Does the Equivalence between Gravitational Mass and Energy Survive for a Quantum Body?

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We consider the simplest quantum composite body, a hydrogen atom, in the presence of a weak external gravitational field. We show that passive gravitational mass operator of the atom in the post-Newtonian approximation of general relativity does not commute with its energy operator, taken in the absence of the field. Nevertheless, the equivalence between the expectations values of passive gravitational mass and energy is shown to survive at a macroscopic level for stationary quantum states. Breakdown of the equivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states can be experimentally detected by studying unusual electromagnetic radiation, emitted by the atoms, supported and moved in the Earth gravitational field with constant velocity, using spacecraft or satellite.

1 Introduction

Formulation of a successful quantum gravitation theory is considered to be one of the most important problems in modern physics and the major step towards the so-called "Theory of Everything". On the other hand, fundamentals of general relativity and quantum mechanics are so different that there is a possibility that it will not be possible to unite these two theories in a feasible future. In this difficult situation, it seems to be important to suggest a combination of quantum mechanics and some non-trivial approximation of general relativity. In particular, this is important in the case where such theory can be experimentally tested. To the best of our knowledge, so far only quantum variant of the trivial Newtonian approximation of general relativity has been tested experimentally in the famous COW [1] and ILL [2] experiments. As to such important and nontrivial quantum effects in general relativity as the Hawking radiation [3] and the Unruh effect [4], they are still very far from their direct and unequivocal experimental confirmations.

The notion of gravitational mass of a composite body is known to be non-trivial in general relativity and related to the following paradoxes. If we consider a free photon with energy E and apply to it the so-called Tolman formula for gravitational mass [5], we will obtain $m^g = 2E/c^2$ (i.e., two times bigger value than the expected one) [6]. If a photon is confined in a box with mirrors, then we have a composite body at rest. In this case, as shown in Ref. [6], we have to take into account a negative contribution to m^g from stress in the box walls to restore the Einstein equation, $m^g = E/c^2$. It is important that the later equation is restored only after averaging over time. A role of the classical virial theorem in establishing of the equivalence between averaged over time gravitational mass and energy is discussed in detail in Refs. [7, 8] for different types of classical composite bodies. In particular, for electrostatically bound two bodies with bare masses m_1 and m_2, it is shown that gravitational field is coupled to a combination $3K + 2U$, where K is kinetic energy, U is the Coulomb potential energy. Since the classical virial theorem states that the following time average is equal to zero, $\langle 2K + U \rangle_t = 0$, then we conclude that averaged over time gravitational mass is proportional to the total amount of energy [7, 8]:

$$\langle m^g \rangle_t = m_1 + m_2 + \langle 3K + 2U \rangle_t / c^2 = E/c^2. \qquad (1)$$

2 Goal

The main goal of our paper is to study a quantum problem about passive gravitational mass of a composite body. As the simplest example, we consider a hydrogen atom in the Earth gravitational field, where we take into account only kinetic and Coulomb potential energies of an electron in a curved spacetime. We claim three main results in the paper (see also Refs. [9, 10]). Our first result is that the equivalence between passive gravitational mass and energy in the absence of gravitational field survives at a macroscopic level in a quantum case. More strictly speaking, we show that the expectation value of the mass is equal to E/c^2 for stationary quantum states due to the quantum virial theorem. Our second result is a breakdown of the equivalence between passive gravitational mass and energy at a microscopic level for stationary quantum states due to the fact that the mass operator does not commute with energy operator, taken in the absence of gravitational field. As a result, there exist a non-zero probability that a measurement of passive gravitational mass gives value, which is different from E/c^2, given by the Einstein equation. Our third result is a suggestion of a realistic experiment to detect this inequivalence by measurements of electromagnetic radiation, emitted by a macroscopic ensemble of hydrogen atoms, supported and moved in the Earth gravitational field, using spacecraft or satellite.

