A Generalized Displacement Problem in Elasticity

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By solving a special coupling boundary value problem for vector Helmholtz equations it is shown how the displacement boundary value problem in elasticity can be solved. It is shown that the generalized displacement problem possesses at most one solution.

1 Statement of the Problem

By D_i we denote a bounded domain in $\boldsymbol{R}^3$ with boundary S belonging to the class C^2, and by D_e the unbounded domain $D_e := \boldsymbol{R}^3 \setminus \overline{D_i}$. We assume that the normal vector n on S is directed into the exterior domain D_e. The physical meaning is that D_i is a fixed elastic solid with no volume forces present and D_e represents a homogeneous isotropic linear solid which is characterized by the density $\rho = 1$ (this is no loss of generality) and the Lamé parameters λ and μ. We consider time-harmonic elastic waves with circular frequency ω and it will be assumed that all Lamé constants and the frequency are positive. We assume that the elastic medium D_e is in welded contact to the rigid inclusion D_i, which means that we consider displacement boundary conditions.

To formulate the elasticity problems we introduce the following function spaces. By $C^{0,\alpha}(S)$ and $C^{0,\alpha}_T(S)$ we denote the spaces of Hölder continuous functions and Hölder continuous tangential fields $(0 < \alpha < 1)$, respectively. The space $C^{0,\alpha}_D(S)$ denotes the subspace of Hölder continuous tangential fields possessing Hölder continuous surface divergence in the sense of the limit integral definition given by Müller [1]. Defining the differential operator $\Delta^* := \Delta + \frac{\lambda+\mu}{\mu}\,\mathrm{grad}\,\mathrm{div}$, where Δ is the Laplace operator and λ and μ are the Lamé elastic constants with $\mu > 0$ and $\lambda + 2\mu > 0$. For a positive frequency ω the wavenumbers κ_p and κ_s are defined by $\kappa_p := \omega/\sqrt{\lambda + 2\mu}$ and $\kappa_s := \omega/\sqrt{\mu}$. Now, the time-harmonic exterior displacement problem in elasticity can be formulated as

<u>Problem D</u>: Find a vector field $u \in C^2(D_e) \cap C(\overline{D_e})$ satisfying the time-harmonic elasticity equation

$$\Delta^* u + \kappa_s^2 u = 0, \text{ in } D_e, \tag{1}$$

the welded contact boundary conditions

$$u = f, \text{ on } S, \tag{2}$$

and the Sommerfeld radiation condition

$$(x, \mathrm{grad}\, u_j(x)) - i\kappa_j u_j(x) = o\left(\frac{1}{|x|}\right), \text{ for } |x| \to \infty, \; j = s, p, \tag{3}$$

uniformly for all directions $x := x/|x|$, where

$$u_p := \frac{-1}{\kappa_p^2}\,\mathrm{grad}\,\mathrm{div}\, u \text{ and } u_s := \frac{1}{\kappa_p^2}\,\mathrm{grad}\,\mathrm{div}\, u + u. \tag{4}$$

Here $f \in C^{0,\alpha}(S)$ is a given vector field.

By (a, b) and $[a, b]$ we denote the scalar product and vector product of the vectors a and b, respectively. The vector fields u_s and u_p are known as the rotational and irrotational parts of u, respectively. The rotational part corresponds to a dilatational or compressional wave and the irrotational part corresponds to a shearing wave. The wave numbers κ_s and κ_p are known as the slownesses of the rotational and irrotational waves, respectively.

That Problem D possesses at most one solution has already been discussed by Kupradze [2] and Ahner [3]. The existence of a solution has been shown by Hähner and Hsiao [4].

For any domain $D \subset \boldsymbol{R}^3$ with boundary ∂D we introduce the linear space of vector fields $u : D \to \boldsymbol{R}^3$ by

$$F(D) := \{u \mid u \in C^2(D) \cap C(\overline{D}),\; \mathrm{curl}\, u, \mathrm{div}\, u \in C(\overline{D})\;\}.$$

From the integral representation theorem for solutions of the time-harmonic elasticity equation, known as the Betti formulas [2], we see that the displacement field is analytic. Therefore, by using (4) u can be split into $u = u_p + u_s$. Differentiating both, u_p and u_s, we see that u_p is *curl*-free and that u_s is *divergence*-free. Furthermore, u_j is a solution of the vector Helmholtz equation $\Delta u_j + \kappa_j u_j = 0$, in D_e, for $j = s, p$.

