Gravitational Spectral Shift Exterior to the Sun, Earth and the Other Oblate Spheroidal Planets

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Here, we use our new metric tensor exterior to homogeneous oblate spheroidal mass distributions to study gravitational spectral shift of light in the vicinity of the Sun, Earth and other oblate spheroidal planets. It turns out most profoundly that, this experimentally verified phenomenon holds good in the gravitational field exterior to an oblate spheroid using our approach. In approximate gravitational fields, our obtained theoretical value for the Pound-Rebka experiment on gravitational spectra shift along the equator of the Earth ($2.578{\times}10^{-15}$) agrees satisfactorily with the experimental value of $2.45{\times}10^{-15}$. We also predict theoretical values for the Pound-Rebka experiment on the surface (along the equator) of the Sun and other oblate spheroidal planets.

1 Introduction

According to the General Theory of Gravitation, the rate of a clock is slowed down when it is in the vicinity of a large gravitating mass. Since the characteristic frequencies of atomic transitions are, in effect, clocks, one has the result that the frequency of such a transition occurring, say, on the surface of the Sun, should be lowered by comparison with a similar transition observed in a terrestrial laboratory. This manifests itself as a gravitational red shift in the wavelengths of spectral lines [1]. It has been experimentally and astrophysically observed that there is an increase in the frequency of light (photon) when the source or emitter is further away from the body than the receiver. The frequency of light will increase (shifting visible light towards the blue end of the spectrum) as it moves to lower gravitational potentials (into a gravity well). Also, there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The frequency of light will decrease (shifting visible light towards the red end of the spectrum) as it moves into higher gravitational potentials (out of a gravity well). This was experimentally confirmed in the laboratory by the Pound-Rebka experiment in 1959 (they used the Mossbauer effect to measure the change in frequency in gamma rays as they travelled from the ground to the top of Jefferson Labs at Havard University) [2]. This gravitational phenomenon was later confirmed by astronomical observations [3]. In this article, we verify the validity of our metric tensor exterior to a massive homogeneous oblate spheroid by studying gravitational spectral shift in the vicinity of the Sun, Earth and other oblate spheroidal planets. Basically, we assume that these gravitational sources are time independent and homogeneous distributions of mass within spheroids, characterized by at most two typical integrals of geodesic motion, namely, energy and angular momentum. From an astrophysical point of view, such an assumption, although not necessary, could, however, prove useful, because it is equivalent to the assumption that the gravitational source is changing slowly in time so that partial time derivatives are negligible compared to the spatial ones. We stress that the mass source considered is not the most arbitrary one from a theoretical point of view, but on the other hand, many astrophysically interesting systems are usually assumed to be time independent (or static from another point of view) and axially symmetric continuous sources.

2 Covariant metric tensor exterior to a massive homogeneous oblate spheroid

The covariant metric tensor in the gravitational field of a homogeneous oblate spheroid in oblate spheroidal coordinates (η, ξ, ϕ) has been obtained [4, 5] as;

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta,\xi)\right), \tag{2.1}$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right], \tag{2.2}$$

$$g_{12} \equiv g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1}\right], \tag{2.3}$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right], \tag{2.4}$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2)\,, \tag{2.5}$$

$f(\eta, \xi)$ is an arbitrary function determined by the mass or pressure distribution and hence possesses all the symmetries of the latter, a priori. Let us now recall that for any gravitational field [4–7]

$$g_{00} \cong 1 + \frac{2}{c^2}\Phi \tag{2.6}$$

where Φ is Newton's gravitational scalar potential for the field under consideration. Thus we can then deduce that the unknown function in our field equation can be given approximately as

$$f(\eta, \xi) \cong \Phi(\eta, \xi)\,, \tag{2.7}$$

where $\Phi(\eta, \xi)$ is Newton's gravitational scalar potential exterior to a homogeneous oblate spheroidal mass. It has been shown that [8];

$$\Phi(\eta, \xi) = B_0 Q_0(-i\xi)\, P_0(\eta) + B_2 Q_2(-i\xi)\, P_2(\eta)\,, \tag{2.8}$$

where Q_0 and Q_2 are the Legendre functions linearly independent to the Legendre polynomials P_0 and P_2 respectively; B_0 and B_2 are constants given by

$$B_0 = \frac{4\pi G \rho_0 a^2 \xi_0}{3\Delta_1}$$

and

$$B_2 = \frac{4\pi G \rho_0 a^2 \xi_0}{9\Delta_2} \left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0},$$

where Δ_1 and Δ_2 are defined as

$$\Delta_1 = \left[\frac{d}{d\xi} Q_0(-i\xi)\right]_{\xi=\xi_0}$$

and

$$\Delta_2 = Q_0 \left[\frac{d}{d\xi} P_2(-i\xi)\right]_{\xi=\xi_0} - P_2(-i\xi) \left[\frac{d}{d\xi} Q_2(-i\xi)\right]_{\xi=\xi_0},$$

G is the universal gravitational constant, ρ_0 is the uniform density of the oblate spheroid and a is a constant parameter.

