The Kruskal-Szekeres "Extension": Counter-Examples

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The Kruskal-Szekeres "coordinates" are said to "extend" the so-called "Schwarzschild solution", to remove an alleged "coordinate singularity" at the event horizon of a black hole at $r = 2m$, leaving an infinitely dense point-mass singularity at "the origin" $r = 0$. However, the assumption that the point at the centre of spherical symmetry of the "Schwarzschild solution" is at "the origin" $r = 0$ is erroneous, and so the Kruskal-Szekeres "extension" is invalid; demonstrated herein by simple counter-examples.

1 Introduction

According to the astrophysical scientists the solution for Einstein's static vacuum gravitational field must satisfy the following conditions [1–11]:

(a) It must be static; i.e. all the components of the metric tensor must be independent of time and the geometry must be unchanged under time reversal;

(b) It must be spherically symmetric;

(c) It must satisfy the equations $R_{\mu\nu} = 0$; no matter present;

(d) It must be asymptotically Minkowski spacetime.

The so-called "Schwarzschild solution" (which is not in fact Schwarzschild's solution at all) is (using $c = 1$ and $G = 1$),

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \\ - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{1}$$

The astrophysical scientists merely inspect this line-element and thereby assert that there are singularities at $r = 2m$ and at $r = 0$ [3, 4, 7, 9]; the former they claim to be a "coordinate" or "removable" singularity which denotes the "radius" of an event horizon of a black hole of mass m located at the "real" or "physical" singularity at $r = 0$. They call $r = 2m$ the "Schwarzschild radius" and $r = 0$ "the origin".

It is plainly evident that metric (1) changes its signature from $(+, -, -, -)$ to $(-, +, -, -)$ when $0 < r < 2m$, despite the fact that metric (1) is supposed to be a generalisation of Minkowski spacetime, described by (using $c = 1$),

$$ds^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{2}$$

$$0 \leqslant r < \infty,$$

which has *fixed* signature $(+, -, -, -)$; and so there is in fact no possibility for Minkowski spacetime to change signature from $(+, -, -, -)$ to $(-, +, -, -)$ [5]. Consequently, $0 \leqslant r < 2m$ on Eq. (1) has no counterpart in Minkowski spacetime. Nonetheless, although the astrophysical scientists *deliberately* fix the signature to $(+, -, -, -)$ at the very outset of their derivation of Eq. (1) [1–9, 11, 12], in order to maintain the signature of Minkowski spacetime, they nonetheless allow a change of signature to occur in Eq. (1) to $(-, +, -, -)$ [3, 4, 7, 9, 10, 13, 14] according to their assumption that $0 \leqslant r < \infty$ applies to Eq. (1); in direct violation of their initial construction. They then invoke a complicated "change of coordinates" to make the singularity at $r = 2m$ disappear; the Kruskal-Szekeres coordinates [3, 4, 9, 13, 14]. The astrophysical scientists merely assume that the point at the centre of spherical symmetry of the manifold described by Eq. (1) is located at "the origin", $r = 0$. To justify their assumptions on the variable r, which they evidently conceive of as radial distance in "Schwarzschild" spacetime (e.g. "Schwarzschild radius"), they also claim that because the Riemann tensor scalar curvature invariant (the "Kretschmann scalar"), given by $f = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, is finite at $r = 2m$ and unbounded at $r = 0$, there must be a "real" singularity only at $r = 0$. This argument they apply *post hoc*, without any proof that General Relativity requires such a condition on the Kretschmann scalar.

The assumption that "the origin" $r = 0$ marks the point at the centre of spherical symmetry of the manifold described by (1) is demonstrably false. Furthermore, a geometry is fully determined by its line-element [5, 15], not by arbitrary values assigned to any curvature invariant which is calculated from the line-element itself in the first place. Given a line-element of the form of Eq. (1) the admissible values of its associated curvature invariants and the location of its centre of spherical symmetry are fully fixed by it, and so they cannot be arbitrarily determined by simple inspection and *ad hoc* assumptions.

