LETTERS TO PROGRESS IN PHYSICS

Coordinate Transformations and Metric Extension: a Rebuttal to the Relativistic Claims of Stephen J. Crothers

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The concept of coordinate transformation is fundamental to the theory of differentiable manifolds, which in turn plays a central role in many modern physical theories. The notion of metric extension is also important in these respects. In this short note we provide some simple examples illustrating these concepts, with the intent of alleviating the confusion that often arises in their use. While the examples themselves can be considered unrelated to the theory of general relativity, they have clear implications for the results cited in a number of recent publications dealing with the subject. These implications are discussed.

1 Introduction

Differentiable manifolds play a central role in modern physical theories. Roughly speaking, a differentiable manifold (hereafter manifold) is a topological space whose local equivalence to Euclidean space permits a global calculus. In more precise mathematical terms, a manifold is a topological space M with a collection of coordinate systems that cover all of M. Thus the concept of a coordinate system is fundamental to the notion of manifold.

A coordinate system is defined as a mapping ϕ (with certain properties) from an open set U of a topological space onto an open set $\phi(U)$ of Euclidean space. The open set U is called the coordinate neighborhood of ϕ and the functions $x^1, \ldots, x^n$ on U such that $\phi = \left(x^1, \ldots, x^n\right)$, are the coordinate functions, or more simply the coordinates. A manifold can have an infinite number of equally valid coordinates defined on it.

As an example consider the topological space S^2 (the unit sphere). Further consider the northern and southern hemispheres of the sphere, which are both open subsets of S^2. On each of the hemispheres we can define stereographic coordinates by projecting the respective hemispheres onto two-dimensional Euclidean space. Each of the projections defines a coordinate system, which when taken together cover all of S^2. Thus S^2 is a manifold.

The notion of a metric tensor g on a manifold M is fundamental to the theory of differential geometry (indeed, the metric tensor is alternatively called the first fundamental form). Explicitly, g is a type-(0,2) tensor that defines a scalar product $g(p)$ on the tangent space $T_p(M)$, for each point $p \in M$. On a domain U, corresponding to a particular coordinate system $\{x^1, \ldots, x^n\}$, the components of the metric tensor are $g_{ij} = g(\partial_i, \partial_j)$. It is important to note that the metric components g_{ij} are functions, not tensors. The metric tensor itself is given by $g = g_{ij}\, dx^i \otimes dx^j$, where summation over the indices is implied. It must be stressed that a metric, by virtue of the fact that it is a tensor, is independent of the coordinate system which is used to express the component functions g_{ij}.

The metric tensor can be represented by its line-element ds^2, which gives the associated quadratic form of $g(p)$. We stress that a line-element is *not* a tensor. A line-element can be expressed in terms of a coordinate system as

$$ds^2 = g_{ij}\, dx^i\, dx^j.$$

Representing the metric in a particular coordinate system by the associated quadratic form is equivalent to expressing it as a square matrix with respect to the coordinate basis. For example, on the unit sphere the metric σ is often written in terms of the line-element with respect to spherical coordinates $\{\theta, \varphi\}$ as

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2,$$

or equivalently as the matrix

$$[\sigma]_{\{\theta,\varphi\}} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

It is important when practicing differential geometry to distinguish between coordinate dependent quantities and coordinate invariant quantities. We have already seen some examples of these: the metric tensor is coordinate invariant (as is any tensor), while the line-element is coordinate dependent. Another example of a coordinate dependent quantity are the Christoffel symbols

$$\Gamma^i_{jk} = g^{im}\left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\right)$$

while the scalar curvature (Kretschmann scalar), which is derived from them as

$$f = g^{ab}\left(\partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^d_{ab}\Gamma^c_{cd} - \Gamma^d_{ac}\Gamma^c_{bd}\right),$$

is coordinate invariant. Another example of a coordinate invariant quantity is the metric length of a path in a manifold.

