Smarandache's Minimum Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry

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In this note, we present a proof to the Smarandache's Minimum Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry.

1 Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Here, in this study, we give hyperbolic version of Smarandache minimum theorem in the Einstein relativistic velocity model of hyperbolic geometry. The well-known Smarandache minimum theorem states that if ABC is a triangle and AA', BB', CC' are concurrent cevians at P, then

$$\frac{PA}{PA'} \cdot \frac{PB}{PB'} \cdot \frac{PC}{PC'} \geq 8$$

and

$$\frac{PA}{PA'} + \frac{PB}{PB'} + \frac{PC}{PC'} \geq 6$$

(see [1]).

Let D denote the complex unit disc in complex z-plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leqslant \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1. ***(Ceva's theorem for hyperbolic triangles)***. *If M is a point not on any side of an gyrotriangle ABC in a gyrovector space $(V_s, \oplus, \otimes)$, such that AM and BC meet in A', BM and CA meet in B', and CM and AB meet in C', then*

$$\frac{\gamma_{|AC'|}\,|AC'|}{\gamma_{|BC'|}\,|BC'|} \cdot \frac{\gamma_{|BA'|}\,|BA'|}{\gamma_{|CA'|}\,|CA'|} \cdot \frac{\gamma_{|CB'|}\,|CB'|}{\gamma_{|AB'|}\,|AB'|} = 1,$$

where $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$.

(See [2, p. 564].) For further details we refer to the recent book of A.Ungar [3].

Theorem 2. ***(Van Aubel's theorem in hyperbolic geometry).*** *If the point P does lie on any side of the hyperbolic triangle ABC, and BC meets AP in D, CA meets BP in E, and AB meets CP in F, then*

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|BC|}\,|BC|}{2}\left(\frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{1}{\gamma_{|BD|}\,|BD|}\right) + \frac{\gamma_{|BC|}\,|BC|}{2}\left(\frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|} \cdot \frac{1}{\gamma_{|CD|}\,|CD|}\right).$$

(See [4].)

2 Main result

In this section, we prove Smarandache's minimum theorem in the Einstein relativistic velocity model of hyperbolic geometry.

Theorem 3. *If ABC is a gyrotriangle and AA', BB', CC' are concurrent cevians at P, then*

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}\cdot\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}\geqslant 1,$$

and

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}+\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}+\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}\geqslant 3.$$

Proof. We set

$$|A'C| = a_1, |BA'| = a_2, |B'A| = b_1,$$

$$|B'C| = b_2, |C'B| = c_1, |C'A| = c_2,$$

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}\cdot\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}\cdot\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}= P,$$

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}+\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}+\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}= S.$$

If we use the Van Aubel's theorem in the gyrotriangle ABC (See Theorem 2), then

$$\frac{\gamma_{|AP|}|AP|}{\gamma_{|PA'|}|PA'|}=\frac{\gamma_{|BC|}|BC|}{2}\left(\frac{\gamma_{|AB'|}|AB'|}{\gamma_{|CB'|}|CB'|}\cdot\frac{1}{\gamma_{|BA'|}|BA'|}\right)+\frac{\gamma_{|BC|}|BC|}{2}\left(\frac{\gamma_{|AC'|}|AC'|}{\gamma_{|BC'|}|BC'|}\cdot\frac{1}{\gamma_{|CA'|}|CA'|}\right)$$

$$=\frac{\gamma_a a}{2}\left[\frac{\gamma_{b_1}b_1}{\gamma_{b_2}b_2}\cdot\frac{1}{\gamma_{a_2}a_2}+\frac{\gamma_{c_2}c_2}{\gamma_{c_1}c_1}\cdot\frac{1}{\gamma_{a_1}a_1}\right], \tag{1}$$

and

$$\frac{\gamma_{|BP|}|BP|}{\gamma_{|PB'|}|PB'|}=\frac{\gamma_{|CA|}|CA|}{2}\left(\frac{\gamma_{|BC'|}|BC'|}{\gamma_{|AC'|}|AC'|}\cdot\frac{1}{\gamma_{|CB'|}|CB'|}\right)+\frac{\gamma_{|CA|}|CA|}{2}\left(\frac{\gamma_{|BA'|}|BA'|}{\gamma_{|CA'|}|CA'|}\cdot\frac{1}{\gamma_{|AB'|}|AB'|}\right)$$

$$=\frac{\gamma_b b}{2}\left[\frac{\gamma_{c_1}c_1}{\gamma_{c_2}c_2}\cdot\frac{1}{\gamma_{b_2}b_2}+\frac{\gamma_{a_2}a_2}{\gamma_{a_1}a_1}\cdot\frac{1}{\gamma_{b_1}b_1}\right], \tag{2}$$

and

$$\frac{\gamma_{|CP|}|CP|}{\gamma_{|PC'|}|PC'|}=\frac{\gamma_{|AB|}|AB|}{2}\left(\frac{\gamma_{|CA'|}|CA'|}{\gamma_{|BA'|}|BA'|}\cdot\frac{1}{\gamma_{|AC'|}|AC'|}\right)+\frac{\gamma_{|AB|}|AB|}{2}\left(\frac{\gamma_{|CB'|}|CB'|}{\gamma_{|AB'|}|AB'|}\cdot\frac{1}{\gamma_{|BC'|}|BC'|}\right)$$

$$=\frac{\gamma_c c}{2}\left(\frac{\gamma_{a_1}a_1}{\gamma_{a_2}a_2}\cdot\frac{1}{\gamma_{c_2}c_2}+\frac{\gamma_{b_2}b_2}{\gamma_{b_1}b_1}\cdot\frac{1}{\gamma_{c_1}c_1}\right). \tag{3}$$