In a recent article [9], we obtained a satisfactory approximate expression for equation (2.8) as;

$$\Phi(\eta, \xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right) i - \frac{B_2}{30\xi^3}\left(7+15\xi^2\right)\left(3\eta^2-1\right) i \tag{2.9}$$

with

$$\Phi(\eta, \xi) \approx \frac{B_0}{3\xi^3}\left(1 + 3\xi^2\right) i + \frac{B_2}{30\xi^3}\left(7 + 15\xi^2\right) i$$

and

$$\Phi(\eta, \xi) \approx \frac{B_0}{3\xi^3}\left(1 + 3\xi^2\right) i - \frac{B_2}{15\xi^3}\left(7 + 15\xi^2\right) i$$

as the respective approximate expressions for the gravitational scalar potential along the equator and pole exterior to homogeneous oblate spheroidal bodies. These equations were used to compute approximate values for the gravitational scalar potential exterior to the Sun, Earth and other oblate spheroidal planets [9].

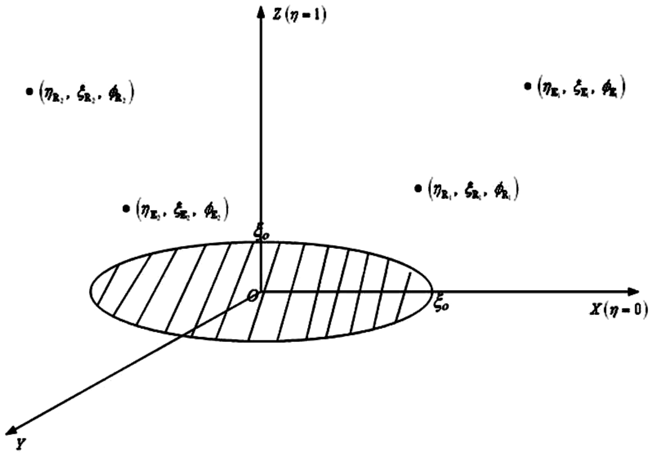

Fig. 1: Emission and reception space points of light (photon).

3 Gravitational spectral shift exterior to oblate spheroidal distributions of mass

Here, we consider a beam of light moving from a source or emitter at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver at a fixed point in the same gravitational field. Einstein's equation of motion for a photon is used to derive an expression for the shift in frequency of a photon moving in the gravitational field of an oblate spheroidal mass.

Now, consider a beam of light moving from a source or emitter (E) at a fixed point in the gravitational field of an oblate spheroidal body to an observer or receiver (R) at a fixed point in the field as shown in Fig. 1.

Let the space time coordinates of the emitter and receiver be $(t_E, \eta_E, \xi_E, \phi_E)$ and $(t_R, \eta_R, \xi_R, \phi_R)$ respectively. It is well known that light moves along a null geodesic given by

$$d\tau = 0\,. \tag{3.1}$$

Thus, the world line element for a photon (light) takes the form

$$c^2 g_{00}\, dt^2 = g_{11}\, d\eta^2 + 2 g_{12}\, d\eta\, d\xi + g_{22}\, d\xi^2 + g_{33}\, d\phi^2. \tag{3.2}$$

Substituting the covariant metric tensor for this gravitational field in equation (3.2) gives

$$\begin{aligned} c^2\left(1 + \frac{2}{c^2} f(\eta, \xi)\right) dt^2 = &-\frac{a^2}{1+\xi^2-\eta^2} \times \\ &\times \left[\eta^2\left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\xi^2\left(1+\xi^2\right)}{(1-\eta^2)}\right] d\eta^2 - \\ &- \frac{2a^2\eta\xi}{1+\xi^2-\eta^2}\left[1 - \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1}\right] d\eta\, d\xi - \\ &- \frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} + \frac{\eta^2\left(1-\eta^2\right)}{(1+\xi^2)}\right] d\xi^2 - \\ &- a^2\left(1+\xi^2\right)\left(1-\eta^2\right) d\phi^2. \end{aligned} \tag{3.3}$$

Now, let u be a suitable parameter that can be used to study the motion of a photon in this gravitational field. Then equation (3.3) can be written as

$$c^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)\left(\frac{dt}{du}\right)^2 = -\frac{a^2}{1+\xi^2-\eta^2}\times \\ \times\left[\eta^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\xi^2\left(1+\xi^2\right)}{(1-\eta^2)}\right]d\eta^2- \\ -\frac{2a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]\left(\frac{d\eta}{du}\frac{d\xi}{du}\right)- \\ -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\eta^2\left(1-\eta^2\right)}{(1+\xi^2)}\right]\times \\ \times\left(\frac{d\xi}{du}\right)^2-a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\left(\frac{d\phi}{du}\right)^2. \quad (3.4)$$

