Orbits in Homogeneous Oblate Spheroidal Gravitational Space-Time

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The generalized Lagrangian in general relativistic homogeneous oblate spheroidal gravitational fields is constructed and used to study orbits exterior to homogenous oblate spheroids. Expressions for the conservation of energy and angular momentum for this gravitational field are obtained. The planetary equation of motion and the equation of motion of a photon in the vicinity of an oblate spheroid are derived. These equations have additional terms not found in Schwarzschild's space time.

1 Introduction

It is well known experimentally that the Sun and planets in the solar system are more precisely oblate spheroidal in geometry [1–6]. The oblate spheroidal geometries of these bodies have corresponding effects on their gravitational fields and hence the motion of test particles and photons in these fields.

It is also well known that satellite orbits around the Earth are governed by not only the simple inverse distance squared gravitational fields due to perfect spherical geometry. They are also governed by second harmonics (pole of order 3) as well as fourth harmonics (pole of order 5) of gravitational scalar potential not due to perfect spherical geometry. Therefore, towards the more precise explanation and prediction of satellite orbits around the Earth, Stern [3] and Garfinkel [4] introduced the method of quadratures for approximating the second harmonics of the gravitational scalar potential of the Earth due to its spheroidal Earth. This method was improved by O'Keefe [5]. Then in 1960, Vinti [6] suggested a general mathematical form of the gravitational scalar potential of the spheroidal Earth and how to estimate some of the parameters in it for use in the study of satellite orbits. Recently [1], an expression for the scalar potential exterior to a homogeneous oblate spheroidal body was derived. Most recently, Ioannis and Michael [3] proposed the Sagnac interferometric technique as a way of detecting corrections to the Newton's gravitational scalar potential exterior to an oblate spheroid.

In this article, we formulate the metric tensor for the gravitational field exterior to massive homogeneous oblate spheroidal bodies as a direct extension of Schwarzschild's metric. This metric tensor is then used to study orbits in homogeneous oblate spheroidal space time.

2 Metric tensor exterior to a homogeneous oblate spheroid

The invariant world line element in the exterior region of all possible static spherical distributions of mass is given [1, 7] as

$$c^2 d\tau^2 = c^2 \left[1 + \frac{2f(r,\theta,\phi)}{c^2}\right] dt^2 - \\ - \left[1 + \frac{2f(r,\theta,\phi)}{c^2}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \quad (2.1)$$

where $f(r,\theta,\phi)$ is a generalized arbitrary function determined by the distribution of mass or pressure and possess all the symmetries of the mass distribution. It is a well known fact of general relativity that $f(r,\theta,\phi)$ is approximately equal to Newton's gravitational scalar potential in the space-time exterior to the mass or pressure distributions within regions of spherical geometry [1, 7]. For a static homogeneous spherical body ("Schwarzschild's body") the arbitrary function takes the form $f(r)$.

Now, let "Schwarzschild's body" be transformed, by deformation, into an oblate spheroidal body in such a way that its density and total mass remain the same and its surface parameter is given in oblate spheroidal coordinates [1] as

$$\xi = \xi_0 \,;\ \text{constant}. \quad (2.2)$$

The general relativistic field equation exterior to a homogeneous static oblate spheroidal body is tensorially equivalent to that of a static homogeneous spherical body ("Schwarzschild's body") [1, 7] hence, is related by the transformation from spherical to oblate spheroidal coordinates. Therefore, to get the corresponding invariant world line element in the exterior region of a static homogeneous oblate spheroidal mass, we first replace the arbitrary function in Schwarzschild's field, $f(r)$ by the corresponding arbitrary function exterior to static homogenous oblate spheroidal bodies, $f(\eta,\xi)$. Thus, the function $f(\eta,\xi)$ is approximately equal to the gravitational potential exterior to a homogeneous spheroid. The gravitational scalar potential exterior to a homogeneous static oblate spheroid [1] is given as

$$f(\eta,\xi) = B_0 Q_0(-i\xi) P_0(\eta) + B_2 Q_2(-i\xi) P_2(\eta) \quad (2.3)$$