If we use the Ceva's theorem in the gyrotriangle ABC (See Theorem 1), we have

$$\frac{\gamma_{|CA'|}\,|CA'|}{\gamma_{|BA'|}\,|BA'|}\cdot\frac{\gamma_{|AB'|}\,|AB'|}{\gamma_{|CB'|}\,|CB'|}\cdot\frac{\gamma_{|BC'|}\,|BC'|}{\gamma_{|AC'|}\,|AC'|}=\frac{\gamma_{a_1}a_1}{\gamma_{a_2}a_2}\cdot\frac{\gamma_{b_1}b_1}{\gamma_{b_2}b_2}\cdot\frac{\gamma_{c_1}c_1}{\gamma_{c_2}c_2}=1. \tag{4}$$

From (1) and (4), we have

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PA'|}\,|PA'|}=\frac{\gamma_a a}{2}\left(\frac{\gamma_{b_1}b_1\gamma_{c_2}c_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}\right)+\frac{\gamma_a a}{2}\left(\frac{\gamma_{b_1}b_1\gamma_{c_2}c_2}{\gamma_{a_1}a_1\gamma_{b_1}b_1\gamma_{c_1}c_1}\right)=\frac{\gamma_a a}{2}\cdot\frac{2\gamma_{b_1}b_1\gamma_{c_2}c_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}$$

$$=\frac{\gamma_a a\gamma_{b_1}b_1\gamma_{c_2}c_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}. \tag{5}$$

Similary we obtain that

$$\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PB'|}\,|PB'|}=\frac{\gamma_b b\gamma_{c_1}c_1\gamma_{a_2}a_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}, \tag{6}$$

and

$$\frac{\gamma_{|CP|}\,|CP|}{\gamma_{|PC'|}\,|PC'|}=\frac{\gamma_c c\gamma_{a_1}a_1\gamma_{b_2}b_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}. \tag{7}$$

From the relations (5), (6) and (7) we get

$$P=\frac{\gamma_a a\gamma_{b_1}b_1\gamma_{c_2}c_2\cdot\gamma_b b\gamma_{c_1}c_1\gamma_{a_2}a_2\cdot\gamma_c c\gamma_{a_1}a_1\gamma_{b_2}b_2}{(\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2)^3}=$$

$$=\frac{\gamma_a a\gamma_b b\gamma_c c}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2} \tag{8}$$

and

$$S=\frac{\gamma_a a\gamma_{b_1}b_1\gamma_{c_2}c_2+\gamma_b b\gamma_{c_1}c_1\gamma_{a_2}a_2+\gamma_c c\gamma_{a_1}a_1\gamma_{b_2}b_2}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}. \tag{9}$$

Because $\gamma_a \geqslant \gamma_{a_2}$, $\gamma_b \geqslant \gamma_{b_2}$, and $\gamma_c \geqslant \gamma_{c_2}$ result

$$\gamma_a\gamma_b\gamma_c \geqslant \gamma_{a_2}\gamma_{b_2}\gamma_{c_2}. \tag{10}$$

Therefore

$$\frac{\gamma_a a\gamma_b b\gamma_c c}{\gamma_{a_2}a_2\gamma_{b_2}b_2\gamma_{c_2}c_2}\geqslant 1. \tag{11}$$

From the relations (8) and (11), we obtain that $P \geqslant 1$. If we use the inequality of arithmetic and geometric means, we obtain

$$S \geqslant 3\sqrt[3]{\frac{\gamma_a a \gamma_{b_1} b_1 \gamma_{c_2} c_2 \cdot \gamma_b b \gamma_{c_1} c_1 \gamma_{a_2} a_2 \cdot \gamma_c c \gamma_{a_1} a_1 \gamma_{b_2} b_2}{(\gamma_{a_2} a_2 \gamma_{b_2} b_2 \gamma_{c_2} c_2)^3}} =$$

$$= 3\sqrt[3]{\frac{\gamma_a a \gamma_b b \gamma_c c}{\gamma_{a_2} a_2 \gamma_{b_2} b_2 \gamma_{c_2} c_2}}. \qquad (12)$$

From the relations (11) and (12), we obtain that $S \geqslant 3$. □

3 Conclusion

The special theory of relativity as was originally formulated by Einstein in 1905, [8], to explain the massive experimental evidence against ether as the medium for propagating electromagnetic waves, and Varičak in 1908 discovered the connection between special theory of relativity and hyperbolic geometry, [9]. The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Einstein relativistic velocity model, Smarandache minimum theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, gamma factor γ_v reduces to 1, so that the gyroinequalities (11) and (12) reduces to the

$$\frac{PA}{PA'} \cdot \frac{PB}{PB'} \cdot \frac{PC}{PC'} \geqslant 1,$$

and

$$\frac{PA}{PA'} + \frac{PB}{PB'} + \frac{PC}{PC'} \geqslant 3,$$

in Euclidean geometry. We observe that the previous inequalities are "weaker" than the inequalities of Smarandache's theorem of minimum.

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