To illustrate the inadmissibility of the methods applied by the astrophysical scientists in their analysis of Eq. (1), I shall adduce counter-examples that satisfy all the required conditions (a)–(d) and their additional assumptions concerning r and the Kretschmann scalar, but nevertheless clearly contradict the claims made by the astrophysical scientists in relation to Eq. (1). By these counter-examples I will demonstrate, by application of the very same methods the astrophysical scientists apply to Eq. (1), that there are "spacetimes" in which the singularity of a "black hole" is encountered before the event horizon, and that this event horizon can be "removed" by application of the Kruskal-Szekeres method. I will also give an example that not only inverts the locations of the event horizon and the singularity, relative to Eq. (1), but also locates them both at places other than the "origin" $r = 0$ at which the metric is well-defined. It is in fact rather easy to generate an infinite number of such counter-examples (but just one is sufficient to invalidate the Kruskal-Szekeres "extension").

These counter-examples amplify the fact that the usual assumption on Eq. (1) that "the origin" $r = 0$, simply by inspection, marks the point at the centre of spherical symmetry of the manifold it describes, is entirely false, and that the additional assumption that the Kretschmann scalar must be unbounded at a "real" or "physical" singularity is also false. This should not really be all that surprising, bearing in mind that the usual assumptions are just that, for which no proofs have ever been produced. It follows that there is no black hole associated with Eq. (1), and that the Kruskal-Szekeres "extension" is fallacious.

It is easily proven that r in Eq. (1) is the inverse square root of the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section [16, 17, 19]. Being directly related to a curvature invariant, its values are fixed by the intrinsic geometry, fixed by the form of the line-element itself, as are all other related curvature invariants.

It must also be remarked that the transition from Minkowski spacetime to Schwarzschild spacetime involves no matter whatsoever. Therefore Schwarzschild spacetime is not in fact a generalisation of the laws of Special Relativity; only a generalisation of the geometry of Minkowski spacetime. The speed of light in vacuum, c, which appears in the Minkowski line-element is not a photon; it is a speed, the maximum speed with which a point is permitted to move in Minkowski spacetime. Similarly, the appearance of the constant c in Schwarzschild spacetime does not imply the presence of a photon there either. A photon must be present *a priori* to assign the speed c to the photon. Neither photons nor masses are present, by construction, in the generalisation of Minkowski spacetime to Schwarzschild spacetime, owing to the equations $R_{\mu\nu} = 0$ according to condition (c). Minkowski spacetime is not Special Relativity — the latter requires the *a priori* presence of matter, the former does not. Schwarzschild spacetime is a spacetime that by construction contains no matter, and hence no sources.

2 Counter-examples

Consider the metric

$$ds^2 = \left(1 - \frac{2m}{2m - r}\right) dt^2 - \left(1 - \frac{2m}{2m - r}\right)^{-1} dr^2 - \\ - (r - 2m)^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{3}$$