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{2.10}$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1} + \frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right] \tag{2.11}$$

$$g_{12} \equiv g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right] \tag{2.12}$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1} + \frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right] \tag{2.13}$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2) \tag{2.14}$$

$$g_{\mu\nu} = 0; \text{ otherwise} \tag{2.15}$$

$$g^{00} = \left[1 + \frac{2}{c^2} f(\eta, \xi)\right]^{-1} \tag{2.16}$$

$$g^{11} = \frac{-\left(1-\eta^2\right)\left(1+\xi^2-\eta^2\right)\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(1-\eta^2)+\xi^2(1+\xi^2)\right]^2} \tag{2.17}$$

$$g^{12} \equiv g^{21} = \frac{-\eta\xi\left(1-\eta^2\right)\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(1-\eta^2)+\xi^2(1+\xi^2)\right]^2} \tag{2.18}$$

$$g^{22} = \frac{-\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)\left[\xi^2\left(1+\xi^2\right)+\eta^2\left(1-\eta^2\right)\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\left[\eta^2(1-\eta^2)+\xi^2(1+\xi^2)\right]^2} \tag{2.19}$$

$$g^{33} = -\left[a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{-1} \tag{2.20}$$

$$g^{\mu\nu} = 0; \text{ otherwise} \tag{2.21}$$

where Q_O and Q_2 are the Legendre functions linearly independent to the Legendre polynomials P_0 and P_2 respectively. B_0 and B_2 are constants.

Secondly, we transform coordinates from spherical to oblate spheroidal coordinates;

$$(ct, r, \theta, \phi) \to (ct, \eta, \xi, \phi) \tag{2.4}$$

on the right hand side of equation (2.1).

From the relation between spherical polar coordinates and Cartesian coordinates as well as the relation between oblate spheroidal coordinates and Cartesian coordinates [8] it can be shown trivially that

$$r(\eta, \xi, \phi) = a(1+\xi^2-\eta^2)^{\frac{1}{2}} \tag{2.5}$$

and

$$\theta(\eta, \xi, \phi) = \cos^{-1}\left[\frac{\eta\xi}{(1+\xi^2-\eta^2)^{\frac{1}{2}}}\right] \tag{2.6}$$

where a is a constant parameter. Therefore,

$$dr = a(1+\xi^2-\eta^2)^{-\frac{1}{2}}(\xi d\xi - \eta d\eta) \tag{2.7}$$

and

$$d\theta = -\frac{\xi(1+\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}(1+\xi^2-\eta^2)}\,d\eta - \\ -\frac{\eta(1-\eta^2)^{\frac{1}{2}}}{(1+\xi^2)^{\frac{1}{2}}(1+\xi^2-\eta^2)}\,d\xi\,. \tag{2.8}$$

Also,

$$\sin^2\theta = \frac{(1+\xi^2)(1-\eta^2)}{(1+\xi^2-\eta^2)}\,. \tag{2.9}$$

Substituting equations (2.5), (2.7), (2.8) and (2.9) into equation (2.1) and simplifying yields the following components of the covariant metric tensor in the region exterior to a

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{d\eta}{d\tau}\right)^2 - 2g_{12}\left(\frac{d\eta}{d\tau}\right)\left(\frac{d\xi}{d\tau}\right) - g_{22}\left(\frac{d\xi}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right)^{\frac{1}{2}} \tag{3.1}$$

$$L = \frac{1}{c}\left[-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)\dot{t}^2 - \frac{a^2\xi^2}{1+\xi^2}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\dot{\xi}^2 + a^2(1+\xi^2)\dot{\phi}^2\right]^{\frac{1}{2}} \tag{3.2}$$

static homogeneous oblate spheroid in oblate spheroidal coordinates (2.10)–(2.15).

The covariant metric tensor, equations (2.10) to (2.15) is the most fundamental geometric parameter required to study general relativistic mechanics in static homogeneous oblate spheroidal gravitational fields. The covariant metric tensor obtained above for gravitational fields exterior to oblate spheroidal masses has two additional non-zero components g_{12} and g_{21} not found in Schwarzschild field [7]. Thus, the extension from Schwarzschild field to homogeneous oblate spheroidal gravitational fields has produced two additional non-zero tensor components and hence this metric tensor field is unique. This confirms the assertion that oblate spheroidal gravitational fields are more complex than spherical fields and hence general relativistic mechanics in this field is more involved [6].