Suppose now that we have two different sets of coordinates defined on an open set $U \subset M$. That is to say that we have two mappings ϕ_1 and ϕ_2 that act from U onto two (possibly different) open sets V_1 and V_2 in Euclidean space. It is apparent that we can change from one coordinate system to the other with the maps $\phi_2 \circ \phi_1^{-1}$ or $\phi_1 \circ \phi_2^{-1}$. Such maps define a change of coordinates or *coordinate transformation*. Alternatively if we have a mapping ζ from V_1 into V_2 and a coordinate system (mapping) ϕ from U onto V_1, then the mapping $\zeta \circ \phi$ also defines a coordinate system. In this context ζ is the coordinate transformation. Coordinate invariant quantities, such as the metric, the scalar curvature and lengths, do not change under the action of a coordinate transformation

In what follows we illustrate these concepts by means of some simple examples and discuss some of their implications.

2 Some simple examples

We begin by illustrating the concept of coordinate transformation with a simple example in ordinary Euclidean 3-space (E^3). Suppose that (r, θ, φ) are the usual spherical coordinates on E^3 and consider the spherically symmetric line-element

$$ds^2 = r^2\,dr^2 + r^2\,d\Omega^2, \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ is the usual shorthand for the line-element on the unit sphere S^2.

Defining a new radial coodinate ρ by $2\rho = r^2$, the line-element can be written in terms of the coordinates (ρ, θ, φ) as

$$ds^2 = d\rho^2 + 2\rho\,d\Omega^2. \tag{2}$$

Note that if ρ is held constant then the line-element reduces to the standard line-element for a sphere of radius $\sqrt{2\rho} = r$.

Note that the coordinate transformation has changed nothing. The metrics corresponding to the line-elements given by (1) and (2) are exactly the same tensor, they have just been expressed in two different sets of coordinates. To illustrate this consider calculating metric length along a radial line. Specifically, consider the path defined in terms of the (r, θ, ϕ) coordinates by

$$\gamma_a = \{(r, \theta, \varphi) : r \in (0, a),\ \theta = \pi/4,\ \varphi = 0\}.$$

Equivalently, we can define the path in terms of the (ρ, θ, φ) coordinates as

$$\gamma_a = \{(\rho, \theta, \varphi) : \rho \in (0, a^2/2),\ \theta = \pi/4,\ \varphi = 0\}.$$

Thus calculating the metric length of the path γ_a with respect to the line-element (1) we find

$$L(\gamma_a) = \int_{r=0}^{r=a} r\,dr = \frac{a^2}{2},$$

while if we calculate it with respect to the line-element (2) we find that

$$L(\gamma_a) = \int_{\rho=0}^{\rho=a^2/2} d\rho = \frac{a^2}{2}\,.$$

This confirms that the metric length does not depend on the particular coordinate expression (line-element) representing the metric.

This example also illustrates another interesting property of the metric corresponding to (1) or (2). If we set $\rho = b$, where b is a constant, the line-element (2) reduces to the 2D line-element:

$$ds^2 = 2b\,d\Omega^2.$$

This is the line-element of a 2-sphere with a radius of curvature of $\sqrt{2b}$, i.e. the Gaussian curvature is $1/2b$. However, calculating the metric distance d from the origin ($\rho = 0$) to this sperical shell ($\rho = b$), we find that

$$d = \int_0^b d\rho = b.$$

Hence, the metric radius and the radius of curvature are not equal in general. Repeating the calulation with (1) yields the same result.

As another example consider the two-dimensional, non-Euclidean metric

$$ds_1^2 = -x^2\,dt^2 + dx^2, \tag{3}$$

where it is assumed that $t \in (-\infty, \infty)$ and $x \in (0, \infty)$. In terms of the coordinates $\{t, x\}$ the metric tensor g_1 can therefore be represented as

$$[g_1]_{\{t,x\}} = \begin{pmatrix} -x^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

with a metric determinant of $|g_1| = -x^2$, which suggests that as $x \to 0$ the metric becomes singular.

