Astrophysically Satisfactory Solutions to Einstein's R-33 Gravitational Field Equations Exterior/Interior to Static Homogeneous Oblate Spheroidal Masses

Chifu Ebenezer Ndikilar

Physics Department, Gombe State University, P.M.B. 127, Gombe, Gombe State, Nigeria

E-mail: ebenechifu@yahoo.com

In this article, we formulate solutions to Einstein's geometrical field equations derived using our new approach. Our field equations exterior and interior to the mass distribution have only one unknown function determined by the mass or pressure distribution. Our obtained solutions yield the unknown function as generalizations of Newton's gravitational scalar potential. Thus, our solution puts Einstein's geometrical theory of gravity on same footing with Newton's dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newton's gravitational scalar potential. Our results in this article are of much significance as the Sun and planets in the solar system are known to be more precisely oblate spheroidal in geometry. The oblate spheroidal geometries of these bodies have effects on their gravitational fields and the motions of test particles and photons in these fields.

1 Introduction

After the publication of A. Einstein's geometrical theory of gravitation in 1915/1916, the search for exact solutions to its inherent geometrical field equations for various mass distributions began [1]. Four well known approaches have so far been proposed.

The first approach is to seek a mapping under which the metric tensor assumed a simple form, such as the vanishing of the off-diagonal components.With sufficiently clever assumptions of this sort, it is often possible to reduce the Einstein field equations to a much simpler system of equations, even a single partial differential equation (as in the case of stationary axisymmetric vacuum solutions, which are characterised by the Ernst equation) or a system of ordinary differential equations (this led to the first exact analytical solution — the famous Schwarzschild's solution [2]). A special generalization of the Schwarzschild's metric is the Kerr metric. This metric describes the geometry of space time around a rotating massive body.

The second method assumes that the metric tensor has symmetries-assumed forms of the Killing vectors. This led to the solution found by Weyl and Levi-Civita [3–6]. The third approach required that the metric tensor leads to a particular type of the classifications of Weyl and Riemann — Christoffel tensors. These are often stated interms of Petrov classification of the possible symmetries of the Weyl tensor or the Segre classification of the possible symmetries of the Ricci tensor. This leds to plane fronted wave solutions [3–6]. It is worth remarking that even after the symmetry reductions in the three methods above, the reduced system of equations is often difficult to solve. The fourth approach is to seek Taylor series expansion of some initial value hyper surface, subject to consistent initial value data. This method has not proved successful in generating solutions [3–6].

Recently [7–12], we introduced our own method and approach to formulation of exact analytical solutions as an extension of Schwarzschild's method. In this article, we show how exact analytical solutions of order c^{-2} (where c is the speed of light in vacuum) can be constructed in gravitational fields interior and exterior to static homogeneous oblate spheroids placed in empty space. For the sake of mathematical convenience we choose to use the 3rd (R_{33}) field equation [7].

2 Exterior field equation

The covariant metric tensor in the gravitational field of a static homogeneous oblate spheroid in oblate spheroidal coordinates (η, ξ, ϕ) has been obtained [7, 12] as

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right), \tag{2.1}$$

$$\begin{aligned} g_{11} = &-\frac{a^2}{1+\xi^2-\eta^2} \times \\ &\times \left[\eta^2 \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right], \end{aligned} \tag{2.2}$$

$$g_{12} \equiv g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1 - \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1}\right], \tag{2.3}$$

$$\begin{aligned} g_{22} = &-\frac{a^2}{1+\xi^2-\eta^2} \times \\ &\times \left[\xi^2 \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right], \end{aligned} \tag{2.4}$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2), \tag{2.5}$$

$$g_{\mu\nu} = 0; \ \text{otherwise}, \tag{2.6}$$

$$g^{00} = \left[1 + \frac{2}{c^2}\, f\left(\eta, \xi\right)\right]^{-1} \tag{2.7}$$

$$g^{11} = \frac{-\left(1-\eta^2\right)\left(1+\xi^2-\eta^2\right)\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2} \tag{2.8}$$

$$g^{12} \equiv g^{21} = \frac{-\eta\xi\left(1-\eta^2\right)\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)\left[1-\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2} \tag{2.9}$$

$$g^{22} = \frac{-\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)\left[\xi^2\left(1+\xi^2\right)+\eta^2\left(1-\eta^2\right)\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\right]}{a^2\left(1+\frac{2}{c^2}\, f\left(\eta,\xi\right)\right)^{-1}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2} \tag{2.10}$$

$$g^{33} = -\left[a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{-1} \tag{2.11}$$

