Aspects of Stability and Quantum Mechanics

Robert Carroll

University of Illinois, Urbana, IL 61801, USA

E-mail: rcarroll@math.uiuc.edu

We comment on some work of Ruslov and Vlasenko indicating how stable Hamiltonian systems can be quantized under certain assumptions about the perturbations.

1 Introduction

In [7] we indicated some results of Rusov and Vlasenko [56, 57] involving Hamiltonian stability and quantization which we summarize here with a somewhat different interpretation. In [56, 57] (which are the same modulo typos and conclusions) one indicates how the work of Chetaev [9–11] (based in particular on classical results of Poincaré [52] and Lyapunov [39]) allow one to relate stability of classical systems to quantum mechanics in certain situations. We review here some of the arguments (cf. also [7, 55, 60] for additional material on the Poincaré-Chetaev equations).

One recalls that holonomic systems involve an agreement of the degrees of freedom with the number of independent variables. Then following [9] consider a holonomic system with Hamiltonian coordinates

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j};\ \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \tag{1.1}$$

and think of perturbations $(\mathbf{1A})$ $q_j = q_j(t) + \xi_j$ and $p_j = p_j(t) + \eta_j$. Denoting then $q_j \sim q_j(t)$ and $p_j \sim p_j(t)$ one has

$$\left.\begin{aligned} \frac{d(q_j+\xi_j)}{dt} &= \frac{\partial H(t, q_i+\xi_i, p_i+\eta_i)}{\partial p_j} \\ \frac{d(p_j+\eta_j)}{dt} &= -\frac{\partial H(t, q_i+\xi_i, p_i+\eta_i)}{\partial q_j} \end{aligned}\right\}. \tag{1.2}$$

Expanding and using (1.1) gives

$$\left.\begin{aligned} \frac{d\xi_j}{dt} &= \sum\left(\frac{\partial^2 H}{\partial p_j \partial q_i}\xi_i + \frac{\partial^2 H}{\partial p_j \partial p_i}\eta_i\right) + X_j \\ \frac{d\eta_j}{dt} &= -\sum\left(\frac{\partial^2 H}{\partial q_j \partial q_i}\xi_i + \frac{\partial^2 H}{\partial q_j \partial p_i}\eta_i\right) + Y_j \end{aligned}\right\}, \tag{1.3}$$

where the X_j, Y_j are higher order terms in ξ, η. The first approximations (with $X_j = Y_j = 0$) are referred to as Poincaré variational equations. Now given stability questions relative to functions Q_s of (t, q, p) one writes

$$\begin{aligned} x_s = Q_s(t, q_i+\xi_i, p_i+\eta_i) - Q_s(t, q_i, p_i) = \\ = \sum\left(\frac{\partial Q_s}{\partial q_i}\xi_i + \frac{\partial Q_s}{\partial p_i}\eta_i\right) + \cdots \end{aligned} \tag{1.4}$$

which implies

$$\frac{dx_s}{dt} = \sum\left(\frac{\partial Q'_s}{\partial q_i}\xi_i + \frac{\partial Q'_s}{\partial p_i}\eta_i\right) + \cdots \tag{1.5}$$

where

$$Q'_s = \frac{\partial Q_s}{\partial t} + \sum\left(\frac{\partial Q_s}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q_s}{\partial p_i}\frac{\partial H}{\partial q_i}\right). \tag{1.6}$$

