Solution of Einstein's Geometrical Gravitational Field Equations Exterior to Astrophysically Real or Hypothetical Time Varying Distributions of Mass within Regions of Spherical Geometry

Chifu Ebenezer Ndikilar* and Samuel Xede Kofi Howusu†

*Physics Department, Gombe State University, P.M.B. 127, Gombe, Gombe State, Nigeria
E-mail: ebenechifu@yahoo.com

†Physics Department, Kogi State University, Anyighba, Kogi State, Nigeria
E-mail: sxkhowusu@yahoo.co.uk

Here, we present a profound and complete analytical solution to Einstein's gravitational field equations exterior to astrophysically real or hypothetical time varying distributions of mass or pressure within regions of spherical geometry. The single arbitrary function f in our proposed exterior metric tensor and constructed field equations makes our method unique, mathematically less combersome and astrophysically satisfactory. The obtained solution of Einstein's gravitational field equations tends out to be a generalization of Newton's gravitational scalar potential exterior to the spherical mass or pressure distribution under consideration.

1 Introduction

After the publication of Einstein's geometrical gravitational field equations in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began [1]. In recent publications [2–4], we have presented a standard generalization of Schwarzschild's metric to obtain the mathematically most simple and astrophysically most satisfactory metric tensors exterior to various mass distributions within regions of spherical geometry. Our method of generating metric tensors for gravitational fields is unique as it introduces the dependence of the field on one and only one dependent function f and thus the geometrical field equations for a gravitational field exterior to any astrophysically real or hypothetical massive spherical body has only one unknown f.

In this article, the equation satisfied by the function f in the gravitational field produced at an external point by a time varying spherical mass distribution situated in empty space is considered and an analytical solution for it proposed. A possible astrophysical example of such a distribution is when one considers the vacuum gravitational field produced by a spherically symmetric star in which the material in the star experiences radial displacement or explosion.

2 Gravitational radiation and propagation field equation exterior to a time varying spherical mass distribution

The covariant metric tensor exterior to a homogeneous time varying distribution of mass within regions of spherical geometry [2] is

$$g_{00} = 1 + \frac{2}{c^2} f(t, r), \tag{2.1}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(t, r) \right]^{-1}, \tag{2.2}$$

$$g_{22} = -r^2, \tag{2.3}$$

$$g_{33} = -r^2 \sin^2 \theta, \tag{2.4}$$

$$g_{\mu\nu} = 0; \text{ otherwise}. \tag{2.5}$$

The corresponding contravariant metric tensor for this field, is then constructed trivially using the Quotient Theorem of tensor analysis and used to compute the affine coefficients, given explicitly as

$$\Gamma^0_{00} = \frac{1}{c^3} \left[1 + \frac{2}{c^2} f(t, r) \right]^{-1} \frac{\partial f(t, r)}{\partial t}, \tag{2.6}$$

$$\Gamma^0_{01} \equiv \Gamma^0_{10} = \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t, r) \right]^{-1} \frac{\partial f(t, r)}{\partial r}, \tag{2.7}$$

$$\Gamma^0_{11} = -\frac{1}{c^3} \left[1 + \frac{2}{c^2} f(t, r) \right]^{-3} \frac{\partial f(t, r)}{\partial t}, \tag{2.8}$$

$$\Gamma^1_{00} = \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t, r) \right] \frac{\partial f(t, r)}{\partial r}, \tag{2.9}$$

$$\Gamma^1_{01} \equiv \Gamma^1_{10} = -\frac{1}{c^3} \left[1 + \frac{2}{c^2} f(t, r) \right]^{-1} \frac{\partial f(t, r)}{\partial t}, \tag{2.10}$$

$$\Gamma^1_{11} = -\frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t, r) \right]^{-1} \frac{\partial f(t, r)}{\partial r}, \tag{2.11}$$

$$\Gamma^1_{22} = -r \left[1 + \frac{2}{c^2} f(t, r) \right], \tag{2.12}$$

$$\Gamma^1_{33} = -r \sin^2 \theta \left[1 + \frac{2}{c^2} f(t, r) \right], \tag{2.13}$$

$$R_{00} = \frac{4}{c^6} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-2} \left(\frac{\partial f}{\partial t}\right)^2 - \frac{1}{c^4} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial^2 f}{\partial t^2} -$$
$$- \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t,r)\right] \frac{\partial^2 f}{\partial r^2} - \frac{2}{rc^2} \left(1 + \frac{2}{c^2} f(t,r)\right) \frac{\partial f}{\partial r} \tag{2.18}$$

$$R_{11} = -\frac{4}{c^6} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-4} \left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{c^4} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-3} \frac{\partial^2 f}{\partial t^2} +$$
$$+ \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial^2 f}{\partial r^2} + \frac{2}{rc^2} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f}{\partial r} \tag{2.19}$$

$$R_{22} = \frac{2}{c^2} \left[1 + \frac{2}{c^2} f(t,r)\right] \tag{2.20}$$

$$R_{33} = \frac{2}{c^2} \sin^2\theta \left(r\frac{\partial f}{\partial r} + f(t,r)\right) \tag{2.21}$$