3 Gravitational Mass in Classical Physics

Below, we derive the Lagrangian and Hamiltonian of a hydrogen atom in the Earth gravitational field, taking into account couplings of kinetic and potential Coulomb energies of an electron with a weak centrosymmetric gravitational field. Note that we keep only terms of the order of $1/c^2$ and disregard magnetic force, radiation of both electromagnetic and gravitational waves as well as all tidal and spin dependent effects. Let us write the interval in the Earth centrosymmetric gravitational field, using the so-called weak field approximation [11]:

$$ds^2 = -\left(1+2\frac{\phi}{c^2}\right)(cdt)^2 + \left(1-2\frac{\phi}{c^2}\right)(dx^2 + dy^2 + dz^2),$$
$$\phi = -\frac{GM}{R}, \tag{2}$$

where G is the gravitational constant, c is the velocity of light, M is the Earth mass, R is a distance between a center of the Earth and a center of mass of a hydrogen atom (i.e., proton). We pay attention that to calculate the Lagrangian (and later — the Hamiltonian) in a linear with respect to a small parameter $\phi(R)/c^2$ approximation, we do not need to keep the terms of the order of $[\phi(R)/c^2]^2$ in metric (2), in contrast to the perihelion orbit procession calculations [11] .

Then, in the local proper spacetime coordinates,

$$x' = \left(1 - \frac{\phi}{c^2}\right)x, \quad y' = \left(1 - \frac{\phi}{c^2}\right)y,$$
$$z' = \left(1 - \frac{\phi}{c^2}\right)z, \quad t' = \left(1 + \frac{\phi}{c^2}\right)t, \tag{3}$$

the classical Lagrangian and action of an electron in a hydrogen atom have the following standard forms:

$$L' = -m_e c^2 + \frac{1}{2}m_e(\mathbf{v}')^2 + \frac{e^2}{r'} , \quad S' = \int L' dt', \tag{4}$$

where m_e is the bare electron mass, e and $\mathbf{v}'$ are the electron charge and velocity, respectively; r' is a distance between electron and proton. It is possible to show that the Lagrangian (4) can be rewritten in coordinates (x, y, z, t) as

$$L = -m_e c^2 + \frac{1}{2}m_e\mathbf{v}^2 + \frac{e^2}{r} - m_e\phi - \left(3m_e\frac{\mathbf{v}^2}{2} - 2\frac{e^2}{r}\right)\frac{\phi}{c^2}. \tag{5}$$

Let us calculate the Hamiltonian, corresponding to the Lagrangian (5), by means of a standard procedure, $H(\mathbf{p}, \mathbf{r}) = \mathbf{p}\mathbf{v} - L(\mathbf{v}, \mathbf{r})$, where $\mathbf{p} = \partial L(\mathbf{v}, \mathbf{r})/\partial\mathbf{v}$. As a result, we obtain:

$$H = m_e c^2 + \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + m_e\phi + \left(3\frac{\mathbf{p}^2}{2m_e} - 2\frac{e^2}{r}\right)\frac{\phi}{c^2}, \tag{6}$$

where canonical momentum in a gravitational field is $\mathbf{p} = m_e\mathbf{v}(1 - 3\phi/c^2)$. [Note that, in the paper, we disregard all

tidal effects (i.e., we do not differentiate gravitational potential with respect to electron coordinates, $\mathbf{r}$ and $\mathbf{r}'$, corresponding to a position of an electron in the center of mass coordinate system). It is possible to show that this means that we consider the atom as a point-like body and disregard all effects of the order of $|\phi/c^2|(r_B/R) \sim 10^{-26}$, where r_B is the Bohr radius (i.e., a typical size of the atom).] From the Hamiltonian (6), averaged over time electron passive gravitational mass, $< m_e^g >_t$, defined as its weight in a weak centrosymmetric gravitational field (2), can be expressed as

$$< m_e^g >_t \quad = m_e + \left\langle\frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r}\right\rangle_t\frac{1}{c^2} + \left\langle 2\frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r}\right\rangle_t\frac{1}{c^2}$$
$$= m_e + \frac{E}{c^2}, \tag{7}$$

where $E = \mathbf{p}^2/2m_e - e^2/r$ is an electron energy. We pay attention that averaged over time third term in Eq. (7) is equal to zero due to the classical virial theorem. Thus, we conclude that in classical physics averaged over time passive gravitational mass of a composite body is equivalent to its energy, taken in the absence of gravitational field [7, 8].