$$g^{\mu\nu} = 0 \text{ otherwise} \tag{2.12}$$

$$\begin{aligned}
&- g^{00}g^{00}g_{00,12} - g^{00}g^{11}{}_{,2}\, g_{00,1} - g^{00}g^{12}g_{00,22} - g^{00}g^{12}{}_{,2}\, g_{00,2} - g^{00}g^{21}{}_{,2}\, g_{00,1} - \\
&- g^{00}g^{22}g_{00,22} - g^{00}g^{22}{}_{,2}\, g_{00,2} - g^{00}g^{11}g^{22}g_{00,1}g_{22,1} - g^{00}g^{11}g^{33}g_{00,1}g_{33,1} - \\
&- g^{00}g^{12}g^{12}g_{00,2}g_{12,1} - g^{00}g^{12}g^{33}g_{00,2}g_{33,1} - g^{00}g^{11}g^{21}g_{00,1}g_{11,2} - \\
&- g^{00}g^{21}g^{12}g_{00,1}g_{12,2} - g^{00}g^{21}g^{33}g_{00,1}g_{33,2} - \tfrac{1}{2}\, g^{00}g^{22}g^{33}g_{00,2}g_{33,2} - \\
&- g^{00}g^{11}g_{00,11} - g^{11}g^{00}{}_{,1}\, g_{00,1} - g^{11}g^{12}{}_{,1}\, g_{11,2} - 2g^{11}g^{22}g_{22,11} - g^{11}g^{22}{}_{,1}\, g_{22,1} - \\
&- g^{11}g^{33}{}_{,1}\, g_{33,1} + 4g^{11}g^{21}g_{11,12} + g^{11}g^{21}{}_{,2}\, g_{11,1} + 4g^{11}g^{22}g_{12,12} - 2g^{11}g^{22}g_{11,22} + \\
&+ g^{11}g^{22}{}_{,2}\, g_{11,2} - g^{11}g^{11}g^{22}g_{11,2}g_{12,1} + g^{00}g^{11}g^{22}g_{00,2}g_{12,1} - g^{00}g^{11}g^{22}g_{00,2}g_{11,2} + \\
&+ g^{11}g^{22}g^{22}g_{12,1}g_{22,1} - \tfrac{1}{2}\, g^{11}g^{22}g^{22}g_{11,2}g_{22,1} + g^{11}g^{22}g^{33}g_{12,1}g_{33,2} - \\
&- \tfrac{1}{2}\, g^{11}g^{22}g^{33}g_{11,2}g_{33,2} + \tfrac{1}{2}\, g^{11}g^{21}g^{22}g_{11,1}g_{22,1} - g^{11}g^{21}g^{21}g_{11,2}g_{11,2} + \\
&+ g^{11}g^{21}g^{12}g_{11,2}g_{12,1} - g^{22}g^{00}g_{00,12} - g^{22}g^{00}{}_{,2}\, g_{00,1} - g^{22}g^{11}{}_{,2}\, g_{00,1} - g^{22}g^{11}{}_{,2}\, g_{11,2} - \\
&- g^{22}g^{12}{}_{,2}\, g_{22,1} - g^{22}g^{33}g_{33,22} - g^{22}g^{33}{}_{,2}\, g_{33,2} + g^{22}g^{12}{}_{,1}\, g_{22,2} + g^{22}g^{11}{}_{,1}\, g_{12,2} - \\
&- g^{22}g^{11}{}_{,1}\, g_{22,1} + \tfrac{1}{2}\, g^{00}g^{22}g^{22}g_{00,2}g_{22,2} - g^{12}g^{12}g^{22}g_{11,2}g_{22,2} + \\
&+ \tfrac{1}{2}\, g^{00}g^{12}g^{22}g_{00,1}g_{22,2} + \tfrac{1}{2}\, g^{11}g^{12}g^{22}g_{11,1}g_{22,2} + \tfrac{1}{2}\, g^{12}g^{22}g^{33}g_{22,2}g_{33,1} - \\
&- g^{11}g^{22}g^{22}g_{12,2}g_{22,1} + g^{00}g^{11}g^{22}g_{00,1}g_{12,2} + g^{11}g^{11}g^{22}g_{11,1}g_{12,2} + g^{11}g^{22}g^{33}g_{12,2}g_{33,1} + \\
&+ \tfrac{1}{2}\, g^{11}g^{22}g^{22}g_{11,2}g_{22,2} + g^{12}g^{12}g^{22}g_{12,2} + \tfrac{1}{2}\, g^{12}g^{12}g^{22}g_{22,1}g_{22,1} - \\
&- \tfrac{1}{2}\, g^{12}g^{22}g^{33}g_{12,2}g_{33,2} + \tfrac{1}{2}\, g^{12}g^{22}g^{33}g_{22,1}g_{33,2} + g^{33}g^{11}{}_{,1}\, g_{33,1} + g^{33}g^{12}{}_{,1}\, g_{33,2} + \\
&+ 2g^{21}g^{33}g_{33,12} - g^{33}g^{21}{}_{,2}\, g_{33,1} - g^{22}g^{33}g_{33,22} - g^{33}g^{22}{}_{,2}\, g_{33,2} - g^{11}g^{22}g^{33}g_{22,1}g_{33,1} - \\
&- g^{12}g^{12}g^{33}g_{12,1}g_{33,2} - \tfrac{1}{2}\, g^{12}g^{22}g^{33}g_{22,1}g_{33,2} - g^{21}g^{21}g^{33}g_{12,2}g_{33,1} - \\
&- \tfrac{1}{2}\, g^{21}g^{22}g^{33}g_{22,2}g_{33,1} - \tfrac{1}{2}\, g^{00}g^{22}g^{33}g_{00,2}g_{33,2} - \tfrac{1}{2}\, g^{11}g^{22}g^{33}g_{11,2}g_{33,2} + \\
&+ g^{00}g^{12}g_{00,12} + 2g^{12}g^{00}{}_{,2}\, g_{00,1} + 2g^{12}g^{11}{}_{,2}\, g_{11,1} + 4g^{12}g^{12}g_{12,12} + 4g^{12}g^{12}{}_{,2}\, g_{12,1} - 2g^{12}g^{12}g_{11,22} - \\
&- 2g^{12}g^{12}{}_{,2}\, g_{11,2} + 2g^{12}g^{33}{}_{,2}\, g_{33,1} + 2g^{12}g^{11}{}_{,1}\, g_{11,2} + 2g^{12}g^{12}g_{22,11} + 2g^{12}g^{12}{}_{,1}\, g_{22,1} + \\
&+ g^{00}g^{11}g^{12}g_{00,1}g_{11,2} + g^{00}g^{12}g^{12}g_{00,1}g_{22,1} + g^{00}g^{12}g^{21}g_{00,2}g_{11,2} + g^{11}g^{12}g^{21}g_{11,2}g_{11,2} + \\
&+ g^{12}g^{12}g^{12}g_{11,2}g_{22,1} + g^{12}g^{21}g^{33}g_{11,2}g_{33,2} + g^{12}g^{12}g^{33}g_{22,1}g_{33,1} - g^{11}g^{12}g^{21}g_{11,1}g_{12,2} - \\
&- g^{12}g^{21}g^{12}g_{11,1}g_{22,2} - g^{12}g^{21}g^{22}g_{12,1}g_{22,2} - g^{11}g^{12}g^{22}g_{11,2}g_{22,1} + g^{12}g^{12}g^{22}g_{11,2}g_{22,2} = 0
\end{aligned} \tag{2.18}$$