$$R_{\alpha\beta} = 0; \text{ otherwise} \tag{2.22}$$

$$R = \frac{8}{c^6} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-3} \left(\frac{\partial f}{\partial t}\right)^2 - \frac{2}{c^4} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-2} \frac{\partial^2 f}{\partial t^2} - \frac{2}{c^2} \frac{\partial^2 f}{\partial r^2} - \frac{8}{rc^2} \frac{\partial f}{\partial r} - \frac{4f(t,r)}{r^2 c^2} \tag{2.23}$$

$$\nabla^2 f(t,r) + \frac{\partial}{\partial t} \left\{\frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-2} \frac{\partial f(t,r)}{\partial t}\right\} = 0 \tag{2.25}$$

$$\nabla^2 f(t,r) + \frac{1}{c^2} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-2} \frac{\partial^2 f(t,r)}{\partial t^2} - \frac{4}{c^4} \left[1 + \frac{2}{c^2} f(t,r)\right]^{-3} \left(\frac{\partial f(t,r)}{\partial t}\right)^2 = 0 \tag{2.26}$$

$$\Gamma^2_{12} \equiv \Gamma^2_{21} \equiv \Gamma^3_{13} \equiv \Gamma^3_{31} = r^{-1}, \tag{2.14}$$

$$\Gamma^2_{33} = -\frac{1}{2} \sin 2\theta, \tag{2.15}$$

$$\Gamma^3_{23} \equiv \Gamma^3_{32} = \cot\theta, \tag{2.16}$$

$$\Gamma^\alpha_{\beta\gamma} = 0; \text{ otherwise}. \tag{2.17}$$

The Riemann-Christoffel or curvature tensor for the gravitational field is then constructed and the Ricci tensor obtained from it as (2.18)–(2.22).

From the Ricci tensor, we construct the curvature scalar R as (2.23).

Now, with the Ricci tensor and the curvature scalar, Einstein's gravitational field equations for a region exterior to a time varying spherical mass distribution is eminent. The field equations are given generally as

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0. \tag{2.24}$$

Substituting the expressions for the Ricci tensor, curvature scalar and the covariant metric tensor; the R_{22} and R_{33} equations reduce identically to zero. The R_{00} and R_{11} field equations reduce identically to the single equation (2.25), or

equivalently (2.26).

It is interesting and instructive to note that to the order of c^0, the geometrical wave equation (2.26) reduces to

$$\nabla^2 f(t,r) + \frac{\partial^2 f(t,r)}{\partial t^2} = 0. \tag{2.27}$$

Equation (2.27) admits a wave solution with a phase velocity v given as

$$v = i \text{ m s}^{-1}, \tag{2.28}$$

where $i = \sqrt{-1}$. Thus, such a wave exists only in imagination and is not physically or astrophysically real.

It is also worth noting that, to the order of c^2, the geometrical wave equation (2.26) reduces, in the limit of weak gravitational fields, to

$$\nabla^2 f(t,r) + \frac{1}{c^2} \frac{\partial^2 f(t,r)}{\partial t^2} = 0 \tag{2.29}$$

and equation (2.28) is the wave equation of a wave propagating with an imaginary speed ic in vacuum.

We now, present a profound and complete analytical solution to the field equation (2.26).

$$\frac{\partial^2}{\partial r^2} f\left(t, r\right) + \frac{2}{r}\frac{\partial}{\partial r} f\left(t, r\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left\{\left[1 - \frac{4}{c^2} f\left(t, r\right) + \frac{12}{c^4} f^2\left(t, r\right) + \ldots\right]\frac{\partial}{\partial t} f\left(t, r\right)\right\} = 0 \tag{3.1}$$

$$\frac{\partial^2}{\partial r^2} f\left(t, r\right) + \frac{2}{r}\frac{\partial}{\partial r} f\left(t, r\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f\left(t, r\right) + \frac{4}{c^4} f\left(t, r\right)\frac{\partial^2}{\partial t^2} f\left(t, r\right) + \frac{4}{c^4}\left[\frac{\partial}{\partial t} f\left(t, r\right)\right]^2 + \ldots = 0 \tag{3.2}$$

$$\frac{\partial^2}{\partial r^2} f\left(t, r\right) = \sum_{n=0}^{\infty}\left[R_n''\left(r\right) - \frac{2ni\omega}{c} R_n'\left(r\right) + \frac{n^2 i^2 \omega^2}{c^2} R_n\left(r\right)\right]\exp ni\omega\left(t - \frac{r}{c}\right) \tag{3.4}$$

$$\frac{2}{r}\frac{\partial}{\partial r} f\left(t, r\right) = \sum_{n=0}^{\infty}\frac{2}{r}\left(R_n'\left(r\right) - \frac{ni\omega}{r}\right)\exp ni\omega\left(t - \frac{r}{c}\right) \tag{3.5}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} f\left(t, r\right) = \frac{1}{c^2}\sum_{n=0}^{\infty} n^2 i^2 \omega^2 R_n \exp\ ni\omega\left(t - \frac{r}{c}\right) \tag{3.6}$$