Given $1 \leqslant s \leqslant 2k$ and $1 \leqslant i, j \leqslant k$ one can express the ξ_i, η_i in terms of x_s and write $(\mathbf{1B})$ $(dx_s/dt) = X_s$ (normal form) with $X_s(0) = 0$. For equations $(\mathbf{1B})$ with $1 \leqslant s \leqslant n$, for sufficiently small perturbations ϵ_j, ϵ'_j one assumes there exists some system of initial values x_{s^0} with $\sum x^2_{s^0} < A$ for an arbitrarily small A (with perturbations ϵ_j, $\epsilon'_j \leqslant E_j$, E'_j). Further for arbitrarily small E_j, E'_j one assumes it is possible to find A as above such that there exists one or more values ϵ_j, ϵ'_j with absolute values $\leqslant E_j$, E'_j. Under these conditions the initial values of x_s play the same role for stability as the ϵ_j, ϵ'_j and one assumes this to hold. One assumes also convergent power series for the X_s etc. Then Lyapunov stability means that for arbitrary small A there exists λ such that for all perburbations x_{s^0} satisfying $\sum x^2_{s^0} \leqslant \lambda$ and for all $t \geqslant t_0$ one has $\sum x^2_s < A$ (i.e. the unperturbed motion is stable). Next one considers $t \geqslant t_0$ and $\sum x^2_s \leqslant H$ and looks for a sign definite (Lyapunov) function V (with $V' = \partial_t V + \sum_1^n X_j(\partial V/\partial x_j)$ then sign definite of opposite sign or zero). If such a function exists the unperturbed motion is stable (see [9] for proof).

We pick up the story now in [10] where relations between optics and mechanics are also illuminated (but not considered here). Take a holonomic mechanical system with coordinates q_i and conjugate momenta p_i with n degrees of freedom. Assume the holonomic constraints are independent of time and the forces acting on the system are represented by a potential function $U(q_i)$. Let $(\mathbf{1C})$ $T = \frac{1}{2}\sum_{i,j} g_{ij}p_ip_j$ denote the kinetic energy where the $g_{ij} = g_{ji}$ are not dependent explicitly on time. Hamilton's equations have the form

$$2T = \sum g_{ij}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j} = 2(U + E) \tag{1.7}$$

where E represents a kinetic energy constant (the sign of U is changed in Section 2). Here the integral of (1.7) is $(\mathbf{1D})$ $S(q_i, \alpha_i) + c$ with the α_i constants and $(\mathbf{1E})$ $||\partial^2 S/\partial q_i \partial\alpha_j|| \neq 0$ while $(\mathbf{1F})$ $E = E(\alpha_i)$. According to the Hamilton-Jacobi theory the general solution of the motion equations is given via $(\mathbf{1G})$ $p_i = \partial S/\partial q_i$ and $\beta_i = -t(\partial E/\partial\alpha_i) + \partial S/\partial\alpha_i$ where the β_i are constants. In order to determine a stable solution one looks at the Poincaré variations

$$\left.\begin{aligned}\frac{d\xi_i}{dt} &= \sum_j \left(\frac{\partial^2 H}{\partial q_j \partial p_i}\,\xi_j + \frac{\partial^2 H}{\partial p_j \partial p_i}\,\eta_j\right)\\ \frac{d\eta_i}{dt} &= -\sum \left(\frac{\partial^2 H}{\partial q_j \partial q_i}\,\xi_j + \frac{\partial^2 H}{\partial p_j \partial q_i}\,\eta_j\right)\end{aligned}\right\}, \tag{1.8}$$

where H should be defined here via $(\mathbf{1H})$ $H = T - U$. For a stable unperturbed motion the differential equations for Poincaré variations (1.8) must be reducible by nonsingular transformation to a system of linear differential equations with constant coefficients all of whose characteristic values must be zero (recall that the Lyapunov characteristic value $X[f]$ of f is $X[f] = -\overline{\lim}_{t\to\infty}[\log(|f(t)|)/t]$ — cf. [39,40]). In such perturbed motion, because of $(\mathbf{1G})$ one has (recall $p_i \sim \partial S/\partial q_i$)

$$\eta_i = \sum_j \frac{\partial^2 S}{\partial q_i \partial q_j}\,\xi_j \;\; (i = 1, \cdots, n)\,. \tag{1.9}$$

Hence

$$\frac{d\xi_i}{dt} = \sum_{j,s} \xi_s \frac{\partial}{\partial q_s}\left(g_{ij}\frac{\partial S}{\partial q_j}\right) \;\; (i = 1, \cdots, n)\,. \tag{1.10}$$

