On a Formalization of Cantor Set Theory for Natural Models of the Physical Phenomena

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This article presents a set theory which is an extension of ZFC. In contrast to ZFC, a new theory admits absolutely non-denumerable sets. It is feasible that a symbiosis of the proposed theory and Vdovin set theory will permit to formulate a (presumably) non-contradictory axiomatic set theory which will represent the core of Cantor set theory in a maximally full manner as to the essence and the contents of the latter. This is possible due to the fact that the generalized principle of choice and the generalized continuum hypothesis are proved in Vdovin theory. The theory, being more complete than ZF and more natural according to Cantor, will allow to construct and study (in its framework) only natural models of the real physical phenomena.

This paper is dedicated to the memory of
Alexander M. Vdovin (1949–2007)

I. It is generally accepted that the (presumably) non-contradictory Zermelo-Fraenkel set theory ZF with the axiom of choice is the most accurate and complete axiomatic representation of the core of Cantor set theory. However, it is acknowledged [3, p. 109], that "Cantor's set theory is so copious as to admit absolutely non-denumerable sets while axiomatic set theory [in particular, ZFC] is so limited [Skolem's paradox] that every non-denumerable set becomes denumerable in a higher system or in an absolute sense". An axiomatic set theory defined here and abbreviated as ZFK admits absolutely non-denumerable sets, as it does Cantor theory.

It is feasible that a symbiosis of the proposed theory and Vdovin set theory [1, 2] will permit to formulate a (presumably) non-contradictory axiomatic set theory which will represent the core of Cantor set theory in a maximally full manner as to the essence and the contents. This is possible due to the fact that the generalized principle of choice and the generalized continuum hypothesis are proved in Vdovin theory.

II. Our definition of ZFK will be based on the traditional (classical) concept of formalized theory explained in [4]. But ZFK is a theory which is axiomatic not completely in the traditional sense, so the syntactic aspects of this theory will be described with references to the principal interpretation of ZFK.

Formulae of ZFK are formulae of the signature $\langle \in, S \rangle$, where $\in$ — is a two-place predicate symbol for denoting the (standard) membership relation on the collection S_k of all Cantor's (intuitive) sets, and S — is a null-place functional symbol (a constant) denoting the family of all axiomatized sets, and in the ZFK formulae containing the symbol "S", the latter symbol is always placed to the right of the symbol "$\in$".

In what follows, we use the conventional notation and abbreviations of ZF. In particular, the relativization of a formula φ to the family S is denoted by $[\varphi]^S$. Besides, depending on the context, records "$\in$" and "S" denote either the signature symbols or denoted by them the relation and the family, respectively. Cantor's (intuitive) sets of S_k will be called k-sets, and the axiomatized sets of S will be simply called as sets.

The axioms of ZFK are divided into two groups: G and G_k. The axioms of group G describe the axiomatized sets, and the axioms of group G_k characterize the relationship between Cantor's (intuitive) sets and the axiomatized sets.

The axioms of group G are the axioms of ZFC (formulae of the signature $\langle \in \rangle$), with exception of the axiom of empty set, which are relativized to the family S.

The axioms of group G_k:

1) Axiom of embedding S into S_k

$$\forall x \in S \; \exists y \, (y = x).$$

2) Axiom of (absolutely) empty set

$$\exists x \in S \; \forall y \, (y \notin x).$$

3) Axiom of transitivity of S in S_k

$$\forall x \in S \, \forall y \, (y \in x \rightarrow y \in S).$$

4) Axiom (schema) of generalization

$$[\varphi]^S \rightarrow \varphi,$$

where φ — is a formula of ZFK.

5) Axiom (schema) of mappings to S_k

$$\forall t \, (\forall v, w_1, w_2 (\varphi(v, w_1, t) \,\&\, \varphi(v, w_2, t) \rightarrow w_1 = w_2) \rightarrow$$
$$\rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists v \in x \, \exists w (z = \langle v, w \rangle \,\&\, \varphi(v, w, t)))),$$

where φ — is a formula of ZFK and the variable y does not occur free in φ.

6) Axiom of general replacement

$$\forall x\,(\mathrm{map}(x)\,\&\,\mathrm{dom}\,(x) \in S\ \&\,\mathrm{rang}\,(x) \subseteq S\ \rightarrow$$
$$\rightarrow \mathrm{rang}\,(x) \in S\ \&\, x \in S\,),$$

where map(x) is the formula

$$\forall z\,(z \in x \rightarrow \exists v, w\,(z = \langle v, w\rangle))\,\&\,\forall v, w_1, w_2\,(\langle v, w_1\rangle \in$$
$$x\,\&\,\langle v, w_2\rangle \in x \rightarrow w_1 = w_2),$$

and k-sets dom(x) and rang(x) satisfy

$$\forall v\,(v \in \mathrm{dom}\,(x) \leftrightarrow \exists\, w\,(\langle v, w\rangle \in x))$$

and

$$\forall w\,(w \in \mathrm{rang}\,(x) \leftrightarrow \exists\, v\,(\langle v, w\rangle \in x)).$$

The logic underlying ZFK is the calculus of predicates in the language of ZFK.

