Einstein's Planetary Equation: An Analytical Solution

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Einstein's planetary equation can be solved by the method of successive approximations.This yields two linearly independent solutions. An analytical solution is presented for this equation. This solution produces eight linearly independent mathematical solutions, two of which are given approximately by the well-known method of successive approximations.

1 Introduction

Einstein's planetary equation is given [1] by

$$\frac{d^2u}{d\phi^2} + u - \frac{k}{l^2} = \frac{3k}{c^2}u^2 \tag{1}$$

where ϕ and u are the instantaneous angular and reciprocal radial displacements of the planet in the fixed plane of motion, with the Sun as origin, l is the constant angular momentum per unit mass [2] and

$$k = GM \tag{2}$$

where M is the rest mass of the Sun, G is the universal gravitational constant and c is the speed of light in vacuum. The method of successive approximations yields the solution of equation (1) [1] as:

$$r(\phi) = \frac{1}{u(\phi)} = \frac{\left(1-\epsilon_0^2\right)a_0}{1+\epsilon_0\cos\left[\left(1-\frac{3k^2}{c^2l^2}\right)\phi+\alpha\right]} \tag{3}$$

where ϵ_0 is the eccentricity, a_0 the semi-major axis and α is the epoch. The second solution of equation (1) obtained from the method of successive approximations is the solution (3) with sine instead of cosine. The effect revealed by these two approximate solutions is an anomalous precession of the planetary orbit in which the perihelion advances by an angle per revolution Δ given [1] by

$$\Delta = \frac{6\pi k^2}{c^2l^2}. \tag{4}$$

In this article, Einstein's planetary equation (1) is solved analytically.

2 Analytical Solution

Suppose the analytical solution of equation (1) is in the form of a Taylor or Laurent series given as

$$u(\phi) = \sum_{n=0}^{\infty} A_n\, exp\left\{ni\left(\omega\phi + \phi_0\right)\right\} \tag{5}$$

where A_n, ω and ϕ_0 are constants. Then, substituting (5) into (1), applying the linear independence of the exponential functions and equating corresponding coefficients on both sides yields the following system of equations:

$$\frac{3k}{c^2}A_0^2 - A_0 + \frac{k}{l^2} = 0 \tag{6}$$

$$\omega^2 = 1 - \frac{6k}{c^2}A_0 \tag{7}$$

$$A_1 = arbitrary\ constant \tag{8}$$

$$A_2 = \frac{3k}{c^2}\left(1 - 2^2\omega^2 - \frac{6k}{c^2}A_0\right)^{-1}A_1^2 \tag{9}$$

$$A_3 = \frac{18k^2}{c^4}\left[\left(1-2^2\omega^2-\frac{6k}{c^2}A_0\right)\left(1-3^2\omega^2-\frac{6k}{c^2}A_0\right)\right]^{-1}A_1^3 \tag{10}$$

and so on. Equation (6) is a binomial in A_0 and has two possible roots given by

$$A_{0^-} = \frac{c^2}{6k}\left[1-\left(1-\frac{12k^2}{c^2l^2}\right)^{1/2}\right] \tag{11}$$

and

$$A_{0^+} = \frac{c^2}{6k}\left[1+\left(1-\frac{12k^2}{c^2l^2}\right)^{1/2}\right] \tag{12}$$

It follows from substituting (11) into (7) that they are two possible values of the parameter ω given as:

$$\omega_1 = \left\{1-\left[1-\left(1-\frac{12k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2} \tag{13}$$

and

$$\omega_2 = -\left\{1-\left[1-\left(1-\frac{12k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2} \tag{14}$$

Similarly, by substituting (12) into (7) other two possible values of the parameter are obtained as:

$$\omega_3 = \left\{1-\left[1+\left(1-\frac{12k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2} \tag{15}$$

and

$$\omega_4 = -\left\{1 - \left[1 + \left(1 - \frac{12k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2}. \quad (16)$$