Note here that (1.8) involves $\sum g_{ij} p_i p_j - U$ so

$$(\bigstar)\;\; \frac{\partial H}{\partial p_i} = \sum g_{ij} p_j; \quad \frac{\partial H}{\partial q_j} = \sum \frac{\partial g_{ij}}{\partial q_j}\, p_i p_j - \frac{\partial U}{\partial q_j}$$

and (1.10) says

$$(\bigstar\bigstar)\;\; \frac{d\xi_i}{dt} = \sum \xi_s \left(\frac{\partial g_{ij}}{\partial q_s}\frac{\partial S}{\partial q_j} + g_{ij}\frac{\partial^2 S}{\partial q_s \partial q_j}\right) = \sum \xi_s \frac{\partial g_{ij}}{\partial q_s}\frac{\partial S}{\partial q_j} + \sum g_{ij}\eta_j\,.$$

The second term here is $[\partial^2 H/\partial p_i \partial p_j]\,\eta_j$ and we want to identify the term $\xi_s(\partial g_{ij}/\partial q_s)(\partial S/\partial q_j)$ with $\partial^2 H/\partial q_s \partial p_i \xi_s$. However we can see that $\partial U/\partial p_i = 0$ so $\xi_s(\partial^2 H/\partial q_s \partial p_i) = \xi_s(2\partial^2 T/\partial q_s \partial p_i) = \xi_s(\partial g_{ij}/\partial q_s)p_j$ confirming (1.10). Here the q_i, α_i are represented by their values in an unperturbed motion. Now for a stable unperturbed motion let (1.10) be reducible by a nonsingular linear transformation $(\mathbf{1I})$ $x_i = \sum \gamma_{ij}\xi_j$ with a constant determinant $\Gamma = ||\gamma_{ij}||$. If ξ_{ir} $(r = 1, \cdots, n)$ are a normal system of independent solutions of (1.10) then $(\mathbf{1J})$ $x_{ir} = \sum_j \gamma_{ij}\xi_{jr}$ will be the solution for the reduced system. For a stable unperturbed motion all the characteristic values of the solutions x_{ir} $(i = 1, \cdots, n)$ are zero and consequently

$$||x_{sr}|| = C^* = ||\gamma_{sj}||\,||\xi_{jr}|| = \Gamma C \exp\left[\int \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial S}{\partial q_j}\right) dt\right]. \tag{1.11}$$

Consequently for a stable perturbed motion (cf. [9,39,40])

$$\sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial S}{\partial q_j}\right) = 0\,. \tag{1.12}$$

2 Stability approach

Following Rusov and Vlasenko one writes an integral of the Hamilton-Jacobi (HJ) equation in the form $(\mathbf{2A})$ $S = f(t, q_i, \alpha_i) + A$ $(i = 1, \cdots, n)$ with the α_i arbitrary constants. The general solution is then $(\mathbf{2B})$ $p_i = \partial S/\partial q_i$ with $\beta_i = \partial S/\partial \alpha_i$ where the β_i are new constants of integration. The canonical equations of motion are $dq_i/dt = \partial H/\partial p_i$ and $dp_i/dt = -\partial H/\partial q_i$ where H is the Hamiltonian and under perturbations of the α_i, β_i one writes $\xi_i = \delta q_i = q_i - q_i(t)$ and $\eta_i = \delta p_i = p_i - p_i(t)$ and derives equations of first approximation

$$\left.\begin{aligned}\frac{d\xi_i}{dt} &= \sum \frac{\partial^2 H}{\partial q_j \partial p_i}\,\xi_j + \sum \frac{\partial^2 H}{\partial p_j \partial p_i}\,\eta_j\\ \frac{d\eta_i}{dt} &= -\sum \frac{\partial^2 H}{\partial q_j \partial q_i}\,\xi_j - \sum \frac{\partial^2 H}{\partial p_j \partial q_i}\,\eta_j\end{aligned}\right\} \tag{2.1}$$

as in (1.8). By differentiating in t one obtains then $(\mathbf{2C})$ $C = \sum(\xi_s\eta'_s - \eta_s\xi'_s)$ where C is a constant. Also for given ξ_s, η_s there is always at least one solution ξ'_s, η'_x for which $C \neq 0$. Stability considerations (as in form. 1.1) then lead via $(\bigstar)$ $\eta_i = \sum(\partial^2 S/\partial q_i \partial q_j)\xi_j$ and $(\mathbf{2D})$ $H = \frac{1}{2}\sum g_{ij} p_i p_j + U = T + U$ to

$$\frac{d\xi_i}{dt} = \sum \xi_s \frac{\partial}{\partial q_s}\left(g_{ij}\frac{\partial S}{\partial q_j}\right) \tag{2.2}$$

