m}{\partial x^j} - \frac{\partial A_k}{\partial x^i}\frac{\partial A_j}{\partial x^m} - \frac{\partial A_k}{\partial x^j}\frac{\partial A_i}{\partial x^m}\right), \quad (28)$$

$$h^{(2)}_{ij} = \overset{\circ}{g}{}^{km}\left(2\frac{\partial A_i}{\partial x^k}\frac{\partial A_j}{\partial x^m} - \frac{\partial A_i}{\partial x^k}\frac{\partial A_m}{\partial x^j} - \frac{\partial A_j}{\partial x^k}\frac{\partial A_m}{\partial x^i}\right). \quad (29)$$

Notice, that not only F_{ij} and L_A but also the tensors $h^{(1)}_{ij}$, $h^{(2)}_{ij}$ are gradient invariant, that is they don't change with transformations

$$A_i \to A_i + \frac{\partial f(x)}{\partial x^i}, \quad (30)$$

where $f(x)$ is an arbitrary scalar function.

Let

$$h_{ij} \equiv h^{(A_k)}_{ij} = \chi(x)\, h^{(1)}_{ij} + [1 - \chi(x)]\, h^{(2)}_{ij}, \quad (31)$$

where $\chi(x)$ is some scalar function. Then in the first order approximation we get

$$L_1 = 2\, \overset{\circ}{g}{}^{ij}\, \overset{\circ}{g}{}^{km} \left(\frac{A_k}{\partial x^i}\frac{\partial A_j}{\partial x^m}\right) \equiv L_A, \quad (32)$$

and the first order approximation for the field $A_i(x)$ gives Maxwell equations

$$\overset{\circ}{g}{}^{ij}\frac{\partial^2}{\partial x^i \partial x^j} A_k - \frac{\partial}{\partial x^k}\left(\overset{\circ}{g}{}^{ij}\frac{\partial A_j}{\partial x^i}\right) = 0. \quad (33)$$

For Lorentz gauge

$$\overset{\circ}{g}{}^{ij}\frac{\partial A_j}{\partial x^i} = 0, \quad (34)$$

the equations Eqs. (33) take the form

$$\Box A_k = 0. \quad (35)$$

It is possible that Eq. (31) is not the most general form for tensor h_{ij} which in the first order approximation gives the field equations coinciding with Maxwell equations.

To obtain Maxwell equations not for the free field but for the field with sources $j_i(x)$, one should add to $h^{(A_k)}_{ij}$ Eq. (31) the following tensor

$$h^{(j_k)}_{ij} = \left(\frac{16\pi}{c}\right)\cdot\frac{1}{2}\left(A_i j_j + A_j j_i\right). \quad (36)$$

This means that the metric tensor Eq. (3) with tensor

$$h_{ij} = h^{(Max)}_{ij} \equiv h^{(A_k)}_{ij} + h^{(j_k)}_{ij} \quad (37)$$

describes the weak electromagnetic field with source $j_k(x)$. We must bear in mind that we use the geometrical approach to the field theory, and we have to consider $j_k(x)$ to be given and not obtained from the field equations.

So, the metric tensor Eq. (3) with tensor

$$h_{ij} = \mu h^{(A_k)}_{ij} + \gamma h^{(grav)}_{ij}, \quad (38)$$

where μ, γ are the fundamental constants in frames of the unique pseudo-Riemannian geometry describes simultaneously the free electromagnetic field and the free gravitational field. To include the sources, $j_k(x)$, of the electromagnetic field, the metric tensor must either include not only $j_k(x)$ but the partial derivatives of this field too or the field $j_k(x)$ must be expressed by the other fields as shown below.