The contravariant metric tensor for this gravitational field is found to be given explicitly as (2.16)–(2.21).

It can be shown that the coefficients of affine connection for the gravitational field exterior to a homogenous oblate spheroidal mass are given in terms of the metric tensors for the gravitational field as

$$\Gamma^0_{01} \equiv \Gamma^0_{10} = \frac{1}{2}\,g^{00}g_{00,1}\,, \tag{2.22}$$

$$\Gamma^0_{02} \equiv \Gamma^0_{20} = \frac{1}{2}\,g^{00}g_{00,2}\,, \tag{2.23}$$

$$\Gamma^1_{00} = -\frac{1}{2}\,g^{11}g_{00,1} - \frac{1}{2}\,g^{12}g_{00,2}\,, \tag{2.24}$$

$$\Gamma^1_{11} = \frac{1}{2}\,g^{11}g_{11,1} + \frac{1}{2}\,g^{12}\left(2g_{12,1} - g_{11,2}\right)\,, \tag{2.25}$$

$$\Gamma^1_{12} \equiv \Gamma^1_{21} = \frac{1}{2}\,g^{11}g_{11,2} + \frac{1}{2}\,g^{12}g_{22,1}\,, \tag{2.26}$$

$$\Gamma^1_{22} = \frac{1}{2}\,g^{11}\left(2g_{12,2} - g_{22,1}\right) + \frac{1}{2}\,g^{12}g_{22,2}\,, \tag{2.27}$$

$$\Gamma^1_{33} = -\frac{1}{2}\,g^{11}g_{33,1} - \frac{1}{2}\,g^{12}g_{33,2}\,, \tag{2.28}$$

$$\Gamma^2_{00} = -\frac{1}{2}\,g^{21}g_{00,1} - \frac{1}{2}\,g^{22}g_{00,2}\,, \tag{2.29}$$

$$\Gamma^2_{11} = \frac{1}{2}\,g^{21}g_{11,1} + \frac{1}{2}\,g^{22}\left(2g_{12,1} - g_{11,2}\right)\,, \tag{2.30}$$

$$\Gamma^2_{12} \equiv \Gamma^2_{21} = \frac{1}{2}\,g^{21}g_{11,2} + \frac{1}{2}\,g^{22}g_{22,1}\,, \tag{2.31}$$

$$\Gamma^2_{22} = \frac{1}{2}\,g^{21}\left(2g_{12,2} - g_{22,1}\right) + \frac{1}{2}\,g^{22}g_{22,2}\,, \tag{2.32}$$

$$\Gamma^2_{33} = -\frac{1}{2}\,g^{21}g_{33,1} - \frac{1}{2}\,g^{22}g_{33,2}\,, \tag{2.33}$$

$$\Gamma^3_{13} \equiv \Gamma^3_{31} = \frac{1}{2}\,g^{33}g_{33,1}\,, \tag{2.34}$$

$$\Gamma^3_{23} \equiv \Gamma^3_{32} = \frac{1}{2}\,g^{33}g_{33,2}\,, \tag{2.35}$$

$$\Gamma^\delta_{\alpha\beta} = 0;\ \text{otherwise}, \tag{2.36}$$

where comma as in usual notation denotes partial differentiation with respect to $\eta(1)$ and $\xi(2)$.

3 Conservation of total energy and angular momentum

Many physical theories start by specifying the Lagrangian from which everything flows. We would adopt the same attitude with gravitational fields exterior to homogenous oblate spheroidal masses. The Lagrangian in the space time exterior to our mass or pressure distribution is defined explicitly in oblate spheroidal coordinates using the metric tensor as (3.1) [7, 9], where τ is the proper time.

For orbits confined to the equatorial plane of a homogenous oblate spheroidal mass [1, 8]; $\eta \equiv 0$ (or $d\eta \equiv 0$) and substituting the explicit expressions for the components of metric tensor in the equatorial plane yields (3.2), where the dot denotes differentiation with respect to proper time.