(note in Section 1 $H \sim T - U$ following [10] but we take now $U \to -U$ to agree with [56, 57] — the sign of U is not important here). According to [56, 57], based on results of Chetaev [10] (as portrayed in Section 1), it results that $L = \sum(\partial/\partial q_i)[g_{ij}(\partial S/\partial q_j)] = 0$ (as in form. 1.12) for stability (we mention e.g. [9–11,39,40,45] for stability theory, Lyapunov exponents, and all that).

REMARK 2.1. One also notes in [56, 57] that a similar result occurs for $(\bullet\bullet)$ $U \to U^* = U + Q$ for some natural Q and the stability condition (1.12) itself provides the natural introduction of quantization (see below). The perturbation relation in (1.9) is irrelevant to this feature (which we did not realize previously) and the quantum perturbations introduced via Q will satisfy the Heisenberg uncertainty principle as desired (cf. [3]). ■

Now one introduces a function $(\mathbf{2E})$ $\psi = A\exp(ikS)$ in (1.12) where k is constant and A is a real function of the coordinates q_i only. There results

$$\frac{\partial S}{\partial q_j} = \frac{1}{ik}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_j}\right) \tag{2.3}$$

so that (1.12) becomes

$$\sum_{i,j} \frac{\partial}{\partial q_i}\left[g_{ij}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_i}\right)\right] = 0\,. \tag{2.4}$$

On the other hand for the perturbed motion (with $U \to$

$\to U^* = U + Q$) the HJ equation can be written in the form

$$\frac{1}{2k^2}\sum_{i,j} g_{ij}\left[\frac{1}{\psi}\frac{\partial\psi}{\partial q_i} - \frac{1}{A}\frac{\partial A}{\partial q_i}\right]\left[\frac{1}{\psi}\frac{\partial\psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_j}\right] = = \partial_t S + U + Q \qquad (2.5)$$

with $\partial_t S$ obtained via (**2E**). Adding (2.4) and (2.5) yields

$$\frac{1}{2k^2\psi}\sum_{i,j}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial\psi}{\partial q_j}\right) - \frac{1}{2k^2 A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right) - - \frac{1}{k^2 A}\sum_{i,j} g_{ij}\frac{\partial A}{\partial q_j}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_i} - \frac{1}{A}\frac{\partial A}{\partial q_i}\right) - - \frac{1}{ikA\psi}[A\,\partial_t\psi - \psi\,\partial_t A] - U - Q = 0 \qquad (2.6)$$

as a necessary stability condition (in the first approximation). Note (2.6) will not contain Q if A is defined via

$$\frac{1}{2k^2 A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right) + + \frac{i}{kA}\sum_{i,j} g_{ij}\frac{\partial A}{\partial q_j}\frac{\partial S}{\partial q_i} - \frac{1}{ikA}\partial_t A + Q = 0 \qquad (2.7)$$

which means

$$\left.\begin{aligned} Q &= -\frac{1}{2k^2 A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right) \\ \partial_t A &= -\sum_{i,j} g_{ij}\frac{\partial A}{\partial q_j}\frac{\partial S}{\partial q_i} \end{aligned}\right\}. \qquad (2.8)$$

A discussion of the physical content of (2.8) appears in [56, 57] and given (2.8) the stability condition (2.6) leads to

$$\frac{i}{k}\partial_t\psi = -\frac{1}{2k^2}\sum_{i,j}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial\psi}{\partial q_j}\right) + U\psi \qquad (2.9)$$

which is of course a SE for $k = 1/\hbar$ (this is the place where quantum mechanics somewhat abruptly enters the picture — see Remark 2.1). In fact for kinetic energy (**2F**) $T = \frac{1}{2m}\left[p_1^2 + + p_2^2 + p_3^2\right]$ (2.9) leads to