It follows from equation (9) that A_2 has eight possible values given by

$$A_{2^1} = \frac{3k}{c^2}\left(1 - 2^2\omega_1^2 - \frac{6k}{c^2}A_{0^+}\right)^{-1} A_1^2 \quad (17)$$

$$A_{2^2} = \frac{3k}{c^2}\left(1 - 2^2\omega_1^2 - \frac{6k}{c^2}A_{0^-}\right)^{-1} A_1^2 \quad (18)$$

$$A_{2^3} = \frac{3k}{c^2}\left(1 - 2^2\omega_2^2 - \frac{6k}{c^2}A_{0^+}\right)^{-1} A_1^2 \quad (19)$$

$$A_{2^4} = \frac{3k}{c^2}\left(1 - 2^2\omega_2^2 - \frac{6k}{c^2}A_{0^-}\right)^{-1} A_1^2 \quad (20)$$

$$A_{2^5} = \frac{3k}{c^2}\left(1 - 2^2\omega_3^2 - \frac{6k}{c^2}A_{0^+}\right)^{-1} A_1^2 \quad (21)$$

$$A_{2^6} = \frac{3k}{c^2}\left(1 - 2^2\omega_3^2 - \frac{6k}{c^2}A_{0^-}\right)^{-1} A_1^2 \quad (22)$$

$$A_{2^7} = \frac{3k}{c^2}\left(1 - 2^2\omega_4^2 - \frac{6k}{c^2}A_{0^+}\right)^{-1} A_1^2 \quad (23)$$

$$A_{2^8} = \frac{3k}{c^2}\left(1 - 2^2\omega_4^2 - \frac{6k}{c^2}A_{0^-}\right)^{-1} A_1^2 \quad (24)$$

Similarly, it follows from (10) that A_3 has eight possible values. The above sequence may be continued to derive the eight possible corresponding values for each of the constants A_4, A_5, ... in terms of the arbitrary constant A_1 . This sequence implies eight mathematically possible analytical solutions of Einstein's planetary equation of the form:

$$\begin{aligned} u(\phi) \;=\; & A_0 + A_1 exp\left[i\left(\omega\phi + \phi_0\right)\right] + \\ & f_2(A_1) exp\left[2i\left(\omega\phi + \phi_0\right)\right] + ... \\ & f_n exp\left[ni\left(\omega\phi + \phi_0\right)\right] + ... \end{aligned} \quad (25)$$

where ϕ_0 and A_1 are arbitrary.

Now, consider the first exact analytical solution corresponding to equations (12) and (14). In this case, it follows from (9) that

$$A_2 = f_2(A_1) = -\frac{k}{c^2}\left(1 - \frac{6k}{c^2}A_{0^-}\right)^{-1} A_1^2 \quad (26)$$

and

$$A_3 = f_3(A_1) \quad (27)$$

and in general

$$A_n = f_n(A_1),\ n = 4,\ 5,\ ... \quad (28)$$

In this case, the exact analytical solution of Einstein's planetary equation is a complex function of ϕ which may be written in Cartesian form as

$$u(\phi) = x(\phi) + iy(\phi) \quad (29)$$

where

$$\begin{aligned} x(\phi) \;=\; & A_{0^-} + A_1 \cos\left(\omega_1\phi + \phi_0\right) + \\ & f_2(A_1) \cos 2\left[\left(\omega_1\phi + \phi_0\right)\right] + \ldots \end{aligned} \quad (30)$$

and

$$\begin{aligned} y(\phi) \;=\; & A_{0^-} + A_1 \sin\left(\omega_1\phi + \phi_0\right) + \\ & f_2(A_1) \sin 2\left[\left(\omega_1\phi + \phi_0\right)\right] + \ldots \end{aligned} \quad (31)$$

Therefore it may be expressed in Euler form as

$$u(\phi) = R(\phi)e^{i\Phi(\phi)} \quad (32)$$

where R is the magnitude given by

$$R(\phi) = \left\{x^2(\phi) + y^2(\phi)\right\}^{\frac{1}{2}} \quad (33)$$

and Φ is the argument given by

$$\Phi(\phi) = \tan^{-1}\left\{\frac{y(\phi)}{x(\phi)}\right\}. \quad (34)$$

Hence by definition the instantaneous radial coordinate of the planet from the Sun, r , is given by

