A Theoretical Description of U(5)-SU(3) Nuclear Shape Transitions in the Interacting Boson Model

A.M. Khalaf* and T.M. Awwad†

*Physics Department, Faculty of Science, Al-Azhar University, Egypt. E-mail: Ali-Khalaf43@hotmail.com
†Department of Physics, Faculty of Girls, Ain Shams University, Egypt. E-mail: tawwad12@hotmail.com

We investigated the evaluation of nuclear shape transition from spherical to axially rotational shapes using the Coherent state formalism of the first version of interacting boson model (sd IBM). The validity of such model is examined for rare-earth Nd/Sm/Gd/Dy isotopic chains by analyzing the potential energy surface (PES's). In this region, a change from spherical to well-deformed nuclei is observed when moving from the lighter to heavier isotopes.

1 Introduction

In recent years, the study of quantum phase transition (QPT) is an important topic in the research of nuclear structure. Some evidence of nuclear shape transition have been observed. For instance, several isotopes have been found to undergo shape phase evolution of first order from spherical vibrator to deformed axially symmetric rotor and phase transition of second order from spherical vibrator to deformed $\gamma-$ soft [1–3].

The Hamiltonian describing this transition is a repulsive boson pairing Hamiltonian that has the particularity of being exactly solvable allowing the study of very large systems. The study of phase shape transitions in nuclei can be best done in the interacting boson model (IBM) [4] which reproduces well the data in all transition regions [5–11].

The possible phases that can occur in the IBM have been classified in a triangular Casten diagram [12], the three phases correspond to the breaking of U(6) into its three subalgebras U(5), SU(3) and O(6) [13]. The X(5) critical point symmetry [14] was developed to describe analytically the structure of nuclei at the critical point of the transition from vibrational U(5) to prolate axially symmetric SU(3) shapes. In addition the symmetry E(5) [15, 16] have been introduced to describe the nuclei at the critical point corresponding to second order transition, nuclear examples of which were used [17]. Recently, the critical point in the phase transition from axially deformed to triaxial nuclei called Y(5), has been analyzed [18]. In all these cases, critical points are defined in the context of the collective Bohr Hamiltonian [19].

Since the IBM was formulated from the beginning in terms of creation and annihilation boson operators, its geometric interpretation in terms of shape variables is usually done by introducing a boson condensate with two shape parameters β and γ. The parameter β is related to the axial deformation of the nucleus, while γ measures the deviation from axial symmetry. The equilibrium shape of the nucleus is obtained by minimizing the expectation value of the Hamiltonian in the intrinsic state.

In this paper, we discuss some aspects of the nuclear shape phase transition in even-even nuclei using the IBM with the intrinsic state formalism. The outline of the present paper is as follows: In Section 2, we construct the IBM Hamiltonian in terms of Casimir operators and using coherent state to get the potential energy surface (PES). In section 3, we check that results of the IBM with coherent state to agree for dynamical limits U(5), SU(3) and O(6) in the limit of large N. In section 4 we applied our model to the rare earth Nd/Sm/Gd/Dy isotopic chains which evolve a rapid structural changes from spherical to well-deformed nuclei when moving from lighter to the heavier isotopes.

2 Coherent State Potential Energy Surface

We start by considering a general standard two-body sd IBM Hamiltonian in the Casimir forms as:

$$
\begin{aligned}
H = {} & \epsilon C_1[U(5)] + K_1 C_2[U(5)] \\
& + K_2 C_2[O(5)] + K_3 C_2[O(3)] \\
& + K_4 C_2[SU(3)] + K_5 C_2[O(6)]
\end{aligned}
\tag{1}
$$

Here $C_n[G]$ is the n-rank Casimir operator of the Lie group G, with

$$C_1[U(5)] = \hat{n}_d \tag{2}$$

$$C_2[U(5)] = \hat{n}_d(\hat{n}_d + 4) \tag{3}$$

$$C_2[O(5)] = 4[\frac{1}{10}(\hat{L}\hat{L}) + \hat{T}_3\hat{T}_3] \tag{4}$$

$$C_2[O(3)] = 2(\hat{L}\hat{L}) \tag{5}$$

$$C_2[SU(3)] = \frac{2}{3}\left[2(\hat{Q}\hat{Q}) + \frac{3}{4}(\hat{L}\hat{L})\right] \tag{6}$$