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta A}{A};\;\; \partial_t A = -\frac{1}{m}\sum\frac{\partial A}{\partial x_j}p_j;\;\; k = \frac{1}{\hbar} \qquad (2.10)$$

and (2.9) becomes (note $A = A(q)$)

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + U\psi\,. \qquad (2.11)$$

Going backwards now put the wave following function $\psi = A\exp(iS/\hbar)$ in (2.11) to obtain via (1.12) and (2.8) the Bohmian equations

$$\left.\begin{aligned} \partial_t A &= -\frac{1}{2m}\left[A\Delta S + 2\nabla A\cdot\nabla S\right] = -\nabla A\cdot\frac{\nabla S}{m} \\ \partial_t S &= -\left[\frac{(\nabla S)^2}{2m} + U - \frac{\hbar^2}{2m}\frac{\nabla A}{A}\right] \end{aligned}\right\}, \qquad (2.12)$$

where the quantum potential QP is naturally identified.

If one writes now $P = \psi\psi^* = A^2$ then (2.12) can be rewritten in a familiar form

$$\left.\begin{aligned} &\partial_t P = -\nabla P\cdot\frac{\nabla S}{m} \\ &\partial_t S + \frac{(\nabla S)^2}{2m} + U - \\ &\quad - \frac{\hbar^2}{4m}\left[\frac{\Delta P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] = 0 \end{aligned}\right\}. \qquad (2.13)$$

That P is indeed a probability density is "substantiated" via a least action of perturbation principle attributed to Chetaev [11, 56, 57] which involves (**2G**) $\int Q|\psi^2|dV = \min$ where dV is a volume element with $\int|\psi|^2 dV = 1$ and this condition is claimed to be necessary for stability (one assumes that the influence of perturbative forces generated by Q is proportional to the density of trajectories $|\psi|^2 = A^2$ and dV cannot be a phase space volume element as stated in [56,57]). Write now, using (**2D**)

$$Q = -\partial_t S - U - T = -\partial_t S - U - \frac{1}{2}\sum g_{ij}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j}\,. \qquad (2.14)$$

Then if (**2E**) holds one can show that

$$\frac{1}{2}\sum g_{ij}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j} = -\frac{1}{2k^2\psi^2}\sum g_{ij}\frac{\partial\psi}{\partial q_i}\frac{\partial\psi}{\partial q_j} + + \frac{1}{2k^2A^2}\sum g_{ij}\frac{\partial A}{\partial q_i}\frac{\partial A}{\partial q_j} + \frac{ik}{2k^2A^2}\sum g_{ij}\frac{\partial A}{\partial q_i}\frac{\partial S}{\partial q_j}\,. \qquad (2.15)$$

Then for the first term on the right side substitute its value from the first stability condition (2.4), then insert this relation into (2.15) and put the result into the equation (2.14) corresponding to the variational principle; the result is then (2.6) and consequently the resulting structure expression and the necessary condition for stability coincide with (2.8) and (2.9). This leads one to conclude that stability and (Bohmian) quantum mechanics are two complementary procedures of Hamiltonian theory. The authors cite an impressive list of references related to experimental work related to the analysis in [56,57].

3 The quantum potential

From Sections 1–2 we have seen that a stable Hamiltonian system as indicated gives rise to a quantum Schrödinger equation with quantum potential Q. It seems therefore appropriate to examine this in the light of other manifestations of the QP as in e.g. [3–6, 16–19, 24, 26–28, 30, 36, 37, 53]. We note that following [4] one can reverse some arguments involving the exact uncertainty principle (cf. [3,26–28,53]) to show that any SE described by a QP based on $|\psi|^2 = P$ can be modeled on a quantum model of a classical Hamiltonian H perturbed by a term H_Q based on Fisher information, namely

$$H_Q = \frac{c}{2m}\int\frac{(\nabla P)^2}{P}\,dx = \frac{c}{2m}\int P(\delta p)^2, \qquad (3.1)$$