and the contravariant metric tensor is as shown in formulas (2.7)–(2.12), where $f(\eta, \xi)$ is an unknown function determined by the mass distribution. From this covariant metric tensor, we can then construct our field equations for the gravitational field after formulating the Coefficients of affine connection, Riemann Christoffel tensor, Ricci tensor and the Einstein tensor [7–12]. After the above steps, it can be shown that the exterior R_{33} field equation in this gravitational field is given as;

$$R_{33} - \frac{1}{2} R\, g_{33} = 0 . \tag{2.13}$$

or more explicitly interms of the affine connections, Ricci tensor and covariant metric tensor as;

$$\begin{aligned}&-\Gamma^1_{33}\Gamma^0_{10} - \Gamma^2_{33}\Gamma^0_{20} - \Gamma^1_{33,1} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33}\Gamma^1_{21}\Gamma^3_{31}\Gamma^1_{33} - \\ &\quad - \Gamma^2_{33,2} - \Gamma^1_{33}\Gamma^2_{12} - \Gamma^2_{33}\Gamma^2_{22} + \Gamma^3_{32}\Gamma^2_{33} - \frac{1}{2} R\, g_{\alpha\beta} = 0\end{aligned} \tag{2.14}$$

with the symbols and numbers having their usual meaning and

$$R = g^{00}R_{00} + g^{11}R_{11} + 2g^{12}R_{12} + g^{22}R_{22} + g^{33}R_{33} . \tag{2.15}$$

Now, multiplying equation (2.13) by $2g^{33}$ and using the fact that $g^{33}g_{33} = 1$ yields

$$2g^{33}R_{33} - R = 0 . \tag{2.16}$$

Writing the expression for the curvature scalar,R as in equation (2.15) gives;

$$\begin{aligned}&-g^{00}R_{00} - g^{11}R_{11} - 2g^{12}R_{12} - \\ &\qquad - g^{22}R_{22} + g^{33}R_{33} = 0 .\end{aligned} \tag{2.17}$$

Writing the various terms of the field equation (2.17) explicitly in terms of the metric tensor gives our field equation explicitly as (2.18).

Now, we realize that our covariant metric tensor (2.1)–(2.6) can be written equally as

$$g_{\mu\nu}(\eta, \xi) = h_{\mu\nu}(\eta, \xi) + f_{\mu\nu}(\eta, \xi) , \tag{2.19}$$

where $h_{\mu\nu}$ are the well known pure empty space components and $f_{\mu\nu}$ are the contributions due to the oblate spheroidal mass distribution. Consequently, as the mass distribution decays out; $f_{\mu\nu} \to 0$ and hence $g_{\mu\nu} \to h_{\mu\nu}$. Therefore, the metric tensor reduces to the pure empty space metric tensor as the distribution of mass decays out. Also,

$$g^{\mu\nu}(\eta, \xi) = h^{\mu\nu}(\eta, \xi) + f^{\mu\nu}(\eta, \xi) , \tag{2.20}$$

where $h^{\mu\nu}$ are the well known pure empty space components and $f^{\mu\nu}$ are the contributions due to the oblate spheroidal mass distribution. Thus it can be shown that for this field, the non zero metric components can be written as;

$$h_{00} = 1 , \tag{2.21}$$

$$h_{11} = -\frac{a^2\left(\eta^2 + \xi^2\right)}{1 - \eta^2} , \tag{2.22}$$

$$h_{22} = -\frac{a^2\left(\eta^2 + \xi^2\right)}{1 + \eta^2} , \tag{2.23}$$

$$h_{33} = -a^2\left(1 + \eta^2\right)\left(1 + \xi^2\right) , \tag{2.24}$$

$$f_{00} = \frac{2}{c^2} f , \tag{2.25}$$

$$f_{11} = -\frac{a^2\eta^2}{\left(1 - \eta^2 + \xi^2\right)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^n} f^n , \tag{2.26}$$

$$f_{12} \equiv f_{21} = -\frac{a^2\eta\xi}{\left(1 - \eta^2 + \xi^2\right)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^n} f^n , \tag{2.27}$$

$$f_{22} = -\frac{a^2\xi^2}{\left(1 - \eta^2 + \xi^2\right)} \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^n} f^n , \tag{2.28}$$

also,

$$h^{00} = \frac{1}{h_{00}} , \tag{2.29}$$

$$h^{11} = \frac{1}{h_{11}} , \tag{2.30}$$

$$h^{22} = \frac{1}{h_{22}} , \tag{2.31}$$

$$h^{33} = \frac{1}{h_{33}} , \tag{2.32}$$

$$f^{00} = \sum_{n=1}^{\infty} \binom{-1}{n} \frac{2^n}{c^n} f^n , \tag{2.33}$$

$$f^{11} = -\frac{f_{11}}{(h_{11})^2} + 0\left(c^{-4}\right) , \tag{2.34}$$

$$f^{12} \equiv f^{21} = -\frac{f_{12}}{h_{11}h_{22}} + 0\left(c^{-4}\right) , \tag{2.35}$$