$$C_2[O(6)] = 2\left[N(N+4) - 4(\hat{P}\hat{P})\right] \tag{7}$$

where $\hat{n}_d, \hat{P}, \hat{L}, \hat{Q}, \hat{T}_3$ and $\hat{T}_4$ are the boson number, pairing, angular momentum, quadrupole, octupole and hexadecapole operators defined as:

$$\hat{n}_d = (d^\dagger \tilde{d})^{(0)} \tag{8}$$

$$\hat{P} = \frac{1}{2}(\tilde{d}\tilde{d}) - \frac{1}{2}(\tilde{s}\tilde{s}) \tag{9}$$

$$\hat{L} = \sqrt{10}\left[d^\dagger \times \tilde{d}\right]^{(1)} \tag{10}$$

$$\hat{T}_3 = \left[d^\dagger \times \tilde{d}\right]^{(3)} \tag{11}$$

$$\hat{T}_4 = \left[d^\dagger \times \tilde{d}\right]^{(4)} \tag{12}$$

where $s^\dagger(s)$ and $d^\dagger(\tilde{d})$ are monopole and quadrupole boson creation (annihilation) operators, respectively. The scalar product is defined as

$$\hat{T}_L\hat{T}_L = \sum_M (-1)^M \hat{T}_{L,M}\hat{T}_{L,-M} \tag{13}$$

where $\hat{T}_{L,M}$ corresponds to the M component of the operator $\hat{T}_L$. The operator $\tilde{d}_m(-1)^m d_{-m}$ and $\tilde{s} = s$ are introduced to ensure the correct tensorial character under spatial rotations.

The Connection between the IBM, PES, geometric shapes and phase transitions can be investigated by introducing a coherent, or intrinsic state which is expressed as a boson condensate [20]

$$|N,\beta,\gamma\rangle = \frac{1}{\sqrt{N!}}\left(b_c^\dagger\right)^N |0\rangle \tag{14}$$

with

$$b_c^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left(s^\dagger + \beta\cos\gamma\, d_o^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right). \tag{15}$$

$|0\rangle$ is the boson vacuum and the variables β and γ determine the geometry of nuclear surface. Spherical shapes are characterized by $\beta = 0$ and deformed ones by $\beta > 0$. The angle γ allows one to distinguish between axially deformed nuclei $\gamma = 0°$ for prolate and $\gamma = 60°$ for oblate deformation and triaxial nuclei $0° < \gamma < 60°$.

The expectation values of the Casimir operators equations (2–7) in the ground state equation (14) is:

$$\langle C_1[U(5)]\rangle = \frac{N}{1+\beta^2}\beta^2 \tag{16}$$

$$\langle C_2[U(5)]\rangle = \frac{5N}{1+\beta^2}\beta^2 + \frac{N(N-1)}{(1+\beta^2)^2}\beta^4 \tag{17}$$

$$\langle C_2[O(5)]\rangle = \frac{8N}{1+\beta^2}\beta^2 \tag{18}$$

$$\langle C_2[O(3)]\rangle = \frac{12N}{1+\beta^2}\beta^2 \tag{19}$$

$$\langle C_2[SU(3)]\rangle = \frac{20}{3}N + \frac{4}{3}\frac{N(N-1)}{(1+\beta^2)^2}\cdot$$
$$\left(4\beta^2 + \frac{1}{2}\beta^4 + 2\sqrt{2}\beta^3\cos(3\gamma)\right) \tag{20}$$

$$\langle C_2[O(6)]\rangle = 2N(N+4) - \frac{1}{2}\frac{N(N-1)}{(1+\beta^2)^2}\left(1-\beta^2\right)^2. \tag{21}$$

The PES associated with the IBM Hamiltonian of equation (1) is given by its expectation value in the coherent state and can be written as:

$$V(\beta,\gamma) = a_\circ \frac{N}{1+\beta^2}\beta^2$$
$$+ \frac{N(N-1)}{(1+\beta^2)^2}\left(a_1 + a_2\beta^2 +\right.$$
$$\left. a_3\beta^3\cos(3\gamma) + a_4\beta^4\right) \tag{22}$$

where the coefficients a_i are linear combinations of the parameters of the Hamiltonian and terms which do not depend on β and/or γ have not been included.

3 Shape Structure of the Dynamical Symmetries

The analysis of the three dynamical symmetry limits of the IBM provides a good test of the formalism presented in the previous section.