However, calculating the scalar curvature of the metric we find that $R_{g_1} = 0$, which is independent of x. The metric g_1 therefore defines a flat manifold (N, g_1). The fact that the singularity arises in the coordinate dependent form of the metric, but not in the coordinate invariant scalar curvature, indicates that the apparent singularity may in fact be due solely to a breakdown in the coordinate system $\{t, x\}$ that was chosen to represent the metric, i.e. it may merely be a *coordinate singularity* rather than a true singularity of the manifold described by g_1. A coordinate singularity can be removed by a good choice of coordinates, whereas a true singularity cannot.

Introducing new coordinates $\{T, X\}$, which are defined in terms of the old coordinates $\{t, x\}$ by

$$X = x\cosh t$$
$$T = x\sinh t,$$

the line-element ds_1^2 may be written as

$$ds_1^2 = -dT^2 + dX^2. \tag{5}$$

Note that $t \in (-\infty, \infty)$ and $x \in (0, \infty)$ implies that $T \in (-\infty, \infty)$ and $X \in (0, \infty)$ also.

In terms of the $\{T, X\}$ coordinates, the metric tensor g_1 is represented by

$$[g_1]_{\{T,X\}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

and so the metric determinant is $|g_1| = -1$. The apparent singularity has been removed by invoking a good choice of coordinates.

We note further that even though the line-element (5) was only defined for $X \in (0, \infty)$ there is now nothing stopping us from extending the definition to include $X \in (-\infty, \infty)$. We thus make the distinction between the line-element ds_1^2, defined above, and the line-element ds_2^2 defined as

$$ds_1^2 = -d\tau^2 + d\xi^2, \tag{7}$$

with coordinates $\tau, \xi \in (-\infty, \infty)$. The metric corresponding to the line-element (7), denoted by g_2, defines a manifold (M, g_2) that can be thought of as 2D Minkowski space. By restricting the coordinate ξ to the semi-finite interval $(0, \infty)$ we recover the metric g_1, that is

$$g_2|_{\xi>0} = g_1.$$

It follows that the manifold (N, g_1) is a submanifold of the Minkowski space (M, g_2). Alternatively we say that (M, g_2) is a *coordinate extension* of the manifold (N, g_1). The manifold (N, g_1) is known as the *Rindler wedge* and corresponds to that part of (M, g_2) defined by $|\tau| < \xi$.

3 Implications

In [1] the author notes that the line-element written in terms of coordinates $\{t, r, \theta, \varphi\}$ as

$$ds^2 = A(r)\,dt^2 + B(r)\,dr^2 + C(r)\,d\Omega^2 \tag{8}$$

corresponds to the most general spacetime metric that is static and spherically symmetric. He then goes on to claim that the line-element written in terms of coordinates $\{t, \rho, \theta, \phi\}$ as

$$ds^2 = A^*(\rho)\,dt^2 + B^*(\rho)\,d\rho^2 + \rho^2\,d\Omega^2 \tag{9}$$

does *not* correspond to the most general metric that is static and spherically symmetric*. This claim is false, as we will now demonstrate.

Consider the line-element (9) and define the coordinate transformation $\rho = \sqrt{C(r)}$, where C is some function independent of the functions A^* and B^*. Taking the differential we find that

$$d\rho = \frac{C'(r)}{2\sqrt{C(r)}}\,dr$$

and so the line-element (9) can be written in terms of the coordinates $\{t, r, \theta, \varphi\}$ as

$$ds^2 = E(r)\,dt^2 + D(r)\,dr^2 + C(r)\,d\Omega^2, \tag{10}$$

where

$$E(r) = A^*\left(\sqrt{C(r)}\right) \quad \text{and} \quad D(r) = \frac{B^*\left(\sqrt{C(r)}\right)\,C'(r)^2}{4C(r)}.$$

Since the functions A^* and B^* are independent of the function C, the functions E and D are also independent of the function C. The line-element (10) is identical to (8) and it follows that the metrics represented by (8) and (9) are the same metric (just expressed in terms of different coordinates), and therefore that both line-elements represent the most general static, sperically symmetric spacetime metric.