This motivates us to study the following slightly more general coupling

<u>Problem HD</u>: Find two vector fields $u_s, u_p \in F(D_e)$ satisfying the vector Helmholtz equations

$$\left.\begin{aligned} \Delta u_s + \kappa_s^2 u_s &= 0, \quad \text{in } D_e, \quad \kappa_s \neq 0,\; \Im(\kappa_s) \geq 0, \\ \Delta u_p + \kappa_p^2 u_p &= 0, \quad \text{in } D_e, \quad \kappa_p \neq 0,\; \Im(\kappa_p) \geq 0, \end{aligned}\right\} \tag{5}$$

the coupling boundary conditions

$$\left.\begin{array}{lclcl} [n, u_s] & + & [n, u_p] & = & c, \\ \mathrm{div}\, u_s & & & = & \gamma, \\ & & [[\mathrm{curl}\, u_p, n], n] & = & d, \\ (n, u_s) & + & (n, u_p) & = & \delta, \text{ on } S, \end{array}\right\} \tag{6}$$

and the radiation conditions

$$[\mathrm{curl}\, u_j, \hat{x}] + \hat{x}\,\mathrm{div}\, u_j - i\kappa_j u_j = o(1/|x|), \text{ for } |x| \to \infty, \tag{7}$$

and $j = s, p$, uniformly for all directions $\hat{x} := x/|x|$. Here $c \in C^{0,\alpha}_D(S)$ and $d \in C^{0,\alpha}_T(S)$ are given tangential fields and $\gamma, \delta \in C^{0,\alpha}(S)$ are given functions.

2 Uniqueness

By Problem HDS we denote the special case of Problem HD, with

$$\kappa_p^2 = \frac{\omega^2}{\lambda + 2\mu} \text{ and } \kappa_s^2 = \frac{\omega^2}{\mu}, \tag{8}$$

and the right-hand sides

$$\gamma = 0 \text{ and } d = 0.$$

Now we have the following equivalence

Theorem 3.1: 1) Let u be a solution of Problem D corresponding to the boundary data $f := n\delta - [n, c]$. Then

$$u_p := \frac{-1}{\kappa_p^2} \operatorname{grad} \operatorname{div} u \text{ and } u_s := \frac{1}{\kappa_p^2} \operatorname{grad} \operatorname{div} u + u,$$

is a solution of Problem HDS.

2) Let u_p, u_s be a solution of Problem HDS corresponding to the boundary data $c := [n, f]$, $\gamma = 0$, $d = 0$ and $\delta := (n, f)$. Then $u := u_p + u_s$ is a solution of Problem D.

Proof: We will show only part 2). Let u_p, u_s be a solution of Problem HDS corresponding to the boundary data $c := [n, f]$, $\gamma = 0$, $d = 0$ and $\delta := (n, f)$. Representing u_s via the representation theorem for solutions of the vector Helmholtz equation [6] it can be seen that $\operatorname{div} u_s$ is a solution of the scalar Helmholtz equation $\Delta \operatorname{div} u_s + \kappa_s^2 \operatorname{div} u_s = 0$ in D_e satisfying the homogeneous Dirichlet boundary condition $\operatorname{div} u = 0$ and the Sommerfeld radiation condition. From the uniqueness theorem for the exterior Dirichlet problem [5, 6] we obtain $\operatorname{div} u_s = 0$ in D_e.

Using the integral representation theorem for solutions of the vector Helmholtz equation [6] it can be seen that $\operatorname{curl} u_p$ solves the vector Helmholtz equation $\Delta \operatorname{curl} u_p + \kappa_p^2 \operatorname{curl} u_p = 0$ in D_e, fulfills the homogeneous electric boundary condition $[[\operatorname{curl} u_p, n], n] = 0$ and $\operatorname{div} \operatorname{curl} u_p = 0$, on S, and the radiation condition (7). From the uniqueness theorem for the exterior electric boundary value problem [6] we obtain $\operatorname{curl} u_p = 0$ in D_e.

That $u := u_p + u_s$ is a solution of $\Delta^* u + \kappa_s^2 u = 0$ in D_e, follows by straightforward calculations. Since the cartesian components of every solution of the vector Helmholtz equation satisfying the radiation condition (7) also satisfy the radiation condition of Sommerfeld [6, see Corollary 4.14], we obtain that u fulfills the radiation condition (3).

That u fulfills the boundary conditions (2) is easily seen by

$$\begin{aligned} u &= u_s + u_p = n(n, u_s + u_p) - [n, [n, u_s + u_p]] \\ &= n(n, f) - [n, [n, f]] = f, \text{ on } S. \end{aligned}$$

From the uniqueness theorem for Problem D we obtain the following uniqueness

Theorem 3.2: Problem HD possesses at most one solution if for κ_p and κ_s the condition (8) holds.

Proof: Let u_p, u_s be a solution of the homogeneous Problem HD. As in the proof of Theorem 3.1 we can see that $u := u_s + u_p$ is a solution of Problem D but now to the homogeneous boundary condition. Therefore, by the uniqueness theorem for the exterior displacement problem we derive $u = 0$ in D_e.

Now we have $u_s = -u_p$ in D_e and there holds $\operatorname{div} u_p = 0$ and $\operatorname{curl} u_s = 0$ in D_e. From this we conclude

$$-\kappa_j^2 u_j = \Delta u_j = \operatorname{grad} \operatorname{div} u_j - \operatorname{curl} \operatorname{curl} u_j = 0,$$

and therefore $u_j = 0$ in D_e, for $j = s, p$. This means that Problem HD possesses at most one solution.

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