3.1 The U(5) Symmetry

The Hamiltonian of the vibrational limit $U(5)$ can be written down by putting $k_4 = k_5 = 0$ in equation (1). This has the consequence that in H remain only the terms which conserve both the number of d-bosons and the one of the s-bosons. The Hamiltonian operator of this approximation reads:

$$H[U(5)] = \epsilon C_1[U(5)] + K_1 C_2[U(5)] +$$
$$K_2 C_2[O(5)] + K_3 C_2[O(3)]. \tag{23}$$

This yields the PES

$$E(N,\beta) = \epsilon_d \frac{N}{1+\beta^2}\beta^2 + f\frac{N(N-1)}{(1+\beta^2)^2}\beta^4. \tag{24}$$

This energy functional is $\gamma-$ independent and has a minimum at $\beta = 0$, Special case for $U(5)$ limit, when

$$H = \epsilon C_1[U(5)], \tag{25}$$

$$E(N,\beta) = \epsilon\frac{N}{1+\beta^2}\beta^2. \tag{26}$$

3.2 The SU(3) Symmetry

In the parametrization equation (1), the $SU(3)$ limit corresponds to $\epsilon = K_1 = K_2 = K_5 = 0$ and the Hamiltonian reads:

$$H[SU(3)] = K_3 C_2[O(3)] + K_4 C_2[SU(3)]. \tag{27}$$

This yields the PES

$$E(N,\beta,\gamma) =$$
$$3(4k_3 + k_4)\frac{N}{1+\beta^2}\beta^2 + \frac{4}{3}k_4\left[\frac{N}{1+\beta^2}\left(5 + \frac{11}{4}\beta^2\right)\right.$$
$$\left. + \frac{N(N-1)}{(1+\beta^2)^2}\left(4\beta^2 + 2\sqrt{2}\beta^3\cos(3\gamma) + \frac{1}{2}\beta^4\right)\right]. \tag{28}$$
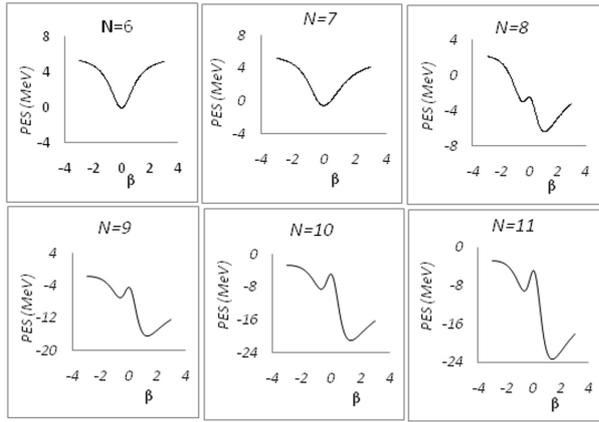
Fig. 1: Calculated PES's as a function of the deformation parameter β in U(5)-SU(3) transition for $^{144-154}$Nd (with $N_\pi = 5$ and $N_\nu = 1 - 6$ neutron bosons) isotopic chain. The total number of bosons N=6-11 and $\chi = -\sqrt{7}/2$).
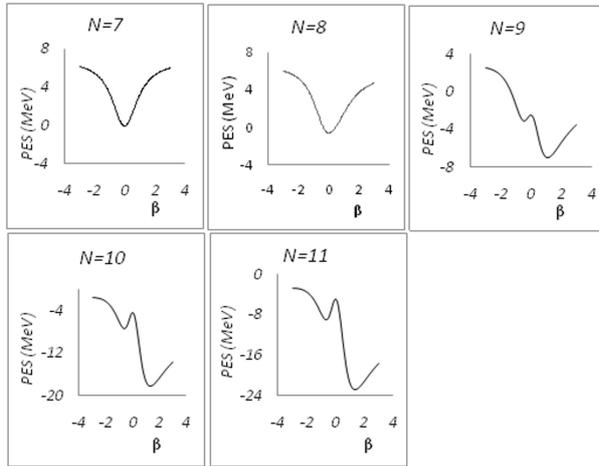

Fig. 2: Calculated PES's as a function of the deformation parameter β in U(5)-SU(3) transition for $^{146-154}$Sm (with $N_\pi = 6$ and $N_\nu = 1 - 5$) isotopic chain. The total number of bosons N=6-11 and $\chi = -\sqrt{7}/2$).

This energy functional has a shape minimum at $\gamma = 0$ and at a value $\beta \neq 0$.