First, it is clear that Eq. (3) satisfies all the conditions (a)–(d), and so metric (3) is as good as metric (1). I now apply to Eq. (3) the very same methods that the astrophysical scientists apply to Eq. (1) and so assume that $0 \leqslant r < \infty$ on Eq. (3), and that "the origin" $r = 0$ marks the point at the centre of spherical symmetry of the manifold. By inspection there are two "singularities"; at $r = 2m$ and at $r = 0$, just as in the case of Eq. (1). When $r > 2m$ the signature of (3) is $(+, -, -, -)$, just as in Eq. (1). When $0 < r < 2m$ the signature is $(-, +, -, -)$, again just as in Eq. (1). Now when $r = 2m$, the coefficient of dt^2 in Eq. (1) is zero, but in Eq. (3) it is undefined. Similarly, when $r = 0$, the coefficient of dt^2 in Eq. (1) is undefined but in Eq. (3) it is zero. Furthermore, when $r = 2m$, the Kretschmann scalar is $f = 3/4m^4$ in Eq. (1) but is undefined in Eq. (3), and when $r = 0$, the Kretschmann scalar is $f = 3/4m^4$ in Eq. (3) but is undefined in Eq. (1). Therefore, according to the methods of the astrophysical scientists there is an infinitely dense point-mass singularity at $r = 2m$ and an event horizon at $r = 0$ in Eq. (3) (or alternatively a singularity of finite density and radius $r = 2m$ so that the event horizon is within the singularity). Thus the singularity is encountered before the event horizon, and the "Schwarzschild radius" of the black hole in Eq. (3) is $r = 0$. Again, following the very same methods that the astrophysical scientists apply to Eq. (1), apply the Kruskal-Szekeres method to remove the "coordinate singularity" at $r = 0$ in Eq. (3) by setting

$$u = \left(1 - \frac{2m - r}{2m}\right)^{\frac{1}{2}} e^{\frac{2m-r}{4m}} \sinh \frac{t}{4m}\,,$$

$$v = \left(1 - \frac{2m - r}{2m}\right)^{\frac{1}{2}} e^{\frac{2m-r}{4m}} \cosh \frac{t}{4m}\,.$$

Then metric (3) becomes,

$$ds^2 = \frac{32m^3}{r - 2m}\, e^{\frac{r-2m}{2m}} \left(du^2 - dv^2\right) + \\ + (r - 2m)^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{4}$$

where r is a function of u and v, by means of

$$\left(\frac{r}{2m}\right) e^{\frac{2m-r}{2m}} = v^2 - u^2.$$

It is now apparent that Eq. (4) is not singular at $r = 0$. The singularity at the event horizon with its "Schwarzschild radius" $r = 0$ has been removed. The metric is singular only at $r = 2m$ where according to the astrophysical scientists there must be an infinitely dense point-mass singularity (or alternatively a singularity of finite density and radius $r = 2m$ so that the event horizon is within the singularity).

In obtaining Eq. (4) I have done nothing more than that which the astrophysical scientists do to Eq. (1), and since (1) and (3) satisfy conditions (a)–(d), the one is as good as the other, and so Eq. (3) is as valid as Eq. (1) insofar as the methods of the astrophysical scientists apply. Thus, the methods employed by the astrophysical scientists are flawed. To amplify this even further, consider the metric,

$$ds^2 = \left(1 - \frac{2m}{4m - r}\right) dt^2 - \left(1 - \frac{2m}{4m - r}\right)^{-1} dr^2 - \\ -(r - 4m)^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{5}$$

It is clear that this metric also satisfies conditions (a)–(d), and so Eq. (5) is as good as eqs. (1) and (3). Once again, applying the very same methods of the astrophysical scientists, assume that $0 \leqslant r < \infty$ and that $r = 0$ is the "origin". Then by inspection there are singularities at $r = 4m$ and at $r = 2m$. For $r > 4m$ the signature of (5) is $(+, -, -, -)$; for $2m < r < 4m$ it is $(-, +, -, -)$ and for $0 \leqslant r < 2m$ it is $(+, -, -, -)$. Now at $r = 4m$ the coefficient of dt^2 is unbounded and at $r = 2m$ it is zero. But at $r = 0$ it is neither zero nor unbounded — the metric is well-defined there. Furthermore, at $r = 4m$ the Kretschmann scalar is unbounded and at $r = 2m$ it is $f = 3/4m^4$, but at $r = 0$ it is $f = 3/256m^4$. Thus, according to the methods of the astrophysical scientists there is an event horizon at $r = 2m$ with "Schwarzschild radius" $r = 2m$, and an infinitely dense point-mass singularity at $r = 4m$ (or alternatively a singularity of finite density and radius $r = 4m$ so that the event horizon is within the singularity). So the singularity is encountered before the event horizon. The "coordinate" event horizon singularity at "Schwarzschild radius" $r = 2m$ can be removed by again applying the Kruskal-Szekeres method, by setting