$$f^{22} = -\frac{f_{22}}{(h_{22})^2} + 0\left(c^{-4}\right) . \tag{2.36}$$

To begin the explicit formulation of the R_{33} field equation we note, first of all, that all the terms of order c^0 cancel out identically since the empty space time metric tensor $h_{\mu\nu}$ independently satisfies the homogeneous R_{33} field equation. Therefore the lowest order of terms we expect in the exterior R_{33} field equation is c^{-2}. Hence in order to formulate the exterior R_{33} field equation of order c^{-2}, let us decompose our covariant metric tensor $g_{\mu\nu}$ into pure empty space part $h_{\mu\nu}$ (of order c^0 only) and the nonempty space part $f_{\mu\nu}$ (of order c^{-2} or higher). Similarly, let our contravariant metric tensor $g^{\mu\nu}$ be decomposed into pure empty space part $h^{\mu\nu}$ (of order c^0 only) and the nonempty space part $f^{\mu\nu}$ (of order c^{-2} or higher). Substituting explicit expressions for equations (2.19) and (2.20) into equation (2.18) and neglecting all terms of order c^0, the exterior R_{33} field equation can be written as (2.37), where the coefficients are given as (2.38)–(2.58).

$$\begin{aligned}
&S_1(\eta,\xi)\, f_{22,11} + S_2(\eta,\xi)\, f_{00,11} + S_3(\eta,\xi)\, f_{12,12} + S_4(\eta,\xi)\, f_{00,12} + S_5(\eta,\xi)\, f_{11,22} + \\
&+ S_6(\eta,\xi)\, f_{00,22} + S_7(\eta,\xi)\, f_{00,1} + S_8(\eta,\xi)\, f_{12,1} + S_9(\eta,\xi)\, f_{22,1} + S_{10}(\eta,\xi)\, f^{11}{}_{,1} + \\
&+ S_{11}(\eta,\xi)\, f^{12}{}_{,1} + S_{12}(\eta,\xi)\, f^{22}{}_{,1} + S_{13}(\eta,\xi)\, f_{00,2} + S_{14}(\eta,\xi)\, f_{11,2} + S_{15}(\eta,\xi)\, f_{12,2} + \\
&+ S_{16}(\eta,\xi)\, f_{22,2} + S_{17}(\eta,\xi)\, f^{12}{}_{,2} + S_{18}(\eta,\xi)\, f^{22}{}_{,2} + S_{19}(\eta,\xi)\, f^{11} + S_{20}(\eta,\xi)\, f^{12} + \\
&+ S_{21}(\eta,\xi)\, f^{22} = 0
\end{aligned} \tag{2.37}$$

$$S_1(\eta,\xi) = -2h^{11}h^{22} \tag{2.38}$$

$$S_2(\eta,\xi) = -h^{11} \tag{2.39}$$

$$S_3(\eta,\xi) = 4h^{11}h^{22} \tag{2.40}$$

$$S_4(\eta,\xi) = -h^{11} - h^{22} \tag{2.41}$$

$$S_5(\eta,\xi) = -2h^{11}h^{22} \tag{2.42}$$

$$S_6(\eta,\xi) = -h^{22} \tag{2.43}$$

$$S_7(\eta,\xi) = -h^{11}{}_{,2} - h^{11}h^{22}h_{22,1} - h^{11}h^{33}h_{33,1} \tag{2.44}$$

$$S_8(\eta,\xi) = h^{11}h^{22}{}_{,2} + h^{11}h^{22}\left(h^{33}h_{33,2} + h^{22}h_{22,1} - h^{11}h_{11,2}\right) \tag{2.45}$$

$$S_9(\eta,\xi) = -h^{11}h^{22}\left(h^{33}h_{33,1} + \tfrac{1}{2}\, h^{22}h_{11,2}\right) - h^{22}h^{11}{}_{,1} - h^{11}h^{22}{}_{,1} \tag{2.46}$$

$$S_{10}(\eta,\xi) = -h^{22}h_{22,1} + h^{33}h_{33,1} \tag{2.47}$$

$$S_{11}(\eta,\xi) = h^{22}h_{22,2} + h^{33}h_{33,2} \tag{2.48}$$

$$S_{12}(\eta,\xi) = -h^{11}h_{22,1} \tag{2.49}$$

$$S_{13}(\eta,\xi) = -h^{22}{}_{,2} - h^{22}h^{33}h_{33,2} + \tfrac{1}{2}\, h^{22}h^{22}h_{22,2} - h^{11}h^{22}h_{11,2} \tag{2.50}$$

$$S_{14}(\eta,\xi) = -h^{11}h^{22}{}_{,2} - h^{22}h^{11}{}_{,2} + \frac{1}{2}\, h^{11}h^{22}\left(-h^{22}h_{22,1} + h^{22}h_{22,2} - h^{33}h_{33,2}\right) \tag{2.51}$$

$$S_{15}(\eta,\xi) = h^{22}h^{11}{}_{,1} + h^{11}h^{22}\left(h^{33}h_{33,1} + h^{11}h_{11,1} - h^{22}h_{22,1}\right) \tag{2.52}$$

$$S_{16}(\eta,\xi) = \tfrac{1}{2}\, h^{11}h^{22}h^{22}h_{11,2} \tag{2.53}$$

$$S_{17}(\eta,\xi) = h^{11}h_{11,1} - h^{22}h_{22,1} - h^{33}h_{33,1} \tag{2.54}$$