Based on the claim of [1], just shown is false, the author goes on to conclude that solutions of the gravitational field equations that are derived from the metric ansatz (9) are particular solutions rather than general solutions. These claims are also false for the same reasons as illustrated above.

The foregoing considerations therefore have bearing on the relativistic arguments contained in [1] and subsequent papers by the author. For example, in [1–8] the author repeatedly makes the following claims:

1. The coordinate ρ, appearing in (9), is not a proper radius;
2. The "Schwarzschild" solution, as espoused by Hilbert and others is different to the Schwarzschild solution obtained originally by Schwarzschild [9];
3. The original Schwarzschild solution is a complete (i.e. inextendible) metric;
4. There are an infinite number of solutions to the static, spherically symmetric solutions to the field equations correponding to a point mass;
5. For line-elements of Schwarzschild form†, the scalar curvature f remains bounded everywhere, and hence there is no "black hole".

We will now address and dismiss each of these claims.

Claim 1. The claim that ρ is not a proper radius stems from a calculation in [1]. The author defines the proper radius as

$$R_p = \int \sqrt{B(r)}\,dr \tag{11}$$

where B is the function appearing in (8). Strictly speaking this is not a radius, per se, but a *function* of the coordinate r. In more precise terms, the proper radius should be defined as the metric length of the radial path γ_a defined by‡

$$\gamma_a = \{(t, r, \theta, \varphi) : r \in (a_1, a_2),\ t, \theta, \varphi = \text{constant}\}.$$

This then implies that the proper radius is defined as

$$R_p = L_1(\gamma_a) = \int_{a_1}^{a_2} \sqrt{B(r)}\,dr. \tag{12}$$

*Note that in [1] the author has used r again instead of ρ. We use the different symbol ρ to avoid confusion.

†Line-elements of "Schwarzschild form" are defined in [2].

‡We believe that this is what the definition in [1] was actually aiming at.

The claim in [1] relates to the fact that R_p, as defined by (11), is equal to r only if $B(r) = 1$. This conclusion is based on an imprecise definition of the proper radius and does not take into account the effect of coordinate transformation. If we work in terms of the coordinates appearing in the line-element (9), which we have already shown represents the same metric as (8), then the path γ_a is defined as

$$\gamma_a = \{(t, \rho, \theta, \varphi) : \rho \in (\rho_1, \rho_2),\ t, \theta, \varphi = \text{constant}\},$$

with $\rho_1 = \sqrt{C(a_1)}$ and $\rho_2 = \sqrt{C(a_2)}$. In terms of the line-element (9) the metric length of γ_a is given by

$$L_2(\gamma_a) = \int_{\sqrt{C(a_1)}}^{\sqrt{C(a_2)}} \sqrt{B^*(\rho)}\, d\rho.$$

Noting the effect of the coordinate transformation, that was established earlier, we then find that

$$\begin{aligned} R_p = L_1(\gamma_a) &= \int_{a_1}^{a_2} \sqrt{B(r)}\, dr \\ &= \int_{a_1}^{a_2} \left[B^*(\sqrt{C(r)}\right]^{1/2} \frac{C'(r)}{2\sqrt{C(r)}}\, dr \\ &= \int_{\sqrt{C(a_1)}}^{\sqrt{C(a_2)}} \sqrt{B^*(\rho)}\, d\rho \\ &= L_2(\gamma_a)\,. \end{aligned}$$

Hence the proper radius does not depend on the form of the line-element. Proper radius (i.e. a metric length) can be equivalently defined in terms of either of the "radial" coordinates r or ρ.

