Resolving Inconsistencies in de Broglie's Relation

Pieter Wagener

Department of Physics, NMMU South Campus, Port Elizabeth, South Africa
E-mail: Pieter.Wagener@nmmu.ac.za

Modern quantum theory is based on de Broglie's relation between momentum and wave-length. In this article we investigate certain inconsistencies in its formulation and propose a reformulation to resolve them.

1 Inconsistencies in de Broglie's relation

Edward MacKinnon made a critical analysis [1–3] of Louis de Broglie's doctoral thesis of 1924 [4]. With this thesis de Broglie is credited with deriving the first relationship between the momentum of a particle and its associated quantum wavelength. MacKinnon's discussion draws some remarkable conclusions. He points out that the most paradoxical feature of de Broglie's thesis is the fact that, although his fundamental argument is essentially relativistic, the only successful applications of his ideas were essentially nonrelativistic. It is well known that his relationship $\lambda = h/mv$ was applied to the Bohr atom and later to the derivation of Schrödinger's equation, both of which are strictly nonrelativistic models. What is not so well known is that the arguments leading to $\lambda = h/mv$ are very much relativistic. De Broglie's problem was to find the relativistic transformation of

$$h\nu_0 = \widetilde{m}_0 c^2, \tag{1}$$

where the relativistic rest mass $\widetilde{m}_0$ and the frequency ν_0 are invariant.

His considerations led him to assign three different frequencies to the same particle:

$$\nu_0 = \frac{\widetilde{m}_0 c^2}{h},$$

the internal frequency in the rest system;

$$\nu_1 = \nu_0 \sqrt{1 - v^2/c^2},$$

the internal frequency as measured by an external observer who sees the system moving with velocity v;

$$\nu = \frac{\nu_0}{\sqrt{1 - v^2/c^2}},$$

the frequency this observer would associate with the particle's total energy.

MacKinnon further points out that de Broglie emphasized the frequency associated with an electron, rather than the wavelength. His wavelength-momentum relationship occurs only once in the thesis, and then only as an approximate expression for the length of the stationary phase waves characterizing a gas in equilibrium. Most of MacKinnon's article is devoted to analyzing the reasons why de Broglie's formula proved successful, despite the underlying conceptual confusion. He finally expresses amazement that this confusion could apparently have gone unnoticed for fifty years.

In addition to MacKinnon's criticism, one can also have doubts about some of the applications of de Broglie's formula in quantum mechanics, particularly to electron diffraction. In standard physics texts [5, p. 567], in order to apply the de Broglie relation, the following assumption is made

$$\widetilde{E}^2 = |\mathbf{p}|^2 c^2 + \widetilde{m}_0^2 c^4 \simeq |\mathbf{p}|^2 c^2. \tag{2}$$

The notation is in accordance with previous articles by the author in this journal [6, 8].

From this equation the momentum of the electron is calculated as $|\mathbf{p}| = \widetilde{E}/c$, and from the de Broglie relation it follows that $\lambda = hc/\widetilde{E}$.

Various explanations are given to support the approximation of (2). The most common is to assume that it is allowed for $\widetilde{E} \gg \widetilde{m}_0 c^2$. Although this assumption satisfies experiment, it is not mathematically or conceptually acceptable. Electron diffraction becomes measurable at high energies and velocities, where relativistic equations are applicable. For these equations to be mathematically consistent all terms must be retained, particularly those in the conservation of energy equation. Another approach is to ignore (2) and to apply a semi-nonrelativistic result, $\widetilde{E} = p^2/2\widetilde{m}$ or $T = \mathbf{p}^2/2\widetilde{m}$ [5, p.147], where $\widetilde{m} = \gamma\widetilde{m}_0$ is the relativistic mass of a particle and T is its kinetic energy. This is clearly untenable because of the high velocities.

Another justification for the approximation is that it works for "experimental purposes" [9]. These assumptions might not be serious to verify predictions expeerimentally, but in the spirit of present attempts to formulate a quantum theory of gravity, these assumptions warrants closer scrutiny.

The use of the above approximation is sometimes subtle and not so apparent. In a popular textbook [10, Problem 12.10] the following equation is given for the conservation of energy in Compton scattering:

$$\frac{hc}{\lambda} + m_0 c^2 = \frac{hc}{\lambda'} + mc^2, \tag{3}$$

where m_0 and m are respectively the rest and final mass of the electron.

The equation is inconsistent since wave and corpuscular expressions are combined in one equation. The expression hc/λ is simply a shortcut for $\sqrt{p^2c^2 + m^2_{\nu_0}c^4}$, where the rest mass of the photon $m_{\nu 0}$ is set to zero and de Broglie's relation is then applied to p. In general, the assumption of $\widetilde{m}_0 = 0$ for a photon has had an uneasy niche in theoretical physics [1].

In a previous paper [6] we presented a unified theory of gravitation and electromagnetism. We show below that the model of that theory resolves the inconsistencies discussed above.