$$u = \left(\frac{4m - r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{4m-r}{4m}} \cosh \frac{t}{4m}$$

$$v = \left(\frac{4m - r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{4m-r}{4m}} \sinh \frac{t}{4m}$$

for $r < 2m$, and

$$u = \left(1 - \frac{4m - r}{2m}\right)^{\frac{1}{2}} e^{\frac{4m-r}{4m}} \sinh \frac{t}{4m}$$

$$v = \left(1 - \frac{4m - r}{2m}\right)^{\frac{1}{2}} e^{\frac{4m-r}{4m}} \cosh \frac{t}{4m}$$

for $r > 2m$.

Metric (5) then becomes

$$ds^2 = \frac{32m^3}{r - 4m} e^{\frac{r-4m}{2m}} \left(du^2 - dv^2\right) + \\ (r - 4m)^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{6}$$

where r is a function of u and v, by means of

$$\left(\frac{2m - r}{2m}\right) e^{\frac{4m-r}{2m}} = u^2 - v^2.$$

It is apparent that Eq. (6) is singular only at $r = 4m$, where, according to the astrophysical scientists, there is an infinitely dense point-mass singularity (or alternatively a singularity of finite density and radius $r = 4m$ so that the event horizon is within the singularity). At the event horizon with "Schwarzschild radius" $r = 2m$, the metric is not singular. At the "origin", $r = 0$ the metric is well-defined, and since Eq.'s (1), (3) and (5) satisfy conditions (a)–(d), any one is as good as any other, and so Eq. (5) is as valid as Eq. (1) insofar as the methods of the astrophysical scientists apply. Since metrics (1), (3) and (5) all satisfy conditions (a)-(d) there is no *a priori* reason to favour one over the other. Moreover, all the faults associated with metrics (3) and (5) are shared by metric (1), insofar as the methods of the astrophysical scientists are concerned, despite them all satisfying the required conditions (a)–(d). Those faults lie in the assumptions of the astrophysical scientists, as applied to all the Schwarzschild spacetime metrics above.

It is of utmost importance to note that Eq. (1) is not in fact Schwarzschild's solution. Here is Schwarzschild's actual solution.

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1} dR^2 - R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right),$$

$$R = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}, \quad 0 < r < \infty, \quad \alpha = const.$$

Here r is not a distance of any kind in the manifold; and it is not the inverse square root of the Gaussian curvature of the spherically symmetric geodesic surface in the spatial section of Schwarzschild's solution — it is a parameter (and so it is also in Eq. (1)). Schwarzschild's solution contains only one singularity, when $r = 0$, and so it precludes the black hole. The so-called "Schwarzschild solution" is a corruption, due to David Hilbert [22, 23], of Schwarzschild's solution, and the solution obtained independently by Johannes Droste [24].

The correct generalised treatment of Schwarzschild geometry is given in [16–21].

3 The usual derivation of the "Schwarzschild solution"

The astrophysical scientists begin with Eq. (2) and propose a generalisation of the form (or equivalent thereof),

$$ds^2 = e^{2\lambda} dt^2 - e^{2\beta} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{7}$$

the exponential functions being introduced to maintain the signature of Minkowski spacetime, $(+, -, -, -)$, thereby ensuring that the coordinates r, θ, φ remain space-like quantities and t remains a time-like quantity [1–9, 11, 12]. Both λ and β are real-valued analytic functions of only the real variable r. Eq. (1) is then obtained in accordance with conditions (a)–(d). Despite the fixed signature of Eq. (7), the astrophysical scientists permit a change of signature in their resultant Eq. (1), in violation of their construction of Eq. (7), by which they produce a black hole by the Kruskal-Szekeres method. Note that the change of signature in Eq. (1) to $(-, +, -, -)$, in violation of the construction of Eq. (7), causes the rôles of the quantities t and r to be exchanged, i.e. t becomes a space-like quantity and r becomes a time-like quantity. This means that all the components of the metric tensor of Eq. (1) become functions of the time-like quantity r: but this is then a non-static metric, in violation of condition (a).