4 Gravitational Mass in Quantum Physics

The Hamiltonian (6) can be quantized by substituting a momentum operator, $\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{r}$, instead of canonical momentum, $\mathbf{p}$. It is convenient to write the quantized Hamiltonian in the following form:

$$\hat{H} = m_e c^2 + \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g\phi, \tag{8}$$

where we introduce passive gravitational mass operator of an electron to be proportional to its weight operator in a weak centrosymmetric gravitational field (2),

$$\hat{m}_e^g = m_e + \left(\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r}\right)\frac{1}{c^2} + \left(2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r}\right)\frac{1}{c^2}. \tag{9}$$

Note that the first term in Eq. (9) corresponds to the bare electron mass, m_e, the second term corresponds to the expected electron energy contribution to the mass operator, whereas the third nontrivial term is the virial contribution to the mass operator. It is important that the operator (9) does not commute with electron energy operator, taken in the absence of the field. It is possible to show that Eqs. (8), (9) can be also obtained directly from the Dirac equation in a curved spacetime, corresponding to a weak centrosymmetric gravitational field (2). For example, the Hamiltonian (8), (9) can be obtained [9, 10] from the Hamiltonian (3.24) of Ref. [12], where different physical problem is considered, by omitting all tidal terms.

Below, we discuss some consequences of Eq. (9). Suppose that we have a macroscopic ensemble of hydrogen atoms with each of them being in a ground state with energy E_1.

Then, as follows from Eq. (9), the expectation value of the gravitational mass operator per one electron is

$$< \hat{m}_e^g > = m_e + \frac{E_1}{c^2} + \left\langle 2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right\rangle \frac{1}{c^2} = m_e + \frac{E_1}{c^2}, \quad (10)$$

where the third term in Eq. (10) is zero in accordance with the quantum virial theorem [13]. Therefore, we conclude that the equivalence between passive gravitational mass and energy in the absence of gravitational field survives at a macroscopic level for stationary quantum states.

Let us discuss how Eqs. (8), (9) break the equivalence between passive gravitational mass and energy at a microscopic level. First of all, we recall that the mass operator (9) does not commute with electron energy operator, taken in the absence of gravitational field. This means that, if we create a quantum state of a hydrogen atom with definite energy, it will not be characterized by definite passive gravitational mass. In other words, a measurement of the mass in such quantum state may give different values, which, as shown, are quantized. Here, we illustrate the above mentioned inequivalence, using the following thought experiment. Suppose that at $t = 0$ we create a ground state wave function of a hydrogen atom, corresponding to the absence of gravitational field,

$$\Psi_1(r, t) = \Psi_1(r) \exp(-iE_1 t/\hbar). \quad (11)$$

In a weak centrosymmetric gravitational field (2), wave function (11) is not anymore a ground state of the Hamiltonian (8), (9), where we treat gravitational field as a small perturbation in an inertial system [7–12]. It is important that for inertial observer, in accordance with Eq. (3), a general solution of the Schrodinger equation, corresponding to the Hamiltonian (8), (9), can be written as

$$\begin{aligned}
\Psi(r, t) = (1 - \phi/c^2)^{3/2} \sum_{n=1}^{\infty} a_n \Psi_n[(1 - \phi/c^2)r] \\
\times \exp[-im_e c^2(1 + \phi/c^2)t/\hbar] \\
\times \exp[-iE_n(1 + \phi/c^2)t/\hbar].
\end{aligned} \quad (12)$$

We pay attention that wave function (12) is a series of eigenfunctions of passive gravitational mass operator (9), if we take into account only linear terms with respect to the parameter ϕ/c^2. Here, factor $1 - \phi/c^2$ is due to a curvature of space, whereas the term $E_n(1 + \phi/c^2)$ represents the famous red shift in gravitational field and is due to a curvature of time. $\Psi_n(r)$ is a normalized wave function of an electron in a hydrogen atom in the absence of gravitational field, corresponding to energy E_n. [Note that, due to symmetry of our problem, an electron from $1S$ ground state of a hydrogen atom can be excited only into nS excited states. We also pay attention that the wave function (12) contains a normalization factor $(1 - \phi/c^2)^{3/2}$.]

In accordance with the basic principles of the quantum mechanics, probability that, at $t > 0$, an electron occupies excited state with energy $m_e c^2(1 + \phi/c^2) + E_n(1 + \phi/c^2)$ is

$$\begin{aligned}
P_n &= |a_n|^2, \\
a_n &= \int \Psi_1^*(r)\Psi_n[(1 - \phi/c^2)r]d^3\mathbf{r} \\
&= -(\phi/c^2) \int \Psi_1^*(r)r\Psi_n'(r)d^3\mathbf{r}.
\end{aligned} \quad (13)$$