Special case for SU(3) limit, when

$$H = a\hat{Q}\,\hat{Q} \tag{29}$$

and if we eliminate the contribution of the one-body terms of the quadrupole -quadrupole interaction, then, the PES reads

$$E(N,\beta,\gamma) = a\frac{N(N-1)}{(1+\beta^2)^2}(4\beta^2 \pm 2\sqrt{2}\beta^3 \cos(3\gamma) + \frac{1}{2}\beta^4). \tag{30}$$

The equilibrium values are obtained by solving

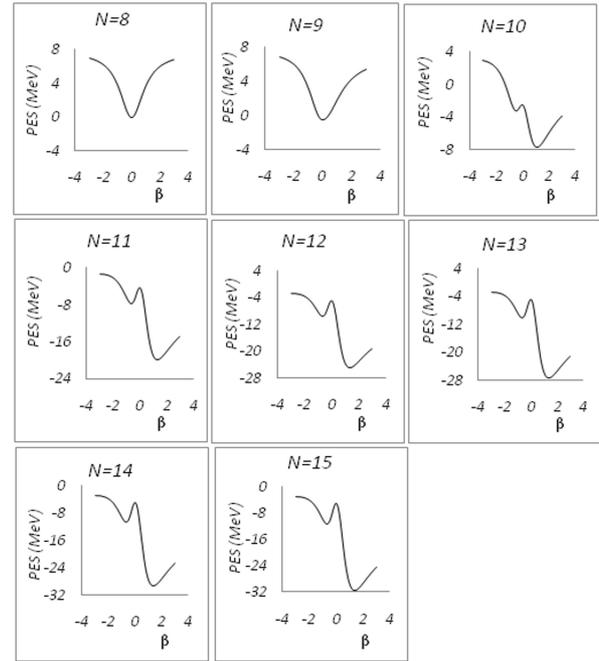$$\frac{\partial E}{\partial \beta} = \frac{\partial E}{\partial \gamma} = 0 \tag{31}$$

Fig. 3: Calculated PES's as a function of the deformation parameter β in U(5)-SU(3) transition for $^{148-162}$Gd (with $N_\pi = 7$ and $N_\nu = 1 - 8$) isotopic chain. The total number of bosons N=6-11 and $\chi = -\sqrt{7}/2$).

to give $\beta_e = \sqrt{2}$ and $\gamma = 0°$ and $\gamma = 60°$.

3.3 The O(6) Symmetry

For the O(6) limit $\epsilon = K_1 = K_2 = 0$ and the Hamiltonian takes the form

$$H[O(6)] = K_2 C_2[O(5)] + K_3 C_2[O(3)] + K_5 C_2[O(6)]. \tag{32}$$

One then obtains the PES

$$E(N,\beta) = 12(2K_2 + K_3)\frac{N}{1+\beta^2}\beta^2 - 2k_5 N(N-1)\left(\frac{1-\beta^2}{1+\beta^2}\right)^2. \tag{33}$$

This energy functional is $\gamma-$independent and has a minimum at a value $|\beta| \neq 0$. For large N, the minimum is at $|\beta| = 1$.

Special case for $O(6)$ limit, when

$$H = a\hat{Q}(\chi)\,\hat{Q}(\chi) \tag{34}$$

$$\chi = 0 \tag{35}$$

and if we eliminate the contribution of the one-body term of the quadrupole-quadrupole interaction, then

$$E(N,\beta) = 4aN(N-1)\left(\frac{\beta}{1+\beta^2}\right)^2 \tag{36}$$
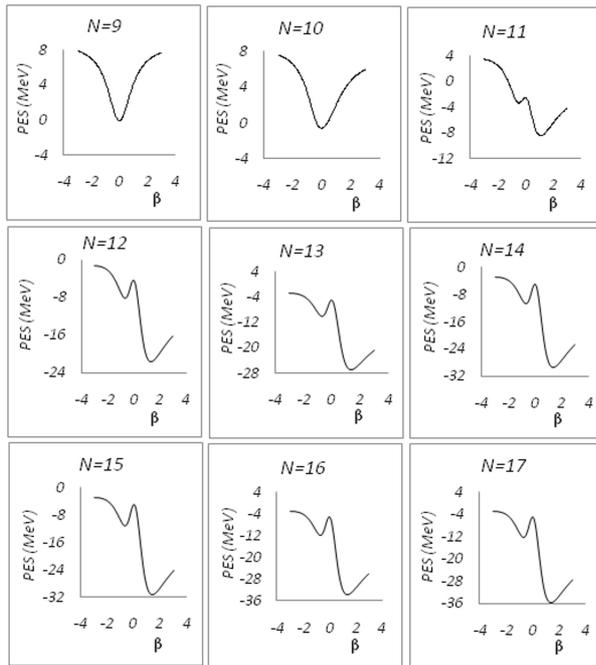
Fig. 4: Calculated PES's as a function of the deformation parameter β in U(5)-SU(3) transition for $^{150-166}$Dy (with $N_\pi = 8$ and $N_\nu = 1 - 9$) isptopic chain. The total number of bosons N=6-11 and $\chi = -\sqrt{7}/2$).