where $\delta p = \nabla P/P$. This does not of course deny the presence of "related" $x \sim q$ oscillations $\delta x \sim \delta q$ and in fact in Olavo [49] (cf. also [3]) Gaussian fluctuations in δq are indicated and related to δp via an exact uncertainty relation $(\mathbf{3A})$ $(\delta p)^2 \cdot (\delta q)^2 = \hbar^2/4$. We note that the arguments establishing exact uncertainty stipulate that the position uncertainty must be entirely characterized by $P = |\psi|^2$ (cf. [3,26–28,53]). Thus the quantum potential generates the quantum perturbations δp and these are essentially unrelated to the $\delta p \sim \eta_i$ of (1.9).

REMARK 3.1. We recall here [29] (cf. also [54]) were it is shown that quantum mechanics can be considered as a classical theory in which a Riemannian geometry is provided with the distance between states defined with natural units determined via Planck's constant (which is the inverse of the scalar curvature). ∎

REMARK 3.2. In [2] one shows that non-relativistic quantum mechanics for a free particle emerges from classical mechanics via an invariance principle under transformations that preserve the Heisenberg inequality. The invariance imposes a change in the laws of classical mechanics corresponding to the classical to quantum transition. Some similarities to the Nottale theory of scale relativity in a fractal spacetime are also indicated (cf. [3, 8, 47, 48]). There are relations here to the Hall-Reginatto treatment which postulates that the non-classical momentum fluctuations are entirely determined by the position probability (as mentioned above). In Brenig's work one derives this from an invariance principle under scale transformations affecting the position and momentum uncertainties and preserving the Heisenberg inequality. One modifies the classical definition of momentum uncertainty in order to satisfy the imposed transformation rules and this modification is also constrained by conditions of causality and additivity of kinetic energy used by Hall-Reginatto. This leads to a complete specification of the functional dependance of the supplementary term corresponding to the modification which turns out to be proportional to the quantum potential. ∎

REMARK 3.3. We note that in work of Grössing (cf. [6, 24]) one deals with subquantum thermal oscillations leading to momentum fluctuations $(\mathbf{3B})$ $\delta p = -(\hbar/2)(\nabla P/P)$ where P is a position probability density with $-\nabla \log(P) = \beta \nabla \mathcal{Q}$ for $\mathcal{Q}$ a thermal term (thus $P = c\exp(-\beta \mathcal{Q})$ where $\beta = 1/kT$ with k the Boltzman constant). This leads also to consideration of a diffusion process with osmotic velocity $\mathbf{u} \propto -\nabla \mathcal{Q}$ and produces a quantum potential

$$Q = \frac{\hbar^2}{4m}\left[\nabla^2 \tilde{\mathcal{Q}} - \frac{1}{D}\,\partial_t \tilde{\mathcal{Q}}\right] \tag{3.2}$$

where $\tilde{\mathcal{Q}} = \mathcal{Q}/kT$ and $D = \hbar/2m$ is a diffusion coefficient. Consequently (cf. [6] one has a Fisher information $(\mathbf{3C})$ $F \propto \beta^2 \int \exp(-\beta \mathcal{Q}(\nabla \mathcal{Q})^2 d^3x$. As in the preceeding discussions the fluctuations are generated by the position probability density and one expects a connection to (Bohmian) quantum mechanics (cf. [3, 12, 18, 19]). ∎

REMARK 3.4. There is considerable literature devoted to the emergence of quantum mechanics from classical mechanics. There have also been many studies of stochastic and hydrodynamic models, or fractal situations, involving such situations and we mention in particular [1, 3–6, 8, 12, 13, 18–20, 23, 24, 26–28, 36, 37, 42–44, 46–49, 53, 58, 59, 61]; a survey of some of this appears in [3]. For various geometrical considerations related to the emergence question see also [14, 15, 25, 30–35, 51, 62] and in connection with chaos we cite e.g. [1, 25, 38, 41, 50, 51, 62, 63]. ∎