Claims 2 and 3. The original Schwarzschild solution obtained in [9] is given as the line-element

$$ds^2 = A(R)dt^2 - A(R)^{-1}dR^2 - R^2 d\Omega^2, \qquad (13)$$

where

$$A(R) = 1 - \frac{\alpha}{R} \quad \text{and} \quad R = (r^3 + \alpha^3)^{1/3}.$$

The coordinate $r \in (0, \infty)$ that appears is the standard spherical radial coordinate. The expression $R = (r^3 + \alpha^3)^{1/3}$ defines a transformation of the radial coordinate r into the auxilliary radial coordinate R. The constant α is related to the value of the mass at the origin [9]. Indeed, by imposing the additional boundary condition at infinity, that the solution be consistent with the predictions of Newtonian gravitational theory, it is found that the constant $\alpha = 2m$, where m is the mass at the origin. The line-element (13) can therefore be written as

$$ds^2 = \left(1 - \frac{2m}{R}\right)dt^2 - \left(1 - \frac{2m}{R}\right)^{-1} dR^2 - R^2 d\Omega^2, \qquad (14)$$

with $R \in (2m, \infty)$. Note that if R and t are held constant (say $R = a$ and $t = t_0$) the line-element reduces to that of a 2-sphere with radius $a > 2m$. The line-element therefore defines a manifold that is foliated by 2-spheres with radii greater than $2m$.

The line-element is of precisely the same form as the line-element derived by Hilbert [10], i.e.

$$ds^2 = \left(1 - \frac{2m}{\rho}\right)dt^2 - \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2 d\Omega^2, \qquad (15)$$

where $\rho \in (0, 2m) \cup (2m, \infty)$. The only difference is that (14) is defined over a subset of the domain over which (15) is defined. To obtain the line-element (15) the radial coordinate has been extended to values less than $2m$ in much the same way that the metric corresponding to (5) was extended to the metric corresponding to (7). The only real difference is that in the case at hand there remains a coordinate singularity at $R = 2m$, and so in terms of the coordinates used, the extended manifold must be viewed as a disjoint union of the regions corresponding to $R < 2m$ and $R > 2m$. Both of the disjoint regions satisfy the static, spherically symmetric field equations. In fact it is well-known that there exist coordinates in which the difficulty at $R = 2m$ can be removed, resulting in a single manifold that satisfies the field equations. As a point of historical interest we note that the extended metric is also known as the "Schwarzschild" metric in honour of Schwarzschild's contribution to the field, despite the fact that his original solution is only a subset of the complete solution.

From the above considerations it clear that the manifold corresponding to the line-element (13) is incomplete. Indeed, in deriving this form of the line-element, Schwarzschild imposed a very specific boundary condition, namely that the line-element is continuous everywhere except at $r = 0$, where $r \in (0, \infty)$ is the standard spherical radial coordinate. Imposition of this boundary condition has significant implications for the solution obtained. In particular, as a consequence of the boundary condtion the coordinate R is shifted away from the origin. Indeed, if $r \in (0, \infty)$ then $R \in (\alpha, \infty)$. Hence the manifold represented by (13) is foliated by 2-spheres of radius greater than $\alpha = 2m$ — the spacetime has a hole in its centre!

Claim 4. In [2] the author derives the general solution for the static, spherically symmetric field due to a point mass as

$$ds^2 = \left(\frac{\sqrt{C_n} - \alpha}{\sqrt{C_n}}\right)dt^2 - \left(\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}\right)\frac{C_n'^2}{4C_n}dr^2 - C_n\, d\Omega^2, \quad (16)$$

where r is the standard radial spherical coordinate and

$$C_n(r) = \left[(r - r_0)^n + \alpha^n\right]^{2/n} \qquad (17)$$

with $r_0 \geqslant 0$ and $n > 0$ arbitrary constants. The author also notes that (16) is only defined for $r > r_0$.

Let us now see the effect of transforming coordinates. Firstly, let $\rho = r - r_0$ so that the coordinate ρ is simply a shifted version of the coordinate r. Taking differentials implies that $d\rho = dr$ and so we may equivalently write the line-element (16) as

$$ds^2 = \left(\frac{\sqrt{C_n} - \alpha}{\sqrt{C_n}}\right) dt^2 - \left(\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}\right) \frac{C_n'^{\,2}}{4C_n} d\rho^2 - C_n\, d\Omega^2, \quad (18)$$

where now

$$C_n(\rho) = [\rho^n + \alpha^n]^{2/n}$$

and the line-element is defined for $\rho > 0$.