Equation (3.4) can be equally written as

$$\frac{dt}{du} = \frac{1}{c}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-\frac{1}{2}}ds\,, \quad (3.5)$$

where ds is defined as

$$ds^2 = -\frac{a^2}{1+\xi^2-\eta^2}\times \\ \times\left[\eta^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\xi^2\left(1+\xi^2\right)}{(1-\eta^2)}\right]d\eta^2- \\ -\frac{2a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right]\left(\frac{d\eta}{du}\frac{d\xi}{du}\right)- \\ -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\eta^2\left(1-\eta^2\right)}{(1+\xi^2)}\right]\times \\ \times\left(\frac{d\xi}{du}\right)^2-a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\left(\frac{d\phi}{du}\right)^2. \quad (3.6)$$

Integrating equation (3.5) for a signal of light moving from emitter E to receiver R gives

$$t_R - t_E = \frac{1}{c}\int_{u_E}^{u_R}\left[\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-\frac{1}{2}}ds\right]du\,. \quad (3.7)$$

The time interval between emission and reception of all light signals is well known to be the same for all light signals in relativistic mechanics (constancy of the speed of light) and thus the integral on the right hand side is the same for all light signals. Consider two light signals designated 1 and 2 then

$$t_R^1 - t_E^1 = t_R^2 - t_E^2 \quad (3.8)$$

or

$$t_R^2 - t_R^1 = t_E^2 - t_E^1\,. \quad (3.9)$$

Thus,

$$\Delta t_R = \Delta t_E\,. \quad (3.10)$$

Hence, coordinate time difference of two signals at the point of emission equals that at the point of reception. From our expression for gravitational time dilation in this gravitational field [10], we can write

$$\Delta\tau_R = \left(1+\frac{2}{c^2}f_R(\eta,\xi)\right)^{\frac{1}{2}}\Delta t_R\,. \quad (3.11)$$

Equations (3.9), (3.10) and (3.11) can be combined to give

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1+\frac{2}{c^2}f_R(\eta,\xi)}{1+\frac{2}{c^2}f_E(\eta,\xi)}\right)^{\frac{1}{2}}. \quad (3.12)$$

Now, consider the emission of a peak or crest of light wave as one event. Let n be the number of peaks emitted in a proper time interval $\Delta\tau_E$, then, by definition, the frequency of the light relative to the emitter, ν_E, is given as

$$\nu_E = \frac{n}{\Delta\tau_E}\,. \quad (3.13)$$

Similarly, since the number of cycles is invariant, the frequency of light relative to the receiver, ν_R, is given as

$$\nu_R = \frac{n}{\Delta\tau_R}\,. \quad (3.14)$$

Consequently,

$$\frac{\nu_R}{\nu_E} = \frac{\Delta\tau_E}{\Delta\tau_R} = \left(1+\frac{2}{c^2}f_E(\eta,\xi)\right)^{\frac{1}{2}}\left(1+\frac{2}{c^2}f_R(\eta,\xi)\right)^{-\frac{1}{2}} \quad (3.15)$$

or

$$\frac{\nu_R}{\nu_E} \approx \left(1+\frac{2}{c^2}f_E(\eta,\xi)\right)\left(1-\frac{2}{c^2}f_R(\eta,\xi)\right) \quad (3.16)$$

or

$$\frac{\nu_R}{\nu_E}-1 \approx \frac{1}{c^2}\left[f_E(\eta,\xi)-f_R(\eta,\xi)\right] \quad (3.17)$$

to the order of c^{-2}. Alternatively, equation (3.17) can be written as

$$z \equiv \frac{\Delta\nu}{\nu_E} \equiv \frac{\nu_R-\nu_E}{\nu_E} \approx \frac{1}{c^2}\left[f_E(\eta,\xi)-f_R(\eta,\xi)\right]. \quad (3.18)$$

It follows from equation (3.18) that if the source is nearer the body than the receiver then $f_E(\eta,\xi) < f_R(\eta,\xi)$ and hence $\Delta\nu < 0$. This indicates that there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The light is said to have undergone a red shift (that is the light moves toward red in the visible spectrum). Otherwise (source further away from body than receiver), the light undergoes a blue shift. Now, consider a signal of light emitted and received along the equator of the homogeneous oblate spheroidal Earth (approximate gravitational field where $f(\eta,\xi) \approx \Phi(\eta,\xi)$. The ratio of the shift