Submitted on December 12, 2008 / Accepted on December 23, 2008

References

1. Benenti G., Casati G., and Strini G. Principles of quantum computation and information. Vols. 1 and 2. World Scientific, 2007.
2. Brenig L. *Jour. Phys. A*, 2007, v. 40, 4567–4584; arXiv: quant-ph/0608025.
3. Carroll R. Fluctuations, information, gravity and the quantum potential. Springer, 2006; On the quantum potential. Arima Publ., 2007.
4. Carroll R. *Teor. i Mat. Fiz.*, 2007, v. 152, 904–914.
5. Carroll R. *Prog. in Phys.*, 2007, v. 4, 22–24 and 2008, v. 1, 21–24 (preprinted in arXiV: math-ph/0703065 and 0710.4351); *Prog. in Phys.*, 2008, v. 2, 89–90 (preprinted in arXiv: math-ph/0712.3251).
6. Carroll R. arXiv: math-ph/0807.1320 and 0807.4158.
7. Carroll R. arXiv: math-ph/0808.2965.
8. Célérier M. and Nottale L. *Jour. Phys. A*, 2004, v. 37, 931; arXiv: hep-th/0112213 and 0210027.
9. Chetaev N. The stability of motion (Ustoichevost dvizhenia). Permagon, 1961; Theoretical mechanics. Mir-Springer, 1989.
10. Chetaev N. *Soviet Applied Math. and Mech.*, 1958, v. 22, 487–489.
11. Chetaev N. *Soviet Applied Math. and Mech.*, 1956, v. 20, 309–314; 1959, v. 23, 425–433; 1960, v. 24, 6–19, 23–32, 33–34.
12. Crowell L. Quantum fluctuations of spacetime. World Scientific, 2005.
13. Davidson M. *Jour. Math. Phys.*, 1979, v. 20, 1865–1869; arXiv: quant-ph/0112063, 0112076, 0112114, 0112099, 0112157, 0211097, and 0602211; *Annales L. de Broglie*, 2004, v. 29, 661–680.
14. Elitzur A., Dolev S., and Kolenda N. (Eds.). Quo vadis quantum mechanics. Springer, 2005.
15. Elze H. arXiv: gr-qc/0512016; quant-ph/0710.2765.
16. Frieden B. and Gatenby R. Exploratory data analysis using Fisher information. Springer, 2007.
17. Frieden B. Physics from Fisher information. Cambridge Univ. Press, 1998; Science from Fisher information. Springer, 2004.