It is well known that the gravitational field is a conservative field. The Euler-Lagrange equations for a conservative system in which the potential energy is independent of the generalized velocities is written as [7, 9];

$$\frac{\partial L}{\partial x^\alpha} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) \tag{3.3}$$

but

$$\frac{\partial L}{\partial x^0} \equiv \frac{\partial L}{\partial t} = 0 \tag{3.4}$$

and thus from equation (3.3), we deduce that

$$\frac{\partial L}{\partial \dot{t}} = \text{constant}. \tag{3.5}$$

From equation (3.3), it can be shown using equation (3.5) that

$$\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)\dot{t} = k, \quad \dot{k} = 0 \tag{3.6}$$

where k is a constant. This is the law of conservation of energy in the equatorial plane of the gravitational field exterior to an oblate spheroidal mass [7, 9].

The law of conservation of total energy, equation (3.6) can also be obtained by constructing the coefficients of affine connection for this gravitational field and evaluating the time equation of motion for particles of non-zero rest masses. The general relativistic equation of motion for particles of non-zero rest masses in a gravitational field are given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \tag{3.7}$$

where $\Gamma^\mu_{\nu\lambda}$ are the coefficients of affine connection for the gravitational field.

Setting $\mu = 0$ in equation (3.7) and substituting the explicit expressions for the affine connections Γ^0_{01} and Γ^0_{02} gives

$$\ddot{t} + \frac{2}{c^2}\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1} \times \times \left(\dot{\eta}\frac{\partial f(\eta,\xi)}{\partial \eta} + \dot{\xi}\frac{\partial f((\eta,\xi))}{\partial \xi}\right)\dot{t} = 0\,. \tag{3.8}$$

Integrating equation (3.8) yields

$$\dot{t} = k\left(1 + \frac{2}{c^2}f(\eta,\xi)\right)^{-1} \tag{3.9}$$

where k is a constant of integration. Thus, the two methods yield same results.

Also, the Lagrangian for this gravitational field is invariant to azimuthal angular rotation and hence angular momentum is conserved, thus;

$$\frac{\partial L}{\partial \phi} = 0 \tag{3.10}$$

and from Lagrange's equation of motion,

$$\left(1 + \xi^2\right)\dot{\phi} = l, \quad \dot{l} = 0 \tag{3.11}$$

where l is a constant. This is the law of conservation of angular momentum in the equatorial plane of the gravitational field exterior to a static homogeneous oblate spheroidal body.

This expression can also be obtained by solving the azimuthal equation of motion for particles of non-zero rest masses in this gravitational field. Setting $\mu = 3$ in equation (3.7) and substituting the relevant affine connection coefficients gives the azimuthal equation of motion as

$$\frac{d}{d\tau}\left(\ln\dot{\phi}\right) + \frac{d}{d\tau}\left(\ln\left(1-\eta^2\right)\right) + + \frac{d}{d\tau}\left(\ln\left(1+\xi^2\right)\right) = 0\,. \tag{3.12}$$

Thus, by integrating equation (3.12), it can be shown that the azimuthal equation of motion for our gravitational field is given as

$$\dot{\phi} = \frac{l}{\left(1-\eta^2\right)\left(1+\xi^2\right)}\,, \tag{3.13}$$

where l is a constant of motion. l physically corresponds to the angular momentum and hence equation (3.13) is the Law of Conservation of angular momentum in this gravitational field [7, 9]. It does not depend on the gravitational potential and is of same form as that obtained in Schwarzschild's Field and Newton's dynamical theory of gravitation [7, 9]. Note that equation (3.13) reduces to equation (3.11) if the particles are confined to move in the equatorial plane of the oblate spheroidal mass.