$$f\left(t, r\right)\frac{\partial^2 f\left(t, r\right)}{\partial t^2} = i^2\omega^2 R_0 R_1 \exp i\omega\left(t - \frac{r}{c}\right) + \left[2^2 i^2\omega^2 R_0 R_2 + i\omega^2 R_1^2\right]\exp 2i\omega\left(t - \frac{r}{c}\right) +$$
$$+ \left[3^2 i^2\omega^2 R_0 R_3 + 2^2 i^2\omega^2 R_1 R_2 + i^2\omega^2 R_1 R_2\right]\exp 3i\omega\left(t - \frac{r}{c}\right) + \ldots \tag{3.7}$$

$$\left(\frac{\partial}{\partial t} f\left(t, r\right)\right)^2 = \left[i\omega R_1\left(r\right)\exp i\omega\left(t - \frac{r}{c}\right) + 2i\omega R_2\left(r\right)\exp 2i\omega\left(t - \frac{r}{c}\right) + \ldots\right]^2 \tag{3.8}$$

$$R_2''\left(r\right) + 2\left(\frac{1}{r} - \frac{2i\omega}{c}\right) R_2'\left(r\right) - \frac{4}{c}\left(\frac{i\omega}{r} + \frac{4\omega}{c^3} R_0\right) R_2\left(r\right) - \frac{8\omega^2}{c^4} R_1^2\left(r\right) = 0\,. \tag{3.13}$$

3 Formulation of analytical solution to Einstein's geometrical gravitational field equation

The field equation for the gravitational field exterior to a time varying mass distribution within regions of spherical geometry are found to be given equally as equation (2.25) or (2.26).

For small gravitational fields (weak fields), the geometrical wave equation (1.1) reduces to (3.1) or equally (3.2).

We now seek a possible solution of equation (3.2) in the form

$$f\left(t, r\right) = \sum_{n=0}^{\infty} R_n\left(r\right)\exp\ ni\omega\left(t - \frac{r}{c}\right), \tag{3.3}$$

where R_n are functions of r only. Thus, by evaluating the first and second partial derivatives of our proposed solution for $f\left(t, r\right)$ in equation (3.3); it can be trivially shown that the separate terms of our expanded field equation (3.2) can be written as (3.4), (3.5), (3.6), (3.7), and (3.8), where the primes on the function R denote differentiation with respect to r. Now, substituting equations (3.4) to (3.8) into our field equation (3.2) and equating coefficients on both sides yields the following:

Equating coefficients of $\exp(0)$ gives

$$R_0'' + \frac{2}{r} R_0' = 0\,. \tag{3.9}$$

Thus, we can conveniently choose the best astrophysical solution for equation (3.9) as

$$R_0\left(r\right) = -\frac{k}{r} \tag{3.10}$$

where $k = GM_0$; by deduction from Schwarzschild's metric and Newton's theory of gravitation; with G being the universal gravitational constant and M_0 the total mass of the spherical body. Thus at this level, we note that the field equation yields a value for the arbitrary function f in our field equal to that in Schwarzschild's field. This is profound and interesting indeed as the link between our solution, Schwarzschild's solution and Newton's dynamical theory of gravitation becomes quite clear and obvious.

Equating coefficients of $\exp i\omega\left(t - \frac{r}{c}\right)$ gives

$$R_1''\left(r\right) + 2\left(\frac{1}{r} - \frac{i\omega}{c}\right) R_1' +$$
$$+ \frac{2\omega}{c}\left(-\frac{i}{r} - \frac{2\omega}{c^3} R_0\right) R_1 = 0\,. \tag{3.11}$$

This is our exact differential equation for R_1 and it determines R_1 in terms of R_0. Thus, the solution admits an exact wave solution which reduces in the order of c^0 to:

$$f\left(t, r\right) \approx -\frac{k}{r}\exp i\omega\left(t - \frac{r}{c}\right). \tag{3.12}$$

Equating coefficients of $\exp 2i\omega\left(t - \frac{r}{c}\right)$ gives (3.13).

This is our exact equation for $R_2(r)$ in terms of $R_0(r)$ and $R_1(r)$. Similarly, all the other unknown functions $R_n(r)$, $n > 2$ are determined in terms of $R_0(r)$ by the other recurrence differential equations. Hence we obtain our unique astrophysically most satisfactory exterior solution of order c^4.

4   Conclusion

Interestingly, we note that the terms of our unique series solution (3.10), (3.11), (3.12) and (3.13) converge everywhere in the exterior space-time. Similarly, all the solutions of the other recurrence differential equations will also converge everywhere in the exterior space-time.

Instructively, we realize that our solution has a unique link to the pure Newtonian gravitational scalar potential for the gravitational field and thus puts Einstein's geometrical gravitational field on same footing with the Newtonian dynamical theory. This method introduces the dependence of geometrical gravitational field on one and only one dependent function f, comparable to one and only one gravitational scalar potential in Newton's dynamical theory of gravitation [4].

Hence, we have obtained a complete solution of Einstein's field equations in this gravitational field. Our metric tensor, which is the fundamental parameter in this field is thus completely defined. The door is thus open for the complete study of the motion of test particles and photons in this gravitational field introduced in the articles [5] and [6].

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