$$S_{18}(\eta,\xi) = -h^{11}h_{11,1} - h^{33}h_{33,2} \tag{2.55}$$

$$\begin{aligned}
&S_{19}(\eta,\xi) = -h^{22}h^{33}h_{33,2} - h^{22}h^{33}h_{33,1} + h^{22}h^{22}h_{11,2}h_{22,2} - \\
&- h^{22}h^{33}h_{11,2}h_{33,2} - \tfrac{1}{2}\, h^{22}h^{22}h_{11,2}h_{22,1} - h^{22}{}_{,2}\, h_{11,2} - 2h^{22}h_{11,22} - h^{33}{}_{,1}\, h_{33,1} - \\
&- h^{22}{}_{,1}\, h_{22,1} - 2h^{22}h_{22,11}
\end{aligned} \tag{2.56}$$

$$\begin{aligned}
&S_{20}(\eta,\xi) = 4h^{11}h_{11,12} + \tfrac{1}{2}\, h^{11}h^{22}h_{11,1}\left(h_{22,1} + h_{22,2}\right) + \\
&+ \tfrac{1}{2}\, h^{22}h^{33}h_{22,1}h_{33,2} + 2h^{33}h_{33,12} - \tfrac{1}{2}\, h^{22}h^{33}h_{22,1} + 2h^{11}{}_{,2}\, h_{11,1} + 2h^{33}{}_{,2}\, h_{33,1} + \\
&+ 2h^{11}{}_{,1}\, h_{11,2} - h^{11}h^{22}h_{11,2}h_{22,1}
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
&S_{21}(\eta,\xi) = -\tfrac{1}{2}\, h^{11}h^{33}h_{33,2} + h^{11}h^{22}h_{11,2}h_{22,2} - h^{11}h^{22}h_{11,2}h_{22,1} - \\
&- \tfrac{1}{2}\, h^{11}h^{33}h_{11,2}h_{33,2} - 2h^{33}h_{33,22} - h^{11}h^{33}h_{33,1} - h^{11}{}_{,2}\, h_{11,2} - h^{33}{}_{,2}\, h_{33,2} - \\
&- h^{11}{}_{,1}\, h_{22,1} - 2h^{11}h_{22,11} - 2h^{11}h_{11,22}\,.
\end{aligned} \tag{2.58}$$

$$K_1(\eta,\xi)\, f_{\eta\eta} + K_2(\eta,\xi)\, f_{\eta\xi} + K_3(\eta,\xi)\, f_{\xi\xi} + K_4(\eta,\xi)\, f_{\eta} + K_5(\eta,\xi)\, f_{\xi} + K_6(\eta,\xi)\, f = 0 \tag{2.59}$$

$$K_1(\eta,\xi) = \frac{2\left(1-\eta^2\right)\left(1-\eta^2+\xi^2\right) - 2a^4\xi^2\left(\eta^2+\xi^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)\left(1-\eta^2+\xi^2\right)} \tag{2.60}$$

$$K_2(\eta,\xi) = \frac{4\left(\eta^2+\xi^2\right)\left(1-\eta^2+\xi^2\right) - 8\eta\xi\left(1-\eta^4\right)}{a^2c^2\left(\eta^2+\xi^2\right)^2\left(1-\eta^2+\xi^2\right)} \tag{2.61}$$

$$K_3(\eta,\xi) = \frac{2\left(1+\eta^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)} \tag{2.62}$$

$$\begin{aligned} K_4(\eta,\xi) = {} & \frac{-8a^2\eta\xi^2 S_1(\eta,\xi) + 2a^2\xi^2 S_9(\eta,\xi)}{c^2\left(1-\eta^2+\xi^2\right)^2} - \frac{8\xi\left(1-\eta^4\right)\left(1-\eta^2-\xi^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)^2\left(1-\eta^2+\xi^2\right)^2} + \\ & + \frac{2a^2\eta^2 S_{10}(\eta,\xi)}{c^2\left(h_{11}\right)^2\left(1-\eta^2+\xi^2\right)} + \frac{2a^2\eta\xi S_{11}(\eta,\xi)}{c^2 h_{11}h_{22}\left(1-\eta^2+\xi^2\right)} - \frac{2a^2\xi^2 S_{12}(\eta,\xi)}{c^2\left(h_{22}\right)^2\left(1-\eta^2+\xi^2\right)} \end{aligned} \tag{2.63}$$

$$\begin{aligned} K_5(\eta,\xi) = {} & \frac{-8\xi\left(1+\eta^2+\xi^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)^2\left(1-\eta^2+\xi^2\right)^2} + \frac{16\eta^2\xi\left(1-\eta^4\right)}{a^2c^2\left(\eta^2+\xi^2\right)^2\left(1-\eta^2+\xi^2\right)} + \frac{2\left[S_9(\eta,\xi)+S_{13}(\eta,\xi)\right]}{c^2} + \\ & + \frac{2a^2\eta\xi\left[-S_8(\eta,\xi) - S_{15}(\eta,\xi) + \xi S_{16}(\eta,\xi)\right]}{c^2\left(h_{11}\right)^2\left(1-\eta^2+\xi^2\right)} + + \frac{2a^2\eta\xi S_{17}(\eta,\xi)}{c^2 h_{11}h_{22}\left(1-\eta^2+\xi^2\right)} - \frac{2a^2\xi^2 S_{17}(\eta,\xi)}{c^2\left(h_{22}\right)^2\left(1-\eta^2+\xi^2\right)} \end{aligned} \tag{2.64}$$