Note that it is possible to demonstrate that for a_1 in Eq. (13) a linear term with respect to gravitational potential, ϕ, is zero, which is a consequence of the quantum virial theorem. Taking into account that the Hamiltonian is a Hermitian operator, it is possible to show that for $n \neq 1$:

$$\begin{aligned}
\int \Psi_1^*(r)r\Psi_n'(r)d^3\mathbf{r} &= \frac{V_{n,1}}{\hbar\omega_{n,1}}, \\
\hbar\omega_{n,1} &= E_n - E_1, \quad n \neq 1,
\end{aligned} \quad (14)$$

where $V_{n,1}$ is a matrix element of the virial operator,

$$V_{n,1} = \int \Psi_1^*(r)\hat{V}(\mathbf{r})\Psi_n(r)d^3\mathbf{r}, \quad \hat{V}(\mathbf{r}) = 2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r}. \quad (15)$$

It is important that, since the virial operator (15) does not commute with the Hamiltonian, taken in the absence of gravitational field, the probabilities (13)–(15) are not equal to zero for $n \neq 1$.

Let us discuss Eqs. (12)–(15). We pay attention that they directly demonstrate that there is a finite probability,

$$P_n = |a_n|^2 = \left(\frac{\phi}{c^2}\right)^2 \left(\frac{V_{n,1}}{E_n - E_1}\right)^2, \quad n \neq 1, \quad (16)$$

that, at $t > 0$, an electron occupies n-th ($n \neq 1$) energy level, which breaks the expected Einstein equation, $m_e^g = m_e + E_1/c^2$. In fact, this means that measurement of passive gravitational mass (i.e., weight in the gravitational field (2)) in a quantum state with a definite energy (11) gives the following quantized values:

$$m_e^g(n) = m_e + E_n/c^2, \quad (17)$$

corresponding to the probabilities (16). [Note that, as it follows from quantum mechanics, we have to calculate wave function (12) in a linear approximation with respect to the parameter ϕ/c^2 to obtain probabilities (16), (22), (23), which are proportional to $(\phi/c^2)^2$. A simple analysis shows that an account in Eq. (12) terms of the order of $(\phi/c^2)^2$ would change electron passive gravitational mass of the order of $(\phi/c^2)m_e \sim 10^{-9}m_e$, which is much smaller than the distance between the quantized values (17), $\delta m_e^g \sim \alpha^2 m_e \sim 10^{-4}m_e$, where α is the fine structure constant.] We also point out that, although the probabilities (16) are quadratic with respect to gravitational potential and, thus, small, the changes of the

passive gravitational mass (17) are large and of the order of $\alpha^2 m_e$. We also pay attention that small values of probabilities (16), $P_n \sim 10^{-18}$, do not contradict the existing Eötvös type measurements [11], which have confirmed the equivalence principle with the accuracy of the order of 10^{-12}-10^{-13}. For our case, it is crucial that the excited levels of a hydrogen atom spontaneously decay with time, therefore, one can detect the quantization law (17) by measuring electromagnetic radiation, emitted by a macroscopic ensemble of hydrogen atoms. The above mentioned optical method is much more sensitive than the Eötvös type measurements and we, therefore, hope that it allows to detect the breakdown of the equivalence between energy and passive gravitational mass, revealed in the paper.

5   Suggested Experiment

Here, we describe a realistic experiment [9, 10]. We consider a hydrogen atom to be in its ground state at $t = 0$ and located at distance R' from a center of the Earth. The corresponding wave function can be written as

$$
\begin{aligned}
\tilde{\Psi}_1(r,t) = &(1 - 2\phi')^{3/2}\Psi_1[(1 - \phi'/c^2)r] \\
&\times \exp[-im_e c^2(1 + \phi'/c^2)t/\hbar] \\
&\times \exp[-iE_1(1 + \phi'/c^2)t/\hbar],
\end{aligned} \tag{18}
$$

where $\phi' = \phi(R')$. The atom is supported in the Earth gravitational field and moved from the Earth with constants velocity, $v \ll \alpha c$, by spacecraft or satellite. As follows from Ref. [7], the extra contributions to the Lagrangian (5) are small in this case in an inertial system, related to a center of mass of a hydrogen atom (i.e., proton). Therefore, electron wave function and time dependent perturbation for the Hamiltonian (8), (9) in this inertial coordinate system can be expressed as

$$
\begin{aligned}
\tilde{\Psi}(r,t) = &(1 - 2\phi')^{3/2}\sum_{n=1}^{\infty}\tilde{a}_n(t)\Psi_n[(1 - \phi'/c^2)r] \\
&\times \exp[-im_e c^2(1 + \phi'/c^2)t/\hbar] \\
&\times \exp[-iE_n(1 + \phi'/c^2)t/\hbar],
\end{aligned} \tag{19}
$$

$$
\hat{U}(\mathbf{r},t) = \frac{\phi(R' + vt) - \phi(R')}{c^2}\left(3\frac{\hat{\mathbf{p}}^2}{2m_e} - 2\frac{e^2}{r}\right). \tag{20}
$$