the equilibrium value is given by $\beta = 1$ corresponding to a γ-unstable deformed shape.

4 Application to Rare-Earth Isotope Chains

Nuclei in the region of Sm are well known examples of U(5)-SU(3) transition going from a vibrational into a rotational behavior. The validity of our model is examined for typical various even-even Nd/Sm/Gd/Dy isotopic chains with total number of bosons from N=6 to N=17.

The set of parameters of the model for each nucleus are adjusted by using a computer simulated search program in order to describe the gradual change in the structure as boson number is varied and to reproduce the properties of the selected states of positive parity excitation ($2_1^+, 4_1^+, 6_1^+, 8_1^+, 0_2^+, 2_3^+, 4_3^+, 2_2^+, 3_1^+$ and 4_2^+) and the two neutron separation energies of all isotopes in each isotopic chain. The best fitting parameters obtained for each nucleus are given explicitly in Tables (1,2).

The PES's versus deformation parameter β for rare earth isotopic chain of nuclei evolving from spherical to axially symmetric well deformed nuclei are illustrated in figures (1-4). A first order shape phase transition with changes in number of bosons when moving from the lighter to heavier isotopes i.e. U(5)-SU(3) transitional region are observed. In our selected region we assumed a value $\chi = -\sqrt{7}/2$ because some Gd isotopes clearly exhibit the character of the SU(3)

dynamical symmetry. Around $N = 90$ these seems to be the $X(5)$critical point symmetry. Each PES displays a relatively similar shape with only a small increase in the sharpness of the potential for increasing boson number.

5 Conclusion

In conclusion, the paper is focused on the properties of quantum phase transition between spherical U(5) and prolate deformed SU(3) in framework of the simple version of interacting boson model IBM-1 of nuclear structure.

The Hamiltonian was studied in the three different limits of the IBM and formed by laking. A systematic study of rare earth Nd/Sm/ Gd/Dy isotope chains was done using the coherent states. Nuclei located at or very close to the first order transition were the N=90 isotones ^{150}Nd, ^{152}Nd, ^{154}Nd and ^{156}Nd. They also follow the $X(5)$ pattern in ground state energies. The geometric character of the nuclei was visualizes by plotting the potential energy surface (PES's). parameters of our model were adjusted for each nucleus by using a computer simulated search program, while the parameter X in the quadrupole operator was restricted to fixed value $x = -\sqrt{7}/2$.

References

1. Jolie J. et al. Triple Point of Nuclear Deformation. *Physical Review Letters*, 2002, v. 89, 182502–182504.

2. Arios J.M., Dukelsky J., and Garcia-Ramos J.E. Quantum Phase Transitions in the Interacting Boson Model: Integrability, Level Repulsion and Level Crossing. *Physical Review Letters*, 2003, v. 91, 162502–162504.

3. Turner P.S. and Rowe D.J. Phase Transitions and Quesidynamical Symmetry in Nuclear Collective Model, II. The Sperical vibrator to gamma-soft rotor Transition in an O(5)-invariant Bohr Model. *Nuclear Physics*, 2005, v. A756, 333–355.

4. Iachello F. and Arima A. The Interacting Boson Model. Cambridge University Press, Cambridge, England, 1987.

5. Scholten O., Iachello F. and Arima A. Interacting Boson Model of Collective Nuclear State III. The Transition from SU(5) to SU(3). *Annals of Physics*, 1978, (N.Y.) v. 115, 325–366.

6. Castonos O., Frank A. and Federman, A. The shape transition in the Sm isotopes and the structure of the IBA hamiltonian. *Physics Letters*, 1979, v. B88, 203–206.