$$\begin{aligned} K_6(\eta,\xi) = {} & \frac{-4a^2\xi^2\left(1+3\eta^2+\xi^2\right)}{c^2\left(1-\eta^2+\xi^2\right)^3} - \frac{8\left(1-\eta^4\right)\left(1-\eta^4-\xi^4-10\eta^2\xi^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)^2\left(1-\eta^2+\xi^2\right)^3} + \frac{8\eta^2\left(1-\eta^2\right)}{a^2c^2\left(\eta^2+\xi^2\right)\left(1-\eta^2+\xi^2\right)^2} + \\ & + \frac{2a^2\left[-\eta\left(1+\eta^2+\xi^2\right)S_8(\eta,\xi) + 2\eta\xi^2 S_9(\eta,\xi) - \eta\left(1-\eta^2-\xi^2\right)S_{15}(\eta,\xi) + 2\xi\left(1-\eta^2\right)S_{16}(\eta,\xi)\right]}{c^2\left(1-\eta^2+\xi^2\right)^2} - \\ & - \frac{-2a^2\eta^2\left(h_{11}\right)^2{}_{,1} S_{10}(\eta,\xi)}{c^2\left(h_{11}\right)^4\left(1-\eta^2+\xi^2\right)} + \frac{2a^2\eta^2\left[2\left(1+\xi^2\right)S_{10}(\eta,\xi) - S_{19}(\eta,\xi)\right]}{c^2\left(h_{11}\right)^2\left(1-\eta^2+\xi^2\right)} + \frac{2a^2\xi\left(1+\eta^2+\xi^2\right)S_{11}(\eta,\xi)}{c^2 h_{11}h_{22}} - \\ & - \frac{2a^2\eta\xi\left[\left(h_{11}h_{22}\right)_{,1} S_{11}(\eta,\xi)\right]}{c^2\left(h_{11}h_{22}\right)^2\left(1-\eta^2+\xi^2\right)} + \frac{2a^2\xi\left[\left(h_{22}\right)^2{}_{,1} S_{12}(\eta,\xi) - \xi\left(h_{22}\right)^2{}_{,2} S_{18}(\eta,\xi)\right]}{c^2\left(h_{22}\right)^4\left(1-\eta^2+\xi^2\right)} + \\ & + \frac{2a^2\left[\left(1-\eta^2-\xi^2\right)S_{17}(\eta,\xi) - \eta\xi\left(1-\eta^2+\xi^2\right)S_{20}(\eta,\xi)\right]}{c^2\left(h_{11}h_{22}\right)\left(1-\eta^2+\xi^2\right)^2} - \\ & - \frac{2a^2\xi\left[\eta\xi S_{12}(\eta,\xi) - 2\left(1-\xi^2\right)^2 S_{18}(\eta,\xi) - \xi S_{21}(\eta,\xi)\right]}{c^2\left(h_{22}\right)^2\left(1-\eta^2+\xi^2\right)^2} \end{aligned} \tag{2.65}$$

Substituting the explicit expressions for the nonempty space parts $f_{\mu\nu}$ and $f^{\mu\nu}$ into equation (2.37), simplifying and grouping like terms yields (2.59), where the terms consisting it are (2.60)–(2.65).

Equation (2.59) is thus our exact explicit R_{33} exterior field equation to the order c^{-2}. We can now conveniently formulate astrophysical solutions for the equation in the next section; which are convergent in the exterior space time of a homogeneous massive oblate spheroid placed in empty space.

3 Formulation of R-33 exterior solution

In the exterior oblate spheroidal space time [7]:

$$\xi \geqslant \xi_0 \ \text{ and } \ -1 \leqslant \eta \leqslant 1; \ \ \xi_0 = constant \tag{3.1}$$

Let us now seek a solution for the R_{33} field equation (2.59) in the form of the power series

$$f(\eta,\xi) = \sum_{n=0}^{\infty} P_n^+(\xi)\,\eta^n. \tag{3.2}$$

where P_n^+ is a function to be determined for each value of n. Substituting the proposed function into the field equation and taking into consideration the fact that $\{\eta^n\}_{n=0}^{\infty}$ is a linearly independent set, we can thus equate the coefficients of η^n on both sides of the obtained equation. From the coefficients of η^0, we obtain the equation

$$\begin{aligned} 0 = {} & K_1(\eta,\xi)\, P_2^+(\xi) + K_2(\eta,\xi)\left[P_1^+(\xi)\right]' + \\ & + K_3(\eta,\xi)\left[P_0^+(\xi)\right]'' + K_4(\eta,\xi)\, P_1^+(\xi) + \\ & + K_5(\eta,\xi)\left[P_0^+(\xi)\right]' + K_6(\eta,\xi)\, P_0^+(\xi) \end{aligned} \tag{3.3}$$

or more explicitly

$$\begin{aligned} & 0 = a^3\xi^3\left(1+\xi^2-a^2\xi^4\right)P_2^+(\xi) + \\ & + 2a^3\xi^3\left(1+\xi^2\right)^2\left[P_1^+(\xi)\right]' + a^3\xi^2\left(1+\xi^2\right)P_1^+(\xi) + \\ & + a^3\xi^3\left(1+\xi^2\right)^2\left[P_0^+(\xi)\right]'' + \left(1+\xi^2\right)\times \\ & \times\left(-1-2a^2\xi^2-\xi^2-a^2\xi^3+4a^2\xi^5\right)\left[P_0^+(\xi)\right]' + \\ & + \left[2a^3\xi\left(4-2\xi^2-a^4\xi^4-a^4\xi^6\right)\right]P_0^+(\xi). \end{aligned} \tag{3.4}$$

Equation (3.4) is the first recurrence differential equation for the unknown functions. All the other recurrence differential equations can thus follow, yielding infinitely many recurrence differential equations that can be used to determine all the unknown functions.

The following profound points can thus be made. Firstly, equation (3.4) determines P_2^+ in terms of P_0^+ and P_1^+, similarly the other recurrence differential equations will determine the other unknown functions $P_3^+, \ldots$ in terms of P_0^+ and P_1^+. Secondly, we note that we have the freedom to choose our arbitrary functions to satisfy the physical requirements or needs of any particular distribution or area of application.

