Gravity and the Conservation of Energy

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The Schwarzschild metric apportions the energy equivalence of a mass into a time component, a space component and a gravitational component. This apportionment indicates there is a source of gravitational energy as well as a limit to the magnitude of gravitational energy.

1 Introduction

Albert Einstein asserted that his field equations are in essence a restatement of the conservation of energy and momentum [1, pp. 145–149]. Every solution of the field equations, therefore, must account for all energy in the system described by the solution. How do solutions to the field equations account for gravitational energy?

This paper explains how within Schwarzschild's solution [2] to Einstein's field equations the effects of gravity can be represented as a velocity and as an apportionment of mass-energy equivalence. This allows an accounting for gravitational energy as part of mass-energy equivalence.

The paper first considers a spacetime without gravity, as described by the Minkowski metric. The Minkowski metric can be rewritten as a summation of velocities and as an apportionment of energy equivalence.

The paper then shows the Schwarzschild metric, which adds a spherical non-rotating mass to the spacetime defined by the Minkowski metric, can also be rewritten as a summation of velocities and as an apportionment of energy equivalence. The apportionment of energy equivalence includes a gravitational component. This indicates gravitational energy has a source and a limit to its magnitude.

2 The Minkowski Metric

The Minkowski metric was originally derived based on Hermann Minkowski's fundamental axiom for space-time set out in an address [3] given in September 1908:

> *The substance at any world-point may always, with the appropriate determination of space and time, be looked upon as at rest.*

Minkowski's fundamental axiom for the space-time continuum indicates that for the substance at a world point (e.g., a particle) there exists a local reference frame, with its own local space and time coordinates, in which the substance is at rest with respect to the local space coordinates (but not with respect to the local time coordinate).

For example, assume the local reference frame for a particle has the local space coordinates (ξ, η, ς) and the local time coordinate τ. For the particle, with respect to the local reference frame,

$$\frac{d\xi}{d\tau} = \frac{d\eta}{d\tau} = \frac{d\varsigma}{d\tau} = 0. \qquad (1)$$

The Minkowski metric is often expressed using Cartesian reference coordinates (x, y, z, t) and the local time coordinate τ, i.e.,

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \qquad (2)$$

The Minkowski metric can also be expressed using spherical coordinates, i.e.,

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - (r sin\theta)^2 d\phi^2. \qquad (3)$$

3 Selection of a reference frame from which to measure velocity

In order to measure velocity in the Minkowski metric (and the Schwarzschild metric) it is important to select and consistently use a reference frame. In the Minkowski metric there are two reference frames to choose from. The first is the local reference frame defined by local coordinates $(\xi, \eta, \varsigma, \tau)$. The other is the reference frame (referred to herein as the coordinate reference frame) defined by reference coordinates (x, y, z, t).

There is a distinct disadvantage to use of the local reference frame to make measurements: in its own local reference frame an object is always at rest, that is, as indicated by (1) there is no spatial velocity, i.e., no change in the values of the local space coordinates (ξ, η, ς) with respect to passage of time as measured by the time coordinate τ.

In the coordinate reference frame, however, there can be a detectable motion through the space coordinates. This is referred to herein as spatial velocity $(\vec{v}_s)$, which is a vector sum of the motion in three dimensions of space, i.e.,

$$\vec{v}_s = \vec{v}_x + \vec{v}_y + \vec{v}_z, \qquad (4)$$

and which has a magnitude v_s where

$$v_s = \left|\vec{v}_s\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}, \qquad (5)$$

as measured by the coordinate reference frame.

Because of this distinct advantage of making measurements from the coordinate reference frame, this is the reference frame that will be consistently used herein to make measurements.