2 A scalar momentum

In the aforementioned paper the following conservation of energy equation, derived in an earlier paper [7], was given for the gravitational model:

$$E = m_0 c^2 \, \frac{e^{2\Phi/c^2}}{\gamma^2} = \text{total energy}\,, \tag{4}$$

where

$$\left.\begin{aligned} \Phi &= \text{ gravitational potential,} \\ m_0 &= \text{gravitational rest mass of a test body} \\ &\quad\ \text{moving about a central mass } M. \end{aligned}\right\} \tag{5}$$

We have generalized the exponential term in this paper to a general potential $\Phi = Rc^2/2r$, where $R = 2GM/c^2$ is the Schwarzschild radius of the central body.

We now define a scalar momentum appropriate to our model.

A constant P_0 with dimensions of linear momentum can be defined in terms of the energy E as

$$P_0^2 = m_0 E. \tag{6}$$

Eq. (4) can then be written as

$$P_0 = \frac{m_0\, c}{\gamma} \exp\frac{\Phi}{c^2}\,, \tag{7}$$

or, if the mass constant m_0 is not required in the energy equation, as

$$E = \frac{P_0\, c}{\gamma} \exp\frac{\Phi}{c^2}\,, \tag{8}$$

$$= P c \exp\frac{\Phi}{c^2}\,, \tag{9}$$

where

$$P = \frac{P_0}{\gamma}. \tag{10}$$

In reference [6] we found the following relationship between the gravitational and electromagnetic energies:

$$E = \widetilde{E}\, e^{\Phi/c^2}, \tag{11}$$

where $\widetilde{E} = \widetilde{m}c^2$ is the energy function of Special Relativity.

Comparing (9) and (11) we get

$$\widetilde{E} = Pc\,. \tag{12}$$

3 Derivation of de Broglie's relation

3.1 Preliminaries

Using the relationship between frequency ν and wavelength λ,

$$c = \lambda\nu = \sigma\omega\,, \tag{13}$$

where

$$\sigma = \frac{\lambda}{2\pi} = \frac{c}{2\pi\nu} = \frac{c}{\omega}\,, \tag{14}$$

we rewrite (12) as

$$\widetilde{E} = P\sigma\omega. \tag{15}$$

Since time does not appear explicitly in the above equation for $\widetilde{E}$, we can write down an equivalent Hamiltonian as

$$\widetilde{H} = P\sigma\omega. \tag{16}$$

This form of the Hamiltonian resembles that of the simple harmonic oscillator, after a canonical transformation with generating function $F = (\widetilde{m}_0/2)\, q^2 \cot Q$, where q and Q are the appropriate canonical variables. The significance of this transformation was first pointed out by Max Born [11, §7].

Briefly, it states that the Hamiltonian of a simple harmonic oscillator, given by

$$\widetilde{H} = \frac{p^2}{2\widetilde{m}_0} + \frac{\widetilde{m}_0\, \omega^2 q^2}{2}\,, \tag{17}$$

can, by a canonical transformation with the above generating function, be expressed as

$$\widetilde{H} = \Lambda\omega, \tag{18}$$

where Λ = constant.

If our system behaves as an oscillator it follows from (16) and (18) that

$$P\sigma = \text{constant}. \tag{19}$$

This result prompts us to provisionally write the constant in (19) as $\hbar$, Planck's constant divided by 2π. This step is taken *a priori*, and its validity will depend on the overall consistency of the subsequent results. Keeping this supposition in mind, we rewrite (19) as

$$P\sigma = \hbar, \tag{20}$$

and (15) as

$$\widetilde{E} = \hbar\omega = h\nu. \tag{21}$$

3.2 The photo-electric effect

Eq. (21), combined with $\widetilde{E} = \widetilde{m}c^2$, gives the photo-electric effect, $\widetilde{m}c^2 = \hbar\omega = h\nu$. Eq. (21) also confirms the use of the constant h in the expression for gravitational redshift,

$$E = \widetilde{E} \exp\frac{R}{2r} = h\nu \exp\frac{R}{2r}\,. \tag{22}$$

Eq. (22) is significant in that it contains both h and G in one relation.

The results of (21) and (22) further confirm the consistency of the derivation of (20).

We emphasize that the ω_0, or ω, used above is an internal property of the test particle; it is not its angular velocity about a central body. We cannot say with certainty what the internal physical structure of the test particle should be; only that if some periodic mechanism exists with respect to the test particle the frequency of that mechanism is controlled by the above equations. This, for example, determines the gravitational redshift. As a model for such a type of test particle we shall simply refer to it as a virtual oscillator.

3.3 Derivation

From (14) and (20),

$$P = \frac{h}{\lambda}\,. \tag{23}$$

Although (23) is similar to the de Broglie relationship between momentum and wavelength, the momentum P is not equal to the classical momentum,

$$\mathbf{p} = \widetilde{m}\mathbf{v}\,. \tag{24}$$

Nevertheless, we shall see that (23) is consistent with the application of the de Broglie relation, and actually resolves some ambiguities in quantum mechanics [1].