If the gravity field is "inserted" in the simplest way as shown in the previous section, then the sources of the electromagnetic field can be expressed by the scalar field as follows

$$j_i(x) = q\frac{\partial\varphi}{\partial x^i}. \quad (39)$$

In this case the first order approximation for Lorentz gauge gives

$$\Box A_k = \frac{4\pi}{c} j_k, \quad (40)$$

$$\Box\varphi = 0. \quad (41)$$

Since the density of the current has the form of Eq. (39), the Eq. (41) gives the continuity equation

$$\overset{\circ}{g}{}^{ij}\frac{\partial j_i}{\partial x^j} = 0. \quad (42)$$

5 Several weak fields

The transition from the weak fields to the strong fields may lead to the transition from the linear equations for the independent fields to the non-linear field equations for the mutually dependent interacting fields $\varphi(x)$ and $\psi(x)$ "including" gravity field in the Minkowski space.

Let

$$h_{ij} = \varepsilon_\varphi \frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} + \varepsilon_\psi\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j}, \quad (43)$$

where $\varepsilon_\varphi, \varepsilon_\psi$ are independent sign coefficients. Then the strict Lagrangian can be written as follows

$$L_{\varphi,\psi} = \sqrt{1 + L_1 + L_2}, \quad (44)$$

and

$$L_1 = \overset{\circ}{g}{}^{ij}\left(\varepsilon_\varphi\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} + \varepsilon_\psi\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j}\right), \quad (45)$$

where

$$L_2 = \varepsilon_\varphi\,\varepsilon_\psi \left[-\left(\frac{\partial\varphi}{\partial x^0}\frac{\partial\psi}{\partial x^1} - \frac{\partial\varphi}{\partial x^1}\frac{\partial\psi}{\partial x^0}\right)^2 - \left(\frac{\partial\varphi}{\partial x^0}\frac{\partial\psi}{\partial x^2} - \frac{\partial\varphi}{\partial x^2}\frac{\partial\psi}{\partial x^0}\right)^2 - \left(\frac{\partial\varphi}{\partial x^0}\frac{\partial\psi}{\partial x^3} - \frac{\partial\varphi}{\partial x^3}\frac{\partial\psi}{\partial x^0}\right)^2 + \left(\frac{\partial\varphi}{\partial x^1}\frac{\partial\psi}{\partial x^2} - \frac{\partial\varphi}{\partial x^2}\frac{\partial\psi}{\partial x^1}\right)^2 + \left(\frac{\partial\varphi}{\partial x^1}\frac{\partial\psi}{\partial x^3} - \frac{\partial\varphi}{\partial x^3}\frac{\partial\psi}{\partial x^1}\right)^2 + \left(\frac{\partial\varphi}{\partial x^2}\frac{\partial\psi}{\partial x^3} - \frac{\partial\varphi}{\partial x^3}\frac{\partial\psi}{\partial x^2}\right)^2 \right]. \tag{46}$$

This formula, Eq. (46), can be obtained from Eq. (9) most easily, if one uses the following simplifying formula

$$\begin{vmatrix} h_{ii\_} & h_{i\_j\_} \\ h_{i\_j\_} & h_{jj\_} \end{vmatrix} = \pm \begin{vmatrix} \frac{\partial\varphi}{\partial x^i} & \frac{\partial\psi}{\partial x^{i-}} \\ \frac{\partial\varphi}{\partial x^{j-}} & \frac{\partial\psi}{\partial x^j} \end{vmatrix}^2 = \pm\left(\frac{\partial\varphi}{\partial x^i}\frac{\partial\psi}{\partial x^j} - \frac{\partial\varphi}{\partial x^j}\frac{\partial\psi}{\partial x^i}\right)^2.$$

In the first order approximation for the Lagrangian, the expression L_1 should be used. Then the field equations give the system of two independent wave equations

$$\left.\begin{aligned} \frac{\partial^2\varphi}{\partial x^0\partial x^0} - \frac{\partial^2\varphi}{\partial x^1\partial x^1} - \frac{\partial^2\varphi}{\partial x^2\partial x^2} - \frac{\partial^2\varphi}{\partial x^3\partial x^3} &= 0 \\ \frac{\partial^2\psi}{\partial x^0\partial x^0} - \frac{\partial^2\psi}{\partial x^1\partial x^1} - \frac{\partial^2\psi}{\partial x^2\partial x^2} - \frac{\partial^2\psi}{\partial x^3\partial x^3} &= 0 \end{aligned}\right\}.$$

Here the fields $\varphi(x)$ and $\psi(x)$ are independent and the superposition principle is fulfilled.