Secondly, define another change of coordinates by $R = \sqrt{C_n(\rho)}$. This is essentially a rescaling of the radial coordinate ρ. Taking differentials we find that

$$dR = \frac{C_n'}{2\sqrt{C_n}}\, d\rho\,.$$

Thus in terms of the coordinate R the line-element may be written as

$$ds^2 = \left(\frac{R - \alpha}{R}\right) dt^2 - \left(\frac{R}{R - \alpha}\right) dR^2 - R^2\, d\Omega^2, \qquad (19)$$

where the coordinate $R > \alpha$.

Hence we have shown that what appeared to be an infinitude of particular solutions are actually just different coordinate expressions of the same solution, which without loss of generality can be expressed in "Schwarzschild coordinates" $\{t, R, \theta, \varphi\}$ by (19). This solution is incomplete, as we have already seen, since the line-element and the corresponding metric are only defined when the coordinate $R > \alpha$. The solution is known as the *exterior* Schwarzschild solution.

Another way of seeing that the metrics corresponding to the line-elements defined by (16) are all the same, is by invoking Birkoff's Theorem [11]. This theorem establishes, with mathematical certainty, that the Schwarzschild solution (exterior, interior or both) is the only solution of the spherically symmetric vacuum field equations*.

Claim 5. In [2] the author notes that the scalar curvature of the metric corresponding to (16) is given by

$$f = \frac{12\,\alpha^2}{C_n^3} = \frac{12\,\alpha^2}{[(r - r_0)^n + \alpha^n]^{6/n}}$$

and that as $r \to r_0$ there is no curvature singularity. He then concludes that a "black hole" singularity cannot exist.

In fact, as we have just seen, the line-element (16) only corresponds to the exterior Schwarzschild solution, which is a manifold foliated by 2-spheres with radial coordinate $R > \alpha$. The calculation in [2] therefore only proves that the exterior solution has no curvature singularity. This is a well known fact. Writing (16) in its equivalent form (19) and extending the coordinate R to obtain the *interior* Schwarzschild solution $(0 < R < \alpha)$, the scalar curvature is given by

$$f = \frac{12\alpha^2}{R^3}\,,$$

from which it is clear that

$$\lim_{R \to 0} f = \infty\,.$$

Hence there is a curvature singularity at $R = 0$. Since the vector ∂_R is timelike for $0 < R < \alpha$, the singularity corresponds to a black hole.

*The assumption of staticity is not actually required, hence all spherically symmetric spacetimes satisfying the vacuum field equations are static.

4 Conclusions

We have presented a number of simple examples which hopefully elucidate the concepts of coordinate transformation and metric extension in differential geometry. Implications of the concepts were also discussed, with particular focus on a number of the relativistic claims of [1–8]. It was proven that each of these claims was false. The claims appear to arise from a lack of understanding of the notions of coordinate transformation and metric (coordinate) extension. Any conclusions contained in [1–8] that are based on such claims should therefore be considered as unproven. In particular, the claim that the black hole "is not consistent at all with general relativity" is completely false.

General relativity is a difficult topic, which is grounded in advanced mathematics (indeed, Einstein himself is quoted as saying something along the lines of "Ever since the mathematicians took hold of relativity, I no longer understand it myself!"). A sound understanding of differential geometry is a prerequisite for understanding the theory in its modern form. Thus to paraphrase Lao Tzu [12] — beware of the half-enlightened master.

Postscript

The article by Stephen J. Crothers in the current issue [13] provides a good illustration of the problems discussed above. For example, in his first "counter-example" he considers a metric which is easily seen to be the Schwarzschild metric written in terms of an 'inverted' radial coordinate. Using x to denote the inverted radial coordinate (denoted by r in [13]), and R to denote the usual Schwarzschild radius, the transformation is $R = 2m - x$. In particular, $R = 0$ corresponds to $x = 2m$, and $R = 2m$ corresponds to $x = 0$. It is thus not surprising that the coordinate singualrity is at $x = 0$ and the point singularity is at $x = 2m$. The other counter-examples in [13] can be dismissed through similar arguments.

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