4 Expressing the Minkowski Metric as a sum of velocities

The Minkowski metric, shown in (2), can be rearranged into the form of a sum of velocities. Since the observer is making measurements from the coordinate reference frame, momentum and energy will need to be measured with respect to changes in the reference time coordinate t. The Minkowski metric is therefore rearranged to show this. Specifically, (2) can be rearranged as

$$c^2dt^2 = c^2d\tau^2 + dx^2 + dy^2 + dz^2, \tag{6}$$

and therefore,

$$c^2 = c^2\left(\frac{d\tau}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \tag{7}$$

which can be reduced to

$$c^2 = c^2\left(\frac{d\tau}{dt}\right)^2 + v_s^2. \tag{8}$$

Let a time velocity v_τ be defined as

$$v_\tau = c\frac{d\tau}{dt}, \tag{9}$$

so that v_τ is a measure of the rate of passage of time as measured by the local time coordinate τ with respect to the rate of the passage of time as measured by the reference time coordinate t. This allows (7) to be rewritten as

$$c^2 = v_\tau^2 + v_s^2. \tag{10}$$

Since the time dimension is regarded as being orthogonal to the space dimensions, (10) can be written in the form of a vector sum, i.e.,

$$c = \left|\vec{v}_\tau + \vec{v}_s\right|. \tag{11}$$

Equation (11) is the Minkowski metric written as a sum of velocities. That is, the vector sum of the velocity in the dimensions of time and space is always equal to the speed of light c.

5 Energy equivalence in the Minkowski metric

The Minkowski metric, like all solutions to Einstein's field equations, describes a matterless field [1, p. 143]. In order to see how the Minkowski metric apportions energy equivalence, it is only necessary to place a particle with mass m anywhere in the field. From (11), a momentum of mass m across four dimensions of time and space can be expressed as

$$mc = \left|m\vec{v}_\tau + m\vec{v}_s\right|. \tag{12}$$

Equation (10) can also be rewritten as

$$mc^2 = mv_\tau^2 + mv_s^2. \tag{13}$$

Equation (13) indicates how the Minkowski metric apportions the energy equivalence [4],

$$E = mc^2, \tag{14}$$

of mass m into an energy component E_τ in the time dimension, where

$$E_\tau = mv_\tau^2, \tag{15}$$

and an energy component in the space dimensions, where

$$E_s = mv_s^2, \tag{16}$$

so that

$$E = mc^2 = E_\tau + E_s. \tag{17}$$

6 The Schwarzschild metric

The full Schwarzschild metric for a spherical non-rotating mass M with a Schwarzschild radius R, is typically expressed with the reference coordinates in the form of spherical coordinates, i.e.,

$$c^2d\tau^2 = c^2\left(1-\frac{R}{r}\right)dt^2 - \frac{dr^2}{(1-R/r)} - r^2d\theta^2 - (r\sin\theta)^2d\phi^2. \tag{18}$$

When $M = 0$ and thus $R = 0$, the Schwarzschild metric reduces to the Minkowski metric.

7 Expressing the Schwarzschild Metric as a sum of velocities

In order to express the Schwarzschild metric as a sum of velocities, a gravitational velocity v_g can be defined using the Newtonian definition of gravitational escape velocity, that is

$$v_g = c\sqrt{\frac{R}{r}}. \tag{19}$$

Likewise because in the Schwarzschild metric space is curved a spatial velocity velocity v_{ss} through curved space can be defined as

$$v_{ss} = \sqrt{\frac{1}{1-R/r}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + (r\sin\theta)^2\left(\frac{d\phi}{dt}\right)^2}. \tag{20}$$

The Schwarzschild metric in (18) can now be expressed as a sum of the velocities v_τ, v_g and v_{ss}. That is, (19) can be rearranged as

$$c^2dt^2 = c^2d\tau^2 + c^2\frac{R}{r}dt^2 + \frac{dr^2}{(1-R/r)} + r^2d\theta^2 + (r\sin\theta)^2d\phi^2, \tag{21}$$

and thus

$$c^2 = c^2\left(\frac{d\tau}{dt}\right)^2 + c^2\frac{R}{r} + \frac{1}{1-R/r}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + (r\sin\theta)^2\left(\frac{d\phi}{dt}\right)^2. \tag{22}$$