18. Garbaczewski P. *Entropy*, 2005, v. 7, 253–299; *Jour. Stat. Phys.*, 2007, v. 123, 315–355.
19. Garbaczewski P. arXiv: cond-mat/0202463, 0211362, 0510533, 0706248, 0604538, and 0703147; quant-ph/0612151.
20. Gell-man M. and Tsallis C. Non-extensive entropy. Oxford Univ. Press, 2004.
21. de Gosson M. arXiv: quant-ph/0808.2774; Maslov classes, metaplectic representations, and Lagrangian quantization. Akad. Verlag, 1997.
22. de Gosson M. and Luef F. *Phys. Lett. A*, 2007, v. 364, 453–457.
23. Granik A. arXiv: quant-ph/0801.3311.
24. Grössing G. *Phys. Lett. A*, 2008, v. 372, 4556–4562; *Found. Phys. Lett.*, 2004, v. 17, 343–362; arXiv: quant-ph/0201035, 0205047, 0404030, 0410236, 0508079, and 0806.4462.
25. Gutzwiller M. Chaos in classical and quantum mechanics. Springer, 1990.
26. Hall M. and Reginatto M. *Jour. Phys. A*, 2002, v. 35, 3289–3303; *Fortschr. Phys.*, 2002, v. 50, 646–651; arXiv: quant-ph/0201084.
27. Hall M., Kumar K., and Reginatto M. *Jour. Phys. A*, 2003, v. 36, 9779–9794.
28. Hall M. arXiv: gr-qc/0408098, quant-ph/0007116; *Jour. Phys. A*, 2004, v. 37, 7799 and 9549 (preprinted in arXiv: quant-ph/0404123 and 0406054).
29. Heslot A. *Amer. Jour. Phys.*, 1983, v. 51, 1096–1102; *Phys. Rev. D*, 1985, v. 31, 1341–1348.
30. Hiley B. Quo vadis quantum mechanics. Springer, 2005, pp. 299–324.
31. Holland P. The quantum theory of motion. Cambridge Univ. Press, 1993.
32. ’t Hooft G. arXiv: hep-th/0707.4568, quant-ph/0604008.
33. Isidro J., Santander J., and de Cordoba P.F. arXiv: hep-th/0808.2351 and 0808.2717; gr-qc/0804.0169.
34. Isidro J. arXiv: hep-th/0110151, 0204178, 0304175, 0407161, 0411015, and 0510075; quant-ph/0307172, 0310092, and 0407159.
35. Isidro J. and de Gosson M. arXiv: hep-th/0608087; quant-ph/0608093.
36. Kaniadakis G. arXiv: quant-ph/0112049.
37. Kaniadakis G. and Scarfone A. arXiv: cond-mat/0303334.
38. Klages R. Microscopic chaos, fractals, and transport in non-equilibrium statistical mechanics. World Scientific, 2007.
39. Lyapunov A. The general problem of stability of motion. Gostekhizdat, 1950; Princeton Univ. Press, 1947.
40. Malkin I. Theory of stability of motion. Gostekhizdat, 1952; Some problems in the theory of nonlinear oscillations. Gostekhizdat, 1956.
41. Magnitskij N. and Sidorov S. New methods for chaotic dynamics. World Scientific, 2006.
42. Nasiri S. arXiv: quant-ph/0511125.
43. Nasiri S., Sobouti Y., and Taati F. arXiv: quant-ph/0605129.
44. Nelson E. Quantum fluctuations. Princeton Univ. Press, 1985; Dynamical theory of Brownian motion. Princeton Univ. Press, 1967.
45. Nemytskii V. and Stepanov V. Qualitative theory of differential equations. Dover, 1989.
46. Nottale L. Fractal space-time and microphysics: Toward a theory of scale relativity. World Scientific, 1993.
47. Nottale L., Célérier M., and Lehner T. arXiv: hep-th/0407093.
48. Nottale L. and Célérier M. arXiv: quant-ph/0711.2418.
49. Olavo L. *Physica A*, 1999, v. 262, 197–214; 1999, v. 267, 260–302; *Phys. Rev. E*, 2001, v. 64, 036125.
50. Ott E. Chaos in dynamical systems. Cambridge Univ. Press, 2002.
51. Pettini M. Geometry and topology in Hamiltonian dynamics and statistical mechanics. Springer, 2007.
52. Poincaré H. New methods of celestial mechanics. Vols. 1–3. Dover, 1957.
53. Reginatto M. arXiv: quant-ph/9909065; gr-qc/0501030; *Phys. Rev. A*, 1998, v. 58, 1775–1778.
54. Rowe D., Ryman A., and Rosensteel G. *Phys. Rev. A*, 1980, v. 22, 2362–2373.
55. Rumyantsev V. *Soviet Applied Math. and Mech.*, 1996, v. 60, 899–909; 1994, v. 58, 373–386.
56. Rusov V. arXiv: quant-ph/0804.1427.
57. Rusov V. and Vlasenko D. arXiv: quant-ph/08064050.
58. Schuch D. *Inter. Jour. Quant. Chem.*, 1989, v. 23, 59–72; 1992, v. 42, 663–683.
59. Soubouti Y. and Nasiri S. *Inter. Jour. Mod. Phys. B*, 1993, v. 7, 3255–3272.
60. Stanuykovitch P., Kolesnikov S., and Moskovkin V. In: *Problems of Space, Time, and Matter Theory*, Atomizdat, 1968.
61. Tsekov R. *Jour. Phys. A*, 1995, v. 28, L557–L561; 2007, v. 40, 10945–10947; arXiv: quant-ph/07111442, 0803.4409, and 0808.0326.
62. Wyatt R. Quantum dynamics with trajectories. Springer, 2005.
63. Zaslavsky G. Hamilton chaos and fractional dynamics. Oxford Univ. Press, 2005.