Using the strict Lagrangian for the two scalar fields Eq. (44) we get a system of two non-linear differential equations of the second order

$$\overset{\circ}{g}{}^{ij}\frac{\partial}{\partial x^i}\left[\frac{\varphi_{,j}\left(1 \pm \overset{\circ}{g}{}^{rs}\psi_{,r}\,\psi_{,s}\right) \mp \psi_{,j}\,\overset{\circ}{g}{}^{rs}\varphi_{,r}\,\psi_{,s}}{\sqrt{1+L_1+L_2}}\right] = 0,$$

$$\overset{\circ}{g}{}^{ij}\frac{\partial}{\partial x^i}\left[\frac{\psi_{,j}\left(1 + \overset{\circ}{g}{}^{rs}\varphi_{,r}\,\varphi_{,s}\right) - \varphi_{,j}\,\overset{\circ}{g}{}^{rs}\varphi_{,r}\,\psi_{,s}}{\sqrt{1+L_1+L_2}}\right] = 0,$$

where comma means the partial derivative. Here the fields $\varphi(x)$, $\psi(x)$ depend on each other, and the superposition principle is not fulfilled. The transition from the last but one equations to the last equations may be regarded as the transition from the weak fields to the strong fields.

6 Non-degenerate polynumbers

Consider a certain system of the non-degenerate polynumbers P_n [5], that is n-dimensional associative commutative non-degenerated hyper complex numbers. The corresponding coordinate space $x^1, x^2, \ldots, x^n$ is a Finsler metric flat space with the length element equal to

$$ds = \sqrt[n]{\overset{\circ}{g}_{i_1 i_2 \ldots i_n}\, dx^{i_1} dx^{i_2} \ldots dx^{i_n}}\,, \tag{47}$$

where $\overset{\circ}{g}_{i_1 i_2 \ldots i_n}$ is the metric tensor which does not depend on the point of the space. The Finsler spaces of this kind can be found in literature (e.g. [6, 7, 8, 9]) but the fact that all the non-degenerated polynumber spaces belong to this type of Finsler spaces was established beginning from the papers [10, 11] and the subsequent papers of the same authors, especially in [5].

The components of the generalized momentum in geometry corresponding to Eq. (47) can be found by the formulas

$$p_i = \frac{\overset{\circ}{g}_{i j_2 \ldots j_n}\, dx^{j_2} \ldots dx^{j_n}}{\left(\overset{\circ}{g}_{i_1 i_2 \ldots i_n}\, dx^{i_1} dx^{i_2} \ldots dx^{i_n}\right)^{\frac{n-1}{n}}}\,. \tag{48}$$

The tangent equation of the indicatrix in the space of the non-degenerated polynumbers P_n can be always written [5] as follows

$$\overset{\circ}{g}{}^{i_1 i_2 \ldots i_n} p_{i_1} p_{i_2} \ldots p_{i_n} - \mu^n = 0\,, \tag{49}$$

where μ is a constant. There always can be found such a basis (and even several such bases) and such a $\mu > 0$ that

$$\left(\overset{\circ}{g}{}^{i_1 i_2 \ldots i_n}\right) = \left(\overset{\circ}{g}_{i_1 i_2 \ldots i_n}\right). \tag{50}$$

Let us pass to a new Finsler geometry on the base of the space of non-degenerated polynumbers P_n. This new geometry is not flat, but its difference from the initial geometry is infinitely small, and the length element in this new geometry is

$$ds = \sqrt[n]{\left[\overset{\circ}{g}_{i_1 i_2 \ldots i_n} + \varepsilon h_{i_1 i_2 \ldots i_n}(x)\right] dx^{i_1} dx^{i_2} \ldots dx^{i_n}}\,, \tag{51}$$

where ε is an infinitely small value. If in the initial space the volume element was defined by the formula

$$dV = dx^{i_1} dx^{i_2} \ldots dx^{i_n}\,, \tag{52}$$

in the new space within the accuracy of ε in the first power we have

$$dV_h \simeq \left[1 + \varepsilon \cdot C_0\, \overset{\circ}{g}{}^{i_1 i_2 \ldots i_n} h_{i_1 i_2 \ldots i_n}(x)\right] dx^{i_1} dx^{i_2} \ldots dx^{i_n}.$$

That is according to [1], the Lagrangian of the weak field in the space with the length element Eq. (51) in the first order approximation is

$$L_1 = \overset{\circ}{g}{}^{i_1 i_2 \ldots i_n} h_{i_1 i_2 \ldots i_n}(x)\,. \tag{53}$$

This expression generalizes formula Eq. (7).