III. It is well known [3, p. 27] that "An axiomatic system is in general constructed in order to axiomatize a certain scientific discipline previously given in a pre-systematic, "naive", or 'genetic' form". ZFK formulated here has been constructed, like ZFC, to axiomatize the "naive" set theory of G. Cantor, or more precisely, to axiomatize its non-contradictory core. But ZFK has a more explicit and tight connection to Cantor set theory than it does ZFC, since ZFK in its principal interpretation defines the collection of all k-sets of S_k (more precisely, $\langle S_k; \in\rangle$) as Cantor pre-axiomatic "world" of sets, and the family S (more precisely, $\langle S\,; \in \cap (S \times S\,)\rangle$, where $S \subseteq S_k$) as the axiomatic fragment of Cantor "world" of sets.

It seems natural that ZFK is non-contradictory if ZFC is non-contradictory. Let us show that it is true.

Suppose that ZFC is a non-contradictory theory. Then, ZFC has a model and, in particular, a standard transitive model $\mathfrak{M} = \langle M; \in \cap (M \times M)\rangle$ such that for any set $m \in M$ absolutely all its subsets belong to the family M. It is clear that the model $\mathfrak{M}$ (the family M) includes absolutely denumerable sets. We consider the family M as the interpretation of the signature symbol "S" and will show that any axiom of ZFK is either true in the model $\mathfrak{M}$ or it does not deny the existence of such a model.

It is natural that all axioms of group G are true in the model $\mathfrak{M}$.

Axioms G_k–1 and G_k–2 affirm an obvious fact: any ZFC-set (a set of the family M) is also a set of Cantor "world" of sets S_k.

Axiom G_k–3 affirms natural transitivity of the family M.

Axiom G_k–4 affirms an obvious fact: any statement concerning sets of the family M is also true for sets of Cantor "world" of sets S_k due to the fact that ZFC is a formalization of the (presumably) non-contradictory core of Cantor set theory.

Axiom G_k–5) is a natural generalization of ZFC axiom of replacement which is true in the model $\mathfrak{M}$.

Axiom G_k–6), in fact, affirms that the model $\mathfrak{M}$ is naturally $\subseteq$-complete in the sense that any subset of the family M belongs to that M if its power is equal to the power of a certain set of M.

IV. Let $x \in S$. Then, a k-set $\{y \mid y \subseteq x\,\&\, y \in S\}$ is denoted by $P(x)$. It is clear that $P(x) \in S$ ($P(x)$ is a set) by axioms of group G and G_k–1).

THEOREM (*ZFK*).

$$\forall x \in S\,\forall\, y\,\big(y \subseteq x \rightarrow y \in P(x)\big).$$

Proof. Let us suppose that the contrary is fulfilled and let k-sets x_0 and y_0 be such that $x_0 \in S$, $y_0 \subseteq x_0$ and $y_0 \notin P(x)$. If $y_0 \in S$, than $y_0 \in P(x)$ by an axiom of group G. Therefore, $y_0 \notin S$. Since $\varnothing \in S$, then $y_0 \neq \varnothing$. Since $y_0 \subseteq x_0 \in S$ and S is transitive in S_k (the axiom G_k–3)) then $y_0 \subseteq S$.

Denote by z_0 some element of a k-set y_0. The axiom G_k–5) says that there is a k-set (k-function) f such that

$$f = \Big\{\langle v, w\rangle \mid v \in x_0, (v \in y_0 \rightarrow w = v), (v \notin y_0 \rightarrow w = z_0)\Big\}.$$

Since map(f), dom(f) $= x_0 \in S$ and rang(f) $= y_0 \subseteq S$, then $y_0 \in S$ by the axiom G_k–6). A contradiction.

V. Let x be a k-set ($x \in S$ or $x \notin S$). Then $P_k(x)$ denotes k-set $\{y \mid y \subseteq x\}$. Since $x \in S_k$, then $P_k(x) \in S_k$ (by the axiom of generalization), i. e. $P_k(x)$ is an element of Cantor pre-axiomatic "world" of sets, whose power by the theorem of G. Cantor is **absolutely** greater than the power of the k-set x.

Let ω be a denumerably infinite set in S. Since $\omega \in S$ then $\omega \in S_k$ (the axiom G_k–1)). It is clear that the k-set $P_k(\omega)$ is absolutely non-denumerable. **THEOREM** says that any k-set y of S_k is such that $y \subseteq \omega$ (i. e. $y \in P_k(\omega)$) is an element of the set $P(\omega)$ of S. Therefore, the equality $P(\omega) = P_k(\omega)$ is always fulfilled. Thus the set $P(\omega)$ is **absolutely non-denumerable** in any axiomatized model of ZFK, i. e. in any model of the type $\langle S\,; \in \cap (S \times S\,)\rangle$.

Thus the concept "The set of all subsets of a set X" which is formalized by the axioms of ZFK is absolute (in view of the **THEOREM**) in the sense that it coincides with Cantor concept "The set of all (absolutely all existing in the Cantor 'world' of sets) subsets of a set X".

VI. Finally it should be noted that a symbiosis of the set theory of Vdovin A. M. and the proposed theory may permit to formulate an axiomatic non-contradictory (presumably) set theory, the only standard model of which will be the most important fragment of Cantor "world"of sets. This is ensured by the fact that Vdovin set theory proves the axioms of ZF, the generalized principle of choice, and the generalized continuum-hypothesis which are natural for Cantor "world" of sets, and the theory presented above proves the absolute character of the concept "The set of all subsets of a set X" which is natural for Cantor "world" of sets, as well.

Since ZF is a generally acknowledged theory and it is applied as a framework for mathematical disciplines used to describe (study) the real physical world, the natural (Cantor-like) character of the future set theory will permit to develop and investigate only natural models of real physical phenomena.

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