There is no matter present in the derivation of Eq. (1) from Eq. (7), since all matter, including sources, is eliminated by construction, according to condition (c), i.e. $R_{\mu\nu} = 0$, and since there is no matter present in Eq. (2) either. It is however claimed by the astrophysical scientists that matter is nonetheless present as a source of the alleged gravitational field "outside a body", and that the field caused by this source, permeating the spacetime "outside" it, in the spacetime of $R_{\mu\nu} = 0$, is Schwarzschild spacetime, obtained from Eq. (7). The constant appearing in the line-element for the "Schwarzschild solution" the astrophysical scientists arbitrarily assign as mass, *post hoc*, by simply inserting Newton's expression for escape velocity: a two-body relation into an alleged one-body problem (their "outside a body"). But it is obviously impossible for Schwarzschild spacetime, which is alleged by the astrophysical scientists by construction to contain one mass in an otherwise totally empty Universe, to reduce to or otherwise contain a relation that is defined in terms of the *a priori* interaction of two masses. Their invalid resort to Newtonian theory is amplified by writing Eq. (1) in terms of c and G explicitly,

$$ds^2 = \left(c^2 - \frac{2Gm}{r}\right)dt^2 - c^2\left(c^2 - \frac{2Gm}{r}\right)^{-1} dr^2 - \\ - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

The term $2Gm/r$ is now immediately recognised as the square of the Newtonian escape velocity from a mass m at radius r. And so the astrophysical scientists assert that for a black hole the "escape velocity" is that of light in vacuum at an event horizon ("Schwarzschild radius") $r_s = 2Gm/c^2$. But escape velocity is a concept that involves *two bodies* - one body escapes from another body. Even though one mass appears in the expression for Newton's escape velocity, it cannot be determined without recourse to a fundamental two-body gravitational interaction by means of Newton's theory of gravitation. The *post hoc* introduction of mass into the "Schwarzschild solution" is thus, inadmissible. Furthermore, the quantity r appearing in Newton's expression for escape velocity is a radial distance, but it is not radial distance in Schwarzschild spacetime because it is not even a distance in Schwarzschild spacetime.

4 Conclusions

The foregoing counter-examples show that the methods used by the astrophysical scientists in analysing Eq. (1), by which they construct the black hole, are invalid. Instead of using the line-element to determine all the intrinsic geometric properties of the manifold, as they should, they instead make false assumptions, by mere inspection, as to the "origin", the geometric identity of the quantity r, the values of the Riemann tensor scalar curvature invariant (the Kretschmann scalar), and the presence of matter. The fact is that the quantity r appearing in all the line-elements discussed herein is not even a distance, let alone a radial one, in any of the line-elements. Moreover, in Eq. (1), $r = 0$ certainly does not mark the "origin" or point at the centre of spherical symmetry of the "Schwarzschild" solution, contrary to the arbitrary assertions of the astrophysical scientists. The identity of the point at the centre of spherical symmetry is also determined from the line-element, by calculation. The astrophysical scientists have never correctly identified the geometric identity of r in Eq. (1). Without knowing the true identity of r, and by making their concomitant additional false assumptions, they have violated the intrinsic geometry of the line-element. It is from these violations that the black hole has been constructed by the astrophysical scientists. There is in truth no solution to Einstein's field equations that predicts the black hole.

Minkowski spacetime is not Special Relativity: there is no matter involved in the transition from Minkowski spacetime to Schwarzschild spacetime, and so Schwarzschild spacetime does not generalise the laws of Special Relativity, and so does not describe Einstein's gravitational field.

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