7 Hyper complex space H_4

In the physical ("orthonormal" [5]) basis in which every point of the space is characterized by the four real coordinates x^0, x^1, x^2, x^3 the fourth power of the length element ds_{H_4} is defined by the formula

$$\begin{aligned}(ds_{H_4})^4 &\equiv \overset{\circ}{g}_{ijkl}\, dx^0 dx^1 dx^2 dx^3 = \\ &= (dx^0 + dx^1 + dx^2 + dx^3)(dx^0 + dx^1 - dx^2 - dx^3)\times \\ &\times (dx^0 - dx^1 + dx^2 - dx^3)(dx^0 - dx^1 - dx^2 + dx^3) = \\ &= (dx^0)^4 + (dx^1)^4 + (dx^2)^4 + (dx^3)^4 + 8dx^0 dx^1 dx^2 dx^3 - \\ &- 2(dx^0)^2(dx^1)^2 - 2(dx^0)^2(dx^2)^2 - 2(dx^0)^2(dx^3)^2 - \\ &- 2(dx^1)^2(dx^2)^2 - 2(dx^1)^2(dx^3)^2 - 2(dx^2)^2(dx^3)^2.\end{aligned} \tag{54}$$

Let us compare the fourth power of the length element ds_{H_4} in the space of polynumbers H_4 with the fourth power of the length element ds_{Min} in the Minkowski space

$$\begin{aligned}(ds_{Min})^4 &= (dx^0)^4 + (dx^1)^4 + (dx^2)^4 + (dx^3)^4 - \\ &- 2(dx^0)^2(dx^1)^2 - 2(dx^0)^2(dx^2)^2 - 2(dx^0)^2(dx^3)^2 - \\ &+ 2(dx^1)^2(dx^2)^2 + 2(dx^1)^2(dx^3)^2 + 2(dx^2)^2(dx^3)^2.\end{aligned} \tag{55}$$

This means

$$\begin{aligned}(ds_{H_4})^4 &= (ds_{Min})^4 + 8dx^0 dx^1 dx^2 dx^3 - \\ &- 4(dx^1)^2(dx^2)^2 - 4(dx^1)^2(dx^3)^2 - 4(dx^2)^2(dx^3)^2,\end{aligned} \tag{56}$$

and in the covariant notation we have

$$\begin{aligned}(ds_{H_4})^4 = \left(\overset{\circ}{g}_{ij}\,\overset{\circ}{g}_{kl} + \frac{1}{3}\,\overset{\circ}{g}{}'_{ijkl} - \overset{\circ}{G}_{ijkl}\right)\times \\ \times\, dx^i dx^j dx^k dx^l\,,\end{aligned} \tag{57}$$

where

$$\overset{\circ}{g}{}'_{ijkl} = \begin{cases} 1\,, & \text{if } i,j,k,l \text{ are all different} \\ 0\,, & \text{else}\end{cases} \tag{58}$$

$$\overset{\circ}{G}_{ijkl} = \begin{cases} 1\,, & \text{if } i,j,k,l \neq 0 \text{ and } i=j\neq k=l, \\ & \qquad\qquad\quad \text{or } i=k\neq j=l, \\ & \qquad\qquad\quad \text{or } i=l\neq j=k \\ 0\,, & \text{else}\end{cases} \tag{59}$$