Let us now recall that for any gravitational field [7, 13],

$$g_{00} \cong 1 + \frac{2}{c^2}\,\Phi \tag{3.5}$$

where Φ is Newton's gravitational scalar potential for the field under consideration. Thus we can then deduce that the unknown function in our field equation can be given approximately as

$$f\left(\eta,\xi\right) \cong \Phi^{+}\left(\eta,\xi\right) \tag{3.6}$$

where $\Phi^{+}\left(\eta,\xi\right)$ is Newton's gravitational scalar potential exterior to a homogeneous oblate spheroidal mass. Recently [14], it has been shown that

$$\Phi^{+}\left(\eta,\xi\right) = B_0 Q_0\left(-i\xi\right) P_0\left(\eta\right) + B_2 Q_2\left(-i\xi\right) P_2\left(\eta\right) \tag{3.7}$$

where Q_0 and Q_2 are the Legendre functions linearly independent to the Legendre polynomials P_0 and P_2 respectively; B_0 and B_2 are constants.

Let us now seek our exact analytical exterior solution (3.4) to be as close as possible to the approximate exterior solution (3.7). Now since the approximate solution possesses no term in the first power of η, let us choose

$$P_0^{+}\left(\xi\right) = B_0 Q_0\left(-i\xi\right) P_0 + B_2 Q_2\left(-i\xi\right) \tag{3.8}$$

and

$$P_1^{+}\left(\xi\right) \equiv 0\,. \tag{3.9}$$

Hence, we can write P_2^+ in terms of P_0^+ as

$$\begin{aligned} P_2^{+}\left(\xi\right) = &-\frac{\left(1+\xi^2\right)^2}{\left(1+\xi^2-a^2\xi^4\right)}\left[P_0^{+}\left(\xi\right)\right]'' - \\ &-\frac{2\left(1+\xi^2\right)\left(3a^2\xi^2+4a^2\xi^5-\xi^2-1\right)}{a^2\xi^3}\left[P_0^{+}\left(\xi\right)\right]' - \\ &-2\left[\frac{1-2a^3\xi^2-a^7\xi^4-a^7\xi^6+a^3}{a^3\xi^2\left(1+\xi^2-a^2\xi^4\right)}\right]P_0^{+}\left(\xi\right). \end{aligned} \tag{3.10}$$

We now remark that the first three terms of our series solution converge everywhere in the exterior space time. We also remark that our solution of order c^0 may be written as

$$f\left(\eta,\xi\right) = \Phi^{+}\left(\eta,\xi\right) + \Phi_0^{+}\left(\eta,\xi\right) \tag{3.11}$$

where $\Phi^{+}\left(\eta,\xi\right)$ is the corresponding Newtonian gravitational scalar potential given by (3.7) and $\Phi_0^{+}\left(\eta,\xi\right)$ is the pure Einsteinian or general relativistic or post Newtonian correction of order c^0.

Hence, we deduce that our exterior analytical solution is of the general form

$$f\left(\eta,\xi\right) = \Phi^{+}\left(\eta,\xi\right) + \Phi_0^{+}\left(\eta,\xi\right) + \sum_{n=1}^{\infty}\Phi_{2n}^{+}\left(\eta,\xi\right). \tag{3.12}$$

4 Formulation of interior R-33 field equation and solution

For the interior space time, Einstein's field equations are well known to be given as;

$$R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} = -\frac{8\pi G}{c^4}\,T_{\mu\nu} \tag{4.1}$$

where $T_{\mu\nu}$ is the energy momentum tensor.

Now, let us assume that the homogeneous mass distribution is a "perfect fluid". Thus, we can define the energy momentum tensor as

$$T_{\mu\nu} = \left(\rho_0 + P_0\right) u_\mu u_\nu - P_0\, g_{\mu\nu} \tag{4.2}$$

where ρ_0 is the proper mass density and P_0 is the proper pressure and u_μ is the velocity four vector. Hence, the five non trivial interior field equations can be written as;

$$R_{00} - \frac{1}{2}\,R\,g_{00} = -\frac{8\pi G}{c^4}\left[\left(\rho_0 + P_0\right) u_0 u_0 - P_0\, g_{00}\right], \tag{4.3}$$

$$R_{11} - \frac{1}{2}\,R\,g_{11} = \frac{8\pi G}{c^4}\,P_0\, g_{11}\,, \tag{4.4}$$

$$R_{12} - \frac{1}{2}\,R\,g_{12} = \frac{8\pi G}{c^4}\,P_0\, g_{12}\,, \tag{4.5}$$

$$R_{22} - \frac{1}{2}\,R\,g_{22} = \frac{8\pi G}{c^4}\,P_0\, g_{22}\,, \tag{4.6}$$

$$R_{33} - \frac{1}{2}\,R\,g_{33} = \frac{8\pi G}{c^4}\,P_0\, g_{33}\,. \tag{4.7}$$

Now, we formulate the solution of (4.7). For the sake of mathematical convenience, we assume in this article that the pressure is negligible compared to the mass density and hence

$$P_0 \cong 0\,. \tag{4.8}$$

Multiplying equation (4.7) by $2g^{33}$ and using the fact that $g^{33}g_{33} = 1$ we obtain precisely as in the section 2;

$$-g^{00}R_{00} - g^{11}R_{11} - g^{22}R_{22} + g^{33}R_{33} - 2g^{12}R_{12} = 0\,. \tag{4.9}$$

Similarly, we obtain the interior equation explicitly as

$$\begin{aligned} &K_1\left(\eta,\xi\right) f_{\eta\eta} + K_2\left(\eta,\xi\right) f_{\eta\xi} + K_3\left(\eta,\xi\right) f_{\xi\xi} + \\ &+ K_4\left(\eta,\xi\right) f_{\eta} + K_5\left(\eta,\xi\right) f_{\xi} + K_6\left(\eta,\xi\right) f = 0\,. \end{aligned} \tag{4.10}$$