We pay attention that in a spacecraft (satellite), which moves with constant velocity, gravitational force, which acts on each hydrogen atom, is compensated by some non-gravitational forces. This causes very small changes of a hydrogen atom energy levels and is not important for our calculations. Therefore, the atoms do not feel directly gravitational acceleration, $\mathbf{g}$, but feel, instead, gravitational potential, $\phi(R' + vt)$, changing with time due to a spacecraft (satellite) motion in the Earth gravitational field. Application of

the time-dependent quantum mechanical perturbation theory gives the following solutions for functions $\tilde{a}_n(t)$ in Eq. (19):

$$
\tilde{a}_n(t) = \frac{\phi(R') - \phi(R' + vt)}{c^2}\frac{V_{n,1}}{\hbar\omega_{n,1}}\exp(i\omega_{n,1}t), \quad n \neq 1, \tag{21}
$$

where $V_{n,1}$ and $\omega_{n,1}$ are given by Eqs. (14), (15); $\omega_{n,1} \gg v/R'$.

It is important that, if excited levels of a hydrogen atom were strictly stationary, then a probability to find the passive gravitational mass to be quantized with $n \neq 1$ (17) would be

$$
\tilde{P}_n(t) = \left(\frac{V_{n,1}}{\hbar\omega_{n,1}}\right)^2\frac{[\phi(R' + vt) - \phi(R')]^2}{c^4}, \quad n \neq 1. \tag{22}
$$

In reality, the excited levels spontaneously decay with time and, therefore, it is possible to observe the quantization law (17) indirectly by measuring electromagnetic radiation from a macroscopic ensemble of the atoms. In this case, Eq. (22) gives a probability that a hydrogen atom emits a photon with frequency $\omega_{n,1} = (E_n - E_1)/\hbar$ during the time interval t. [We note that dipole matrix elements for $nS \to 1S$ quantum transitions are zero. Nevertheless, the corresponding photons can be emitted due to quadrupole effects.]

Let us estimate the probability (22). If the experiment is done by using spacecraft or satellite, then we may have $|\phi(R' + vt)| \ll |\phi(R')|$. In this case Eq. (22) is reduced to Eq. (16) and can be rewritten as

$$
\tilde{P}_n = \left(\frac{V_{n,1}}{E_n - E_1}\right)^2\frac{\phi^2(R')}{c^4} \simeq 0.49 \times 10^{-18}\left(\frac{V_{n,1}}{E_n - E_1}\right)^2, \tag{23}
$$

where, in Eq. (23), we use the following numerical values of the Earth mass, $M \simeq 6 \times 10^{24}$ kg, and its radius, $R_0 \simeq 6.36 \times 10^6$ m. It is important that, although the probabilities (23) are small, the number of photons, N, emitted by macroscopic ensemble of the atoms, can be large since the factor $V_{n,1}^2/(E_n - E_1)^2$ is of the order of unity. For instance, for 1000 moles of hydrogen atoms, N is estimated as

$$
N_{n,1} = 2.95 \times 10^8\left(\frac{V_{n,1}}{E_n - E_1}\right)^2, \quad N_{2,1} = 0.9 \times 10^8, \tag{24}
$$

which can be experimentally detected, where $N_{n,1}$ stands for a number of photons, emitted with energy $\hbar\omega_{n,1} = E_n - E_1$.

6   Summary

To summarize, we have demonstrated that passive gravitational mass of a composite quantum body is not equivalent to its energy due to quantum fluctuations, if the mass is defined to be proportional to a weight of the body. We have also discussed a realistic experimental method to detect this inequivalency. If the corresponding experiment is done, to the best of our knowledge, it will be the first experiment, which directly tests some nontrivial combination of general relativity and quantum mechanics. We have also shown that

the corresponding expectation values are equivalent to each other for stationary quantum states. It is important that our results are due to different couplings of kinetic and potential energy with an external gravitational field. Therefore, the current approach is completely different from that discussed in Refs. [12, 14, 15], where small corrections to electron energy levels are calculated for a free falling hydrogen atom [14, 15] or for a hydrogen atom supported in a gravitational field [12]. Note that phenomena suggested in the paper are not restricted by atomic physics, but also have to be observed in solid state, nuclear, and particle physics.

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