The tangent equation of the indicatrix in the H_4 space can be written in the physical basis as in [5]:

$$\begin{aligned}&(p_0 + p_1 + p_2 + p_3)(p_0 + p_1 - p_2 - p_3)\times \\ &\times (p_0 - p_1 + p_2 - p_3)(p_0 - p_1 - p_2 + p_3) - 1 = 0\,,\end{aligned} \tag{60}$$

where p_i are the generalized momenta

$$p_i = \frac{\partial\, ds_{H_4}}{\partial(dx^i)}\,. \tag{61}$$

Comparing formula Eq. (60) with formula Eq. (61), we have

$$\overset{\circ}{g}{}^{ijkl}\, p_i p_j p_k p_l - 1 = 0\,. \tag{62}$$

Here

$$\overset{\circ}{g}{}^{ijkl} = \overset{\circ}{g}{}^{ij}\,\overset{\circ}{g}{}^{kl} + \frac{1}{3}\,\overset{\circ}{g}{}'^{\,ijkl} - \overset{\circ}{G}{}^{ijkl}\,, \tag{63}$$

and

$$\left.\begin{aligned}\left(\overset{\circ}{g}{}^{ijkl}\right) &= \left(\overset{\circ}{g}_{ijkl}\right) \\ \left(\overset{\circ}{g}{}'^{\,ijkl}\right) &= \left(\overset{\circ}{g}{}'_{ijkl}\right) \\ \left(\overset{\circ}{G}{}^{ijkl}\right) &= \left(\overset{\circ}{G}_{ijkl}\right)\end{aligned}\right\}. \tag{64}$$

To get the Lagrangian for the weak field in the first order approximation, we have to get tensor h_{ijkl} in Eq. (53). In the simplified version it could be splitted into two additive parts: gravitational part and electromagnetic part. The gravitational part can be constructed analogously to Sections 3 and 5 with regard to the possibility to use the two-index number tensors, since now tensors $\overset{\circ}{g}{}^{ijkl}$, h_{ijkl} have four indices. The construction of the electromagnetic part should be regarded in more detail.

Since we would like to preserve the gradient invariance of the Lagrangian and to get Maxwell equations for the free field in the H_4 space, let us write the electromagnetic part of tensor h_{ijkl} in the following way

$$h^{A_k}_{ijkl} = \chi(x)\, h^{(1)}_{ijkl} + [1 - \chi(x)]\, h^{(2)}_{ijkl}\,, \tag{65}$$

where the tensors $h^{(1)}_{ijkl}$, $h^{(2)}_{ijkl}$ are the tensors present in the round brackets in the r.h.s. of formulas Eqs. (28,29). Then

$$\begin{aligned}L_A &= \overset{\circ}{g}{}^{ijkl} h^{A_k}_{ijkl} \equiv \\ &\equiv \overset{\circ}{g}{}^{ij}\,\overset{\circ}{g}{}^{km}\left(\frac{\partial A_k}{\partial x^i}\frac{\partial A_m}{\partial x^j} - \frac{\partial A_k}{\partial x^i}\frac{\partial A_j}{\partial x^m}\right).\end{aligned} \tag{66}$$

To obtain Maxwell equations not for the free field but for the field with a source $j_i(x)$, one should add to the tensor $h^{(A_k)}_{ijkl}$ Eq. (65) the following tensor

$$h^{(j_k)}_{ijkl} = \left(\frac{8}{3\pi}\right)\left(2A_i j_j\,\overset{\circ}{g}_{kl} - A_i\,\overset{\circ}{g}_{jk}\, j_l - j_i\,\overset{\circ}{g}_{jk}\, A_l\,,\right),$$

symmetrized in all indices, that is tensor

$$h_{ijkl} = h^{Max}_{ijkl} \equiv h^{(A_k)}_{ijkl} + h^{(j_k)}_{(ijkl)}$$

describes the weak electromagnetic field with the sources $j_i(x)$, where

$$j_i = \sum_b q_{(a)} \frac{\partial\psi_{(b)}}{\partial x^i}, \tag{67}$$

and $\psi_{(b)}$ are the scalar components of the gravitational field.