7. Cejnar P. and Jolie J. Quantum Phase Transition Studied within the Interacting Boson Model, *Physical Review*, 2000, v. E61, 6237–6247.

8. Cejnar P., Heinze S., and Jolie J. Ground-State Shape Phase Transitions in Nuclei: Thermodynamic analogy and fitie N-effects. *Physical Review*, 2003, v. C68, 034326–034326.

9. Rowe D.J. and Thiamova G. The many relationships between the IBM and the Bohr Model. *Nuclear Physics*, 2005, v. A760, 59–81.

10. Cejnar P., Heinze S. and Dobes J. Thermodynamic analogy for quantum phase transitions at zero temperature. *Physical Review*, 2005, v. C71, 011304R–011309R.

11. Heinze S. et al. Evolution of spectral properties along the O(6)-U(5) transition in the interacting boson model. I. Level dynamics. *Physical Review*, 2006, v. C73, 014306–014316.

12. Casten R.F. Nuclear Structure from a Simple Prespective. Oxford University, Oxford, 1990.

Table 1: Values of the parameter a_o and the total number of boson for the Nd/Sm/Gd/Dy isotopic chain.

No. of Neutrons	$_{60}$Nd	$_{62}$Sm	$_{64}$Gd	$_{66}$Dy
84	1161.91775(6)	1112.0059(7)	1130.70265(8)	1174.6685(9)
86	1082.31775(7)	1078.2059(8)	1160.10265(9)	1223.4685(10)
88	1121.01775(8)	974.0059(9)	1060.40265(10)	1178.2685(11)
90	1078.51735(9)	895.5059(10)	951.80265(11)	1119.7685(12)
92	1011.71775(10)	843.3059(11)	872.90265(12)	1076.8685(13)
94	1071.51775(11)	877.2059(12)	825.80265(13)	1043.5685(14)
96	––	996.9059(13)	813.40265(14)	1029.2685(15)
98	––	1136.7059(14)	827.90265(15)	1025.0685(16)
100	––	––	––	1058.9685(17)

Table 2: Values of the parameters a_1, a_2, a_3, and a_4 describing the IBM Hamiltonian for Nd/Sm/Gd/Dy isotopic chains.

Isotopic chains	a_1	a_2	a_3	a_4
$^{144-154}$Nd	20.93825	−110.4805	−48.51035	−84.10182
$^{146-160}$Sm	13.30225	−85.3005	−41.50433	−61.52960
$^{148-162}$Gd	11.30175	−75.1195	−37.13441	−75.61475
$^{150-166}$Dy	9.66275	−73.8775	−38.57408	−81.01053

13. Arima A. and Iachello F. Interacting Boson Model of Collective States I. The Vibrational Limit. *Annals of Physics*, 1976, (N.Y.) v. 99, 253–317; Arima A. and Iachello F. Interacting Boson Model of Collective States II. The Rotational Limit. *Annals of Physics*, 1978, (N.Y.) v. 111, 201–238.

14. Iachello F. Analytic Prescription of Critical Point Nuclei in a Spherical Axially Deformed Shape Phase Transition. *Physical Review Letters*, 2001, v. 87, 052502–052506.

15. Iachello F. Dynamic Symmetrcies at the Critical Point. *Physical Review Letters*, 2000 v. 85, 3580–3583.

16. Leviatan A. and Ginocchio J.N. Critical Point Symmetry in a Finite System. *Physical Review Letters*, 2005, v. 90, 212501–212505.

17. Casten R.F. and Zamfir N.V. Evidence for a Possible E(5) Symmetry in ^{134}Ba. *Physical Review Letters*, 2000 v. 85, 3584–3586, Casten R.F. and Zamfir N.V. Empirical Realization of a Critical Point Description in Atomic Nuclei. *Physical Reviw Letters*, 2001 v. 87, 052503–052507.

18. Iachello F. Phase Transition an Angle Variables. *Physical Review Letters*, 2003, v. 91, 132502–132507.

19. Bohr A. and Mottelson, Nuclear Structure, Benjamin, New York, 1975, Vol. II.

20. Dieperink A.E.L., Scholten O. and Iachello F. Classical Limit of the Interacting Boson Model. *Physical Reviw Letters*, 1980, v. 44, 1747–1750.

21. Dieperink A.E.L and Scholten O. On Shapes and Shase Transition in the Interacting Boson Model. *Nuclear Physics*, 1980, v. A346, 125–138.

————————