3.4 The relationship between p and P

From (12) and $\widetilde{E} = \widetilde{m}c^2$ we obtain

$$P = \widetilde{m}c. \tag{25}$$

From this we can see that $P = \widetilde{E}/c$ can be regarded as the fourth component of the relativistic four-vector, p_i:

$$p_i = \left(\mathbf{p}, \frac{\widetilde{E}}{c}\right), \quad i = 1,2,3,4, \tag{26}$$

or

$$p_i = (\mathbf{p}, P)\,, \quad i = 1,2,3,4. \tag{27}$$

To find a direct relation between $\mathbf{p}$ and P we note from (24) and (25) that

$$\mathbf{p} = \frac{P\mathbf{v}}{c} \quad \text{or} \quad \mathbf{p}c = P\mathbf{v}\,. \tag{28}$$

The well-known expression of Special Relativity,

$$\widetilde{E}^2 = \mathbf{p}^2c^2 + \widetilde{m}_0^2c^4, \tag{29}$$

can be rewritten, using (28), as

$$\widetilde{E}^2 = P^2v^2 + \widetilde{m}_0^2c^4. \tag{30}$$

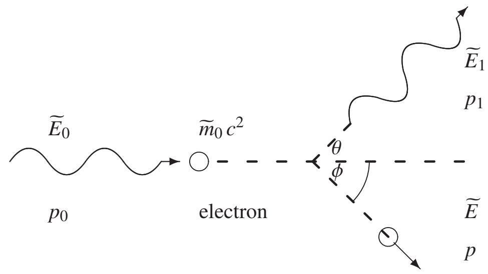

Fig. 1: Compton scattering

4 Applications of de Broglie's relation

The relation of (23), $P = h/\lambda$, is clearly different from the conventional de Broglie relationship. This form is, however, not in conflict with either theory or experiment, but actually simplifies the various formulations.

4.1 Compton scattering

For a photon, $v = c$, and it follows from (23) and (28) that

$$P = |\mathbf{p}| = \frac{h}{\lambda}\,. \tag{31}$$

An advantage of (31) is that, when applied to Compton scattering, it is not necessary to make the assumption $\widetilde{m}_0 = 0$ in (29). It must also be noted that the assumption $\widetilde{m}_0 = 0$ for a photon is not required in our theory; only $v = c$. The paradox of the photon rest mass is resolved in reference [6].

The Compton effect is described schematically in Fig. 1. The equations below follow from this diagram.

Conservation of momentum:

$$p_0 = p_1\cos\theta + p\cos\phi\,, \tag{32}$$

$$p_1\sin\theta = p\sin\phi\,. \tag{33}$$

From (32) and (33),

$$p^2 = p_0^2 + p_1^2 - 2p_0p_1\cos\theta, \tag{34}$$

and applying (31) gives

$$p^2 = \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_1^2} - \frac{2h^2\cos\theta}{\lambda_0\lambda_1}\,. \tag{35}$$

Since

$$\widetilde{E}^2 = p^2c^2 + \widetilde{m}_0^2c^4,$$

it follows that

$$\frac{\widetilde{E}^2}{c^2} - \widetilde{m}_0^2c^2 = \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_1^2} - \frac{2h^2\cos\theta}{\lambda_0\lambda_1}\,. \tag{36}$$

Conservation of energy:

$$\widetilde{E}_0 + \widetilde{m}_0c^2 = \widetilde{E}_1 + \widetilde{E}\,, \tag{37}$$

therefore

$$(\widetilde{E} - \widetilde{m}_0\, c^2)^2 = \widetilde{E}_0^2 + \widetilde{E}_1^2 - 2\,\widetilde{E}_0\,\widetilde{E}_1\,.$$

From (31) and rearranging,

$$\frac{\widetilde{E}^2}{c^2} + \widetilde{m}_0^2\, c^2 - 2\,\widetilde{E}\widetilde{m}_0 = \frac{h^2}{\lambda_0^2} + \frac{h^2}{\lambda_1^2} - \frac{2h^2}{\lambda_0\,\lambda_1}\,. \tag{38}$$

Eq. (38) minus (36), and rearranging:

$$\widetilde{m}_0\, c^2 - \widetilde{E} = -\frac{h^2\,(1-\cos\theta)}{\widetilde{m}_0\,\lambda_0\,\lambda_1}\,. \tag{39}$$

Subsituting (12) and (31) in (39) gives

$$\lambda_1 - \lambda_0 = \frac{h\,(1-\cos\theta)}{\widetilde{m}_0\, c}\,, \tag{40}$$

the standard formulation for Compton scattering.

4.2 Electron diffraction

Another advantage of our formulation applies to electron diffraction. From the results $P = h/\lambda$ and $\widetilde{E} = \hbar\omega$ it follows directly that $\widetilde{E} = Pc$. This obviates the approximation used in standard texts on electron diffraction, i.e. $\widetilde{E}^2 \cong \mathbf{p}^2 c^2$.

5 Conclusion

The above derivation and formulation of de Broglie's relation resolves the inconsistencies in de Broglie's original derivation. It also obviates the questionable approximations made in Compton scattering and electron diffraction.

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