$$r(\phi) = R^{-1}(\phi)\,\ell^{-i\Phi(\phi)}. \quad (35)$$

3 Physical Interpretation of First Analytical Solution

The instantaneous complex radial displacement r of the planet from the Sun is given in terms of the angular displacement Φ as

$$r(\phi) = R^{-1}(\phi)\,\ell^{-i\Phi(\phi)}. \quad (36)$$

Therefore the magnitude of the instantaneous complex radial displacement of the planet from the Sun can be considered to be the real physically measurable instantaneous radial displacement, r_p. Thus,

$$r_p(\phi) = R^{-1}(\phi) = \left\{x^2(\phi) + y^2(\phi)\right\}^{-\frac{1}{2}}. \quad (37)$$

It may be noted from (9) and (10) that for $n > 1$ $f_n(A_1)$ is of order at most c^{-2n}. Therefore as a first approximation let us neglect all terms in $f_n(A_1)$ for $n > 1$. Then it follows from (37) and (31)–(32) that

$$r_p(\phi) = \frac{A}{1 + \varepsilon_1 \cos\left(\omega_1\phi + \phi_0\right)} \quad (38)$$

where

$$A = \frac{1}{A_{0-}}\left(1 + \frac{A_1^2}{A_{0-}^2}\right)^{-\frac{1}{2}} \quad (39)$$

and

$$\varepsilon_1 = \frac{A_1}{A_{0-}}\left(1 + \frac{A_1^2}{A_{0-}^2}\right)^{-1}. \tag{40}$$

Consequently, the orbit is a precesing conic section with eccentricity and hence semi-major axis given by

$$a = \frac{A}{1 - \varepsilon_1^2} \tag{41}$$

and perihelion displacement angle Δ given by

$$\Delta = 2\pi\left(\omega_1^{-1} - 1\right). \tag{42}$$

It follows from (42) and (14) that the perihelion displacement angle from this analytical method is given explicitly as

$$\Delta = \frac{6\pi k^2}{c^2 l^2} + \frac{54\pi k^4}{c^4 l^4}. \tag{43}$$

This is an advance precisely as obtained from the method of successive approximations. The leading term in (43) is identically the same as the leading term of the corresponding advance from the method of successive approximations [1]. Moreso, this analytical method reveals the exact corrections of all orders of c^{-2} to the leading term in (44).

It also follows from (40) and (12) that the orbital eccentricity ε_1 from this analytical method is given explicitly as

$$\begin{aligned}\varepsilon_1 \quad = \quad & \frac{l^2 A_1}{k}\left(1 + \frac{3k^2}{c^2 l^2} + \dots\right)^{-1} \\ & \left[1 + \frac{l^4 A_1^2}{k^2}\left(1 + \frac{3k^2}{c^2 l^2} + \dots\right)^{-2}\right]^{-1}.\end{aligned} \tag{44}$$

Thus, an experimental measurement of the orbital eccentricity ε_1 in equation (45) is sufficient to determine the parameter A_1 that occurs in the exact analytical solution. It also follows from this result that the analytical method in this article reveals post-Newtonian corrections of all order of c^{-2} to the planetary orbital eccentricity which have not been derived from the method of successive approximations.

It also follows from equations (41) and (14) that the orbital semi-major axis from this analytical method is given explicitly as

$$\begin{aligned}a \quad = \quad & \frac{l^2}{\left(1 - \varepsilon_1^2\right)k}\left(1 + \frac{3k^2}{c^2 l^2} + \dots\right)^{-1} \\ & \left[1 + \frac{l^4 A_1^2}{k^2}\left(1 + \frac{3k^2}{c^2 l^2} + \dots\right)^{-2}\right]^{-1}.\end{aligned} \tag{45}$$

Thus, this analytical method reveals post-Newtonian corrections of all orders of c^{-2} to planetary semi-major axis, which have not been derived from the method of successive approximations.

4 Conclusion

This article uncovers an analytical solution to Einstein's planetary equation. The first analytical solution to the order of c^{-2}, reveals post-Newtonian corrections to the orbital eccentricity and semi-major axis of a planet. Moreover, up to the second iterate there is no such correction from the method of successive approximations. Consequently, these unknown corrections to orbital eccentricity revealed by the analytical approach in this article are opened up for experimental investigation.

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