Emi Pt	Recep pt	$z\,(\times 10^{-10})$	Type of shift
ξ_0	ξ_0	0	none
$2\xi_0$	ξ_0	3.454804	blue
$3\xi_0$	ξ_0	4.603165	blue
$4\xi_0$	ξ_0	5.176987	blue
$5\xi_0$	ξ_0	5.521197	blue
$6\xi_0$	ξ_0	5.750643	blue
$7\xi_0$	ξ_0	5.914522	blue
$8\xi_0$	ξ_0	6.037426	blue
$9\xi_0$	ξ_0	6.133016	blue
$10\xi_0$	ξ_0	6.209486	blue

Fig. 2: Ratio of the shift in frequency of light to the frequency of the emitted light at points along equator and received on the surface of the Earth on the equator.

Emi Pt	Recep pt	$z\,(\times 10^{-10})$	Type of shift
ξ_0	ξ_0	0	none
ξ_0	$2\xi_0$	−3.454804	red
ξ_0	$3\xi_0$	−4.603165	red
ξ_0	$4\xi_0$	−5.176987	red
ξ_0	$5\xi_0$	−5.521197	red
ξ_0	$6\xi_0$	−5.750643	red
ξ_0	$7\xi_0$	−5.914522	red
ξ_0	$8\xi_0$	−6.037426	red
ξ_0	$9\xi_0$	−6.133016	red
ξ_0	$10\xi_0$	−6.209486	red

Fig. 3: Ratio of the shift in frequency of light to the frequency of the emitted light at points along equator and received on the surface of the Earth on the equator.

Body	Radial dist. (km)	ξ at pt	Φ_E (N m kg^{-1})	Φ_R (N m kg^{-1})	Predicted shift
Sun	700, 022.5	241.527	-1.9375791×10^{11}	-1.9373218×10^{11}	-2.85889×10^{-21}
Earth	6, 378.023	12.010	-6.2079113×10^{7}	-6.2078881×10^{7}	-2.57800×10^{-15}
Mars	3, 418.5	9.231	-1.2401149×10^{7}	-1.2317966×10^{7}	-9.24256×10^{-20}
Jupiter	71, 512.5	2.641	-1.4968068×10^{9}	-1.4958977×10^{9}	-1.010111×10^{-20}
Saturn	60, 292.5	1.971	-4.8486581×10^{8}	-4.8484869×10^{8}	-1.902222×10^{-21}
Uranus	25, 582.5	3.994	-2.1563913×10^{8}	-2.1522082×10^{8}	-4.647889×10^{-20}
Neptune	24, 782.5	4.304	-2.5243240×10^{8}	-2.5196722×10^{8}	-5.168667×10^{-20}

Fig. 4: Predicted Pound-Rebka shift in frequency along the equator for the Sun, Earth and the other oblate spheroidal planets.

in frequency to the frequency of the emitted light at various points along the equator and received on the equator at the surface of the homogeneous oblate spheroidal Earth can be computed using equation (3.18). This yields Table 1. Also, the ratio of the shift in frequency of light to the frequency of the emitted light on the equator at the surface and received at various points along the equator of the homogeneous oblate spheroidal Earth can be computed. This gives Table 2.

Tables 1, thus confirms our assertion above that there is an increase in the frequency of light when the source or emitter is further away from the body than the receiver. The frequency of light will increase (shifting visible light toward the blue end of the spectrum) as it moves to lower gravitational potentials (into a gravity well). Table 2, also confirms our assertion above that there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The frequency of light will decrease (shifting visible light toward the red end of the spectrum) as it moves to higher gravitational potentials (out of a gravity well). Also, notice that the shift in both cases increases with increase in the distance of separation between the emitter and receiver. The value of the shift is equal in magnitude at the same separation distances for both cases depicted in Tables 1 and 2.

Now, suppose the Pound-Rebka experiment is performed at the surface of the Sun, Earth and other oblate spheroidal planets on the equator. Then, since the gamma ray frequency shift was observed at a height of 22.5m above the surface, we model our theoretical computation and calculate the theoretical value for this shift. This computation yields Table 3.

With these predictions, experimental astrophysicists and astronomers can now attempt carrying out similar experiments on these bodies. Although, the prospects of carrying out such experiments on the surface of some of the planets and Sun are less likely (due to temperatures on their surfaces and other factors); theoretical studies of this type helps us to understand the behavior of photons as they leave or approach these astrophysical bodies. This will thus aid in the development of future instruments that can be used to study these heavenly bodies.

4 Conclusion

The practicability of the findings in this work is an encouraging factor. More so, that in this age of computational precision, the applications of these results is another factor.

Submitted on June 01, 2010 / Accepted on June 05, 2010

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