We now remark that, for the interior field we are required to formulate interior solutions of (4.10) convergent in the range

$$0 \leqslant \xi \leqslant \xi_0\,, \qquad -1 \leqslant \eta \leqslant 1. \tag{4.11}$$

Let us thus seek a series solution of the form;

$$f^-(\eta, \xi) = \sum_{n=0}^{\infty} Z_n^-(\eta)\, \xi^n. \tag{4.12}$$

where Z_n^- are unknown functions to be determined. Now, using the fact that $\{\xi^n\}_{n=0}^{\infty}$ is a linearly independent set, we may equate coefficients on both sides and hence obtain the equations satisfied by Z_n^-. We proceed similarly as in the case of the exterior solution to obtain recurrence differential equations that determine the explicit expression for our exact analytical solution. Equating the coefficients of ξ^0, we obtain the first recurrence differential equation as

$$\begin{aligned} &K_1(\eta, \xi)\left[Z_0^-(\eta)\right]'' + K_2(\eta, \xi)\left[Z_1^-(\eta)\right]' + \\ &+ K_3(\eta, \xi)\, Z_2^-(\eta) + K_4(\eta, \xi)\left[Z_0^-(\eta)\right]' + \\ &+ K_5(\eta, \xi)\, Z_1^-(\eta) + K_6(\eta, \xi)\, Z_0^-(\eta) = 0\,. \end{aligned} \tag{4.13}$$

In a similar manner, the other recurrence differential equations follow.

We can now proceed as in the previous section to choose the most astrophysically satisfactory solution to be as close as possible to the approximate solution. The gravitational scalar potential interior to a homogeneous oblate spheroid is well known [14] to be given as

$$\Phi^-(\eta, \xi) = \left[A_0 - \frac{1}{2}\, A_2 P_2(\eta)\right] - 3/2 A_2 P_2(\eta)\, \xi^2, \tag{4.14}$$

where P_2 is Legendre's polynomial of order 2 and A_0, A_2, are constants.

Since (4.14) converges for all values in the interval (4.11), it is very satisfactory for us to choose;

$$Z_0^-(\eta) = A_0 - \frac{1}{2}\, A_2 P_2(\eta) \tag{4.15}$$

and

$$Z_1^-(\eta) \equiv 0\,. \tag{4.16}$$

Thus the first recurrence differential equation determines Z_2^- in terms of Z_0^-. Similarly, all the other recurrence differential equations will determine all the other functions in terms of Z_0^-. Hence we obtain our unique astrophysically most satisfactory interior solution. It is obvious that this unique solution will converge, precisely as the first two terms. Moreover, it is obvious that our unique solution reduces to the corresponding pure Newtonian gravitational scalar potential in the limit of the first two terms. This solution may be written as

$$f^-(\eta, \xi) = \Phi^-(\eta, \xi) + \Phi_0^-(\eta, \xi) \tag{4.17}$$

where $\Phi^-(\eta, \xi)$ is the corresponding Newtonian gravitational scalar potential given by (4.14) and $\Phi_0^-(\eta, \xi)$ is the pure instructively Einstenian (or general relativistic or post Newtonian correction) of order c^0.

Proceeding exactly as above we may derive all the corresponding solutions of all the other non-trivial interior Einstein's field equations for the sake of mathematical completeness, comparison with those of the R_{33} equation and theoretical applications where and when necessary in Physics. It is clearly obvious how to extend the derivation of the interior Einstein field equations above to include any given pressure function $P_0(\eta, \xi)$, wherever and whenever necessary and useful in physical theory.

5 Conclusions

Interestingly, the single dependent function f in our mathematically most simple and astrophysically most satisfactory solution turns out as the corresponding well known pure Newtonian exterior/interior gravitational scalar potential augmented by hitherto unknown pure Einsteinian (or general relativistic or post-Newtonian) gravitational scalar potential terms of orders c^0, c^{-2}, c^{-4}, … Hence, this article has revealed a hitherto unknown sense in which the exterior/interior Einstein's geometrical gravitational field equations are obtained as a generalization or completion of Newton's dynamical gravitational field equations.

With the formulation of our mathematically most simple and astrophysically most satisfactory solutions in this article, the way is opened up for the formulation and solution of the general relativistic equations of motion for all test particles in the gravitational fields of all static homogeneous distributions of mass within oblate spheroidal regions in the universe. And precisely because these equations contain the pure Newtonian as well as post-Newtonian gravitational scalar potentials all their predictions shall be most naturally comparable to the corresponding predictions from the pure Newtonian theory. This is most satisfactory indeed.

It is now obvious how our work in this article may by emulated to (i) derive a mathematically most simple structure for all the metric tensors in the space times exterior or interior to any distribution of mass within any region having any of the 14 regular geometries in nature, (ii) formulate all the nontrivial Einstein geometrical gravitational field equations and derive all their general solutions and (iii) derive astrophysically most satisfactory unique solutions for application to the motions of all test particles and comparison with corresponding pure Newtonian results and applications. Therefore our goal in this article has been completely achieved: to use the case of a spheroidal distribution of mass to show how the much vaunted Einstein's geometrical gravitational field equations may be solved exactly and analytically for any given distribution of mass within any region having any geometry.

Finally, we conclude that at very long last — 93 years after the publication of the laws of General Relativity by Einstein in 1915 — we have found a method and process for (1) deriving a unique approximate astrophysically most satisfactory solutions for the space times exterior and interior to every distribution of mass within any region having any of the 14 regular geometries in nature, in terms of the corresponding pure Newton's gravitational scalar potential, without even formulating the field equation; and (2) systematically formulating and solving the geometrical gravitational field equations in the space times of all distributions of mass in nature.

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