Using the definition of v_{ss} set out in (20), the definition of v_τ set out in (9) and the definition of v_g set out in (19), allows (22) to be simplified to

$$c^2 = v_\tau^2 + v_g^2 + v_{ss}^2. \tag{23}$$

If a gravitational dimension is regarded as being orthogonal to both the dimensions of curved space and the time dimension, (23) can be written in the form of a vector sum, i.e.,

$$c = \left|\vec{v}_\tau + \vec{v}_g + \vec{v}_{ss}\right|. \tag{24}$$

Equation (24) is the Schwarzschild metric written as a sum of velocities. That is, the vector sum of the velocity in the dimensions of time, space and gravity is always equal to the speed of light c.

8 Using the Schwarzschild metric to apportion energy equivalence

In order to see how the Schwarzschild metric apportions energy equivalence, it is only necessary to place a particle with mass m anywhere in the field. From (24), a momentum of mass m across five dimensions of time, space and gravity can be expressed as

$$mc = \left|m\vec{v}_\tau + m\vec{v}_g + m\vec{v}_{ss}\right|. \tag{25}$$

Equation (10) can also be rewritten as

$$mc^2 = mv_\tau^2 + mv_g^2 + mv_{ss}^2. \tag{26}$$

Equation (26) indicates how the Schwarzschild metric apportions the energy equivalence of mass m into an energy component E_τ, an energy component E_{ss} in the space dimensions, and an energy E_g component where

$$E_g = mv_g^2, \tag{27}$$

so that

$$E = mc^2 = E_\tau + E_g + E_{ss}. \tag{28}$$

9 Reciprocity in the apportionment of energy equivalence

In a system of two particles, one particle having a mass m_1 and a Schwarzschild radius of R_1 and the other particle having a mass m_2 and a Schwarzschild radius of R_2, the Schwarzschild metric allows the energy equivalence of each mass to be apportion into, time, space and gravity components. For example, when spatial coordinates (r_1, θ_1, ϕ_1) are measured with respect to m_1 and local time τ_1 is measured at the location of m_2, the energy equivalence of m_2 can be apportioned using the Schwarzchild metric,

$$c^2 d\tau_1^2 = c^2\left(1 - \frac{R_1}{r_1}\right)dt_1^2 - \frac{dr_1^2}{(1 - R_1/r_1)} - r_1^2 d\theta_1^2 - \\ -(r_1 \sin\theta_1)^2 d\phi_1^2, \tag{29}$$

into the following apportionment of energy equivalence:

$$m_2c^2 = m_2v_{\tau_1}^2 + m_2v_{g_1}^2 + m_2v_{ss_1}^2 = E_{\tau_1} + E_{g_1} + E_{s_1}. \tag{30}$$

Likewise, when spatial coordinates (r_2, θ_2, ϕ_2) are measured with respect to m_2 and local time τ_2 is measured at the location of m_1, the energy equivalence of m_1 can be apportioned using the Schwarzschild metric,

$$c^2 d\tau_2^2 = c^2\left(1 - \frac{R_2}{r_2}\right)dt_2^2 - \frac{dr_2^2}{(1 - R_2/r_2)} - r_2^2 d\theta_2^2 - \\ -(r_2 \sin\theta_2)^2 d\phi_2^2, \tag{31}$$

into the following apportionment of energy equivalence:

$$m_1c^2 = m_1v_{\tau_2}^2 + m_1v_{g_2}^2 + m_1v_{ss_2}^2 = E_{\tau_2} + E_{g_2} + E_{s_2}. \tag{32}$$

10 Implications

The Schwarzschild metric apportions the energy equivalence of a mass into a time component, a spatial component and a gravitational component. This suggests that the source of gravitational energy is the energy equivalence of the mass affected by gravity and therefore that the magnitude of gravitational energy cannot exceed the energy equivalence of that mass. As pointed out by Weller [5, 6], this presents a very significant difficulty for those who view gravity as an unlimited source of energy to perform such tasks as forming black holes and creating universes. This also tends to confirm the assertions of Schwarzschild [7] and Einstein [8] that there is indeed a maximum density of matter.

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