To obtain the unified theory for the gravitational and electromagnetic fields one should take the linear combination of tensor $h^{(Max)}_{ijkl}$ corresponding to the electromagnetic field in the first order approximation, and tensor $h^{(grav)}_{ijkl}$ corresponding to the gravitational field in the first order approximation

$$h_{ijkl} = \mu h^{(Max)}_{ijkl} + \gamma h^{(grav)}_{ijkl}, \tag{68}$$

where μ, γ are constants. Tensor $h^{(grav)}_{ijkl}$ may be, for example, constructed in the following way

$$\begin{aligned} h^{grav}_{ijkl} &= \sum_{a=1}^{N} \varepsilon_{(a)} \frac{\partial\varphi_{(a)}}{\partial x^i}\frac{\partial\varphi_{(a)}}{\partial x^j}\frac{\partial\varphi_{(a)}}{\partial x^k}\frac{\partial\varphi_{(a)}}{\partial x^l} + \\ &+ \sum_{b=1}^{M} \epsilon_{(b)} \frac{\partial\psi_{(b)}}{\partial x^{(i}}\frac{\partial\psi_{(b)}}{\partial x^j} \overset{\circ}{g}_{kl)}, \end{aligned} \tag{69}$$

where $\varepsilon_{(a)}$, $\epsilon_{(b)}$ are the sign coefficients, and $\varphi_{(a)}$, $\psi_{(b)}$ are the scalar fields. The whole number of scalar fields is equal to $(N+M)$.

8 Conclusion

In this paper it was shown that the geometrical approach [1] to the field theory in which there usually appear the non-linear and non-splitting field equations could give a system of independent linear equations for the weak fields in the first order approximation. When the fields become stronger the superposition principle (linearity) breaks, the equations become non-linear and the fields start to interact with each other. We may think that these changes of the equations that take place when we pass from the weak fields to the strong fields are due to the two mechanisms: first is the qualitative change of the field equations for the free fields in the first order approximation; second is the appearance of the additional field sources, that is the generation of the field by the other fields.

In frames of the geometrical approach to the field theory [1] the unification of the electromagnetic and gravitational fields is performed both for the four-dimensional pseudo-Riemannian space with metric tensor $g_{ij}(x)$ and for the four-dimensional curved Berwald-Moor space with metric tensor $g_{ijkl}(x)$.

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References

1. Garas'ko G.I. Field theory and Finsler spaces. *Hyper Complex Numbers in Geometry and Physics*, 2006, v. 3 (2), 6–20.
2. Rashevsky P. K. Geometrical theory of equations with partial derivatives. OGIZ, Moscow-Leningrad, 1947.
3. Garas'ko G.I. Particular stationary solution of the field equation for the space conformally connected with Minkowski space. *Hyper Complex Numbers in Geometry and Physics*, 2007, v. 4 (1), 26–37.
4. Garas'ko G.I. The space conformally connected with Berwald-Moor space. *Hyper Complex Numbers in Geometry and Physics*, 2007, v. 4 (1), 38–51.
5. Pavlov D.G., Garas'ko G.I. Geometry of the non-degenerated polynumbers. *Hyper Complex Numbers in Geometry and Physics*, 2007, v. 4 (1), 3–25.
6. Matsumoto M., Numata S. On Finsler spaces with 1-form metric. *Tensor*, 1978, v. 32, 161–169.
7. Matsumoto M., Numata S. On Finsler spaces with 1-form metric II. Berwald — Moor's metric $L = (y^1 y^2 \dots y^n)^{\frac{1}{n}}$. *Tensor*, 1978, v. 32, 275–277.
8. Matsumoto M., Numata S. On Finsler spaces with a cubic metric, *Tensor*, 1979, v. 33, 153–162.
9. Shimada H. On Finsler spaces with the metric L of m-th root metric. *Tensor*, 1979, v. 33, 365–372.
10. Pavlov D.G. Generalized axioms of the scalar product. *Hyper Complex Numbers in Geometry and Physics*, 2004, v. 1 (1), 6–20.
11. Garas'ko G.I. Generalized analytical functions of the polynumber variable. *Hyper Complex Numbers in Geometry and Physics*, 2004, v. 1 (1), 75–88.