4 Orbits in homogeneous oblate spheroidal gravitational fields

It is well known [7, 9] that the Lagrangian $L = \epsilon$, with $\epsilon = 1$ for time like orbits and $\epsilon = 0$ for null orbits. Setting $L = \epsilon$ in equation (3.2), substituting equations (3.6) and (3.11) and simplifying yields;

$$\frac{a^2\xi^2}{\left(1+\xi^2\right)}\dot{\xi}^2 + \frac{a^2l^2}{\left(1+\xi^2\right)}\left(1 + \frac{2}{c^2}f(\eta,\xi)\right) - - 2\epsilon^2 f(\eta,\xi) = c^2\epsilon^2 + 1\,. \tag{4.1}$$

In most applications of general relativity, we are more interested in the shape of orbits (that is, as a function of the azimuthal angle) than in their time history [7]. Hence, it is instructive to transform equation (4.1) into an equation in terms of the azimuthal angle ϕ. Now, let us consider the following transformation;

$$\xi = \xi(\phi) \text{ and } u(\phi) = \frac{1}{\xi(\phi)}\,, \tag{4.2}$$

thus,

$$\dot{\xi} = -\frac{l}{1+u^2}\frac{du}{d\phi}\,. \tag{4.3}$$

Now, imposing equations (4.2) and (4.3) on equation (4.1) and simplifying yields (4.4). Differentiating equation (4.4) gives (4.5).

For time like orbits $(\epsilon = 1)$, equation (4.5) reduces to (4.6).

This is the planetary equation of motion in this gravitational field. It can be solved to obtain the perihelion precision of planetary orbits. It has additional terms (resulting from the oblateness of the body), not found in the corresponding equation in Schwarzschild's field [7].

Light rays travel on null geodesics $(\epsilon = 0)$ and hence equation (4.5) becomes (4.7).

In the limit of special relativity, some terms in equation (4.7) vanish and the equation becomes (4.8).

Equation (4.7) is the photon equation of motion in the vicinity of a static massive homogenous oblate spheroidal body. The equation contains additional terms not found in the corresponding equation in Schwarzschild's field. The solution of the special relativistic case, equation (4.8) can be used to solve the general relativistic equation, (4.7). This can

$$\frac{1}{\left(1+u^2\right)^3}\left(\frac{du}{d\phi}\right)^2+\frac{u^2}{1+u^2}\left(1+\frac{2}{c^2}f(u)\right)-\frac{2\epsilon^2 f(u)}{a^2l^2}=\frac{c^2\epsilon^2+1}{a^2l^2}\,. \tag{4.4}$$

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)\left(1+\frac{2}{c^2}f(u)\right)=\left(\frac{1+u^2}{acl}\right)^2\left(a^2c^2l^2-\epsilon^2-\epsilon^2u^2\right)\frac{d}{du}f(u)\,. \tag{4.5}$$

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)\left(1+\frac{2}{c^2}f(u)\right)=\left(\frac{1+u^2}{acl}\right)^2\left(a^2c^2u^2-1-u^2\right)\frac{d}{du}f(u)\,. \tag{4.6}$$

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)\left(1+\frac{2}{c^2}f(u)\right)=\frac{u^2}{c^2}\left(1+u^2\right)^2\frac{d}{du}f(u)\,. \tag{4.7}$$

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{\left(u+u^2\right)}{2}\left(u^2-u+2\right)=0\,. \tag{4.8}$$

be done by taking the general solution of equation (4.7) to be a perturbation of the solution of equation (4.8). The immediate consequence of this analysis is that it will produce an expression for the total deflection of light grazing a massive oblate spheroidal body such as the Sun and the Earth.

5 Remarks and conclusion

The immediate consequences of the results obtained in this article are:

1. The equations derived are closer to reality than those in Schwarzschild's gravitational field. In Schwarzschild's space time, the Sun is assumed to be a static perfect sphere. The Sun has been proven to be oblate spheroidal in shape and our analysis agrees perfectly with this shape;
2. The planetary equation of motion and the photon equation of motion have additional spheroidal terms not found in Schwarzschild's field. This equations are opened up for further research work and astrophysical interpretation.
3. In approximate oblate spheroidal gravitational fields, the arbitrary function $f(\eta,\xi)$ can be conveniently equated to the gravitational scalar potential exterior to an oblate spheroid [7]. Thus for these fields, the complete solutions for our equations of motion can be constructed;
4. Einstein's field equations constructed using our metric tensor has only one unknown, $f(\eta,\xi)$. A solution of these field equations will give explicit expressions for the function, $f(\eta,\xi)$ which can then be used in our equations of motion.

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