The Poisson Equation, the Cosmological Constant and Dark Energy

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The Cosmological Constant Λ within the modified form of the Einstein Field Equation (EFE) is now thought to best represent a "dark energy" responsible for a repulsive gravitational effect, although there is no accepted argument for its magnitude or even physical presence. In this work we compare the origin of the Λ argument with the concept of unimodular gravity. A metaphysical interpretation of the Poisson equation during introduction of Λ could account for the confusion.

1 Introduction

In 1916, Einstein introduced his general theory of relativity as a geometrical theory of gravity [4] resulting in the Einstein field equation (EFE),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1}$$

It has been well documented and studied that the EFE did not predict a stable static universe, as it was theorized to be at the time [3]. The equation, however, did accurately predict gravitational redshift, magnitudes of gravitational lensing and account for Mercury's precessing orbit, which the Newtonian equation could not. In order to manufacture an equation that could account for a static universe, but still be empirically accurate, it is often stated that Einstein ad hoc threw in another constant Λ which is known as the cosmological constant. This would have been placed back into the EFE with the metric $g_{\mu\nu}$ as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = G_{\mu\nu}. \tag{2}$$

Once it was discovered that the universe actually appeared to be in a decelerating or coasting expansion mode, Einstein quickly removed the Λ term. Today, though, there is empirical evidence that a very small magnitude Λ exists, but some quantum field theorists estimate it as being over 120 orders of magnitude smaller than their calculations, "probably the worst theoretical prediction in the history of physics" [3]. In addition, the observed small value of Λ requires an extremely high level of arbitrary fine tuning "for no good reason" and is a "cosmologist's worst nightmare come true" [6]. This transformation from a minor but rich interest exploded (5000 papers submitted to date [10]) near the end of the past millennium due to a startling simultaneous discovery of positive acceleration from two teams [7, 8].

The source of this unforeseen positive acceleration has come to be known as "dark energy". The lack of progress in explaining the phenomena led to the creation of a Dark Energy Task Force in 2006 which stated in a report [1]:

> "Most experts believe that nothing short of a revolution in our understanding of fundamental physics will be required to achieve a full understanding of the cosmic acceleration."

This dark energy is currently expected to contribute over 73.4% [5] of the mass-energy of the universe, and there is no sound logical theory for what it is. Consider that this leaves some type of mysterious never-observed particle known as dark matter to contribute another 22.2%, leaving only 4.4% for the normal matter we are familiar with. With this in mind, we propose that it is reasonable to re-examine any argument that has lead us to our current state of physics.

2 Poisson Equation and Gauss' Theorem

The Poisson equation,

$$-\nabla^2 u = \mathbf{f}, \tag{3}$$

is well known to relate the function f as the "source" or "load" of the effect on u of the left hand side. Let us examine what this means *exactly* more in depth and what we can conclude from this tool. As an example, for a function f given on a three dimensional domain denoted by $\Omega \subset \mathbb{R}^3$ we have

$$\alpha u + \beta \frac{\partial u}{\partial \mathbf{n}} = \mathbf{g} \quad \text{on} \quad \partial\Omega. \tag{4}$$

This is a solution u satisfying boundary conditions on the boundary $\partial\Omega$ of Ω. α and β are constants and $\frac{\partial u}{\partial \mathbf{n}}$ represents the directional derivative in the direction normal $\mathbf{n}$ to the boundary $\partial\Omega$ which by convention points outwards. Although if $\alpha = 0$ is referred to as a Neumann boundary condition, even with $\alpha = constant$ the solution is said to only be unique up to this additive constant. Let us examine whether this statement is entirely accurate.

2.1 Graphical Meaning of Poisson Equation

Let us take the divergence of $\mathbf{g}$ so that

$$\nabla \cdot \alpha u + \nabla \cdot \beta \frac{\partial u}{\partial \mathbf{n}} = \nabla \cdot \mathbf{g} \tag{5}$$

and

$$0 + \nabla \cdot \beta \frac{\partial u}{\partial \mathbf{n}} = \nabla \cdot \mathbf{g}. \tag{6}$$

We can see that the presence of αu seems arbitrary since it has no effect. Let us examine a two dimensional slice of scalar values in $\mathbb{R}^3$ to graphically give a better understanding. In Fig. 1 we have an example of Eq. 4 using a Euclidean coordinate system.
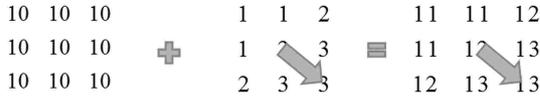

Fig. 1: Two Dimensional Scalar Field

For any derivative of Eq. 5, the constant term of course would result in no vector since there is no directional derivative from αu.

We note that this equation can also be written as

$$\alpha u - \beta \frac{-\partial u}{\partial \mathbf{n}} = \mathbf{g}, \tag{7}$$

shown in Fig. 2, which does not mathematically make a difference but can, however, introduce a question of uniqueness.
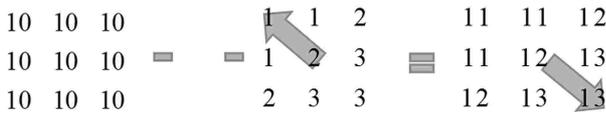

Fig. 2: Alternate Two Dimensional Scalar Field

Let us define the previous scalar field u as u_1 and a second scalar field as u_2. If ξ and γ are constants, then Eq. 8 and Fig. 3 present a dilemma. While there may be no directional derivatives from the constant term, we could also equivalently model this as orthogonal vectors with the sum of 0.

$$\xi u_2 - \gamma \frac{\partial u_2}{\partial \mathbf{n}} = \mathbf{g}_2 \tag{8}$$
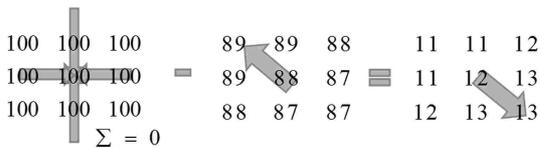

Fig. 3: Second Two Dimensional Scalar Field

From this we can see that there are no unique solutions of u for $\mathbf{g}$ from the Poisson equation, if

$$\alpha u_1 + \beta \frac{\partial u_1}{\partial \mathbf{n}} = \mathbf{g}_1 \tag{9}$$

and

$$\xi u_2 - \gamma \frac{\partial u_2}{\partial \mathbf{n}} = \mathbf{g}_2 \tag{10}$$

but also

$$\nabla \cdot (\alpha u_1 + \beta \frac{\partial u_1}{\partial \mathbf{n}}) = \nabla \cdot \mathbf{g}_1 = \nabla \cdot \mathbf{g} \tag{11}$$

and

$$\nabla \cdot (\xi u_2 - \gamma \frac{\partial u_2}{\partial \mathbf{n}}) = \nabla \cdot \mathbf{g}_2 = \nabla \cdot \mathbf{g} \tag{12}$$

if

$$\beta \frac{\partial u_1}{\partial \mathbf{n}} = -\gamma \frac{\partial u_2}{\partial \mathbf{n}}. \tag{13}$$

2.2 Gauss Theorem

Like our above illustration of the Poisson equation, a misunderstanding of Gauss' Theorem,

$$-\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} = -\int_{\Omega} \nabla^2 u = \int_{\Omega} f \tag{14}$$

could also cause confusion if

$$-\int_{\partial\Omega} \beta \frac{\partial u_1}{\partial \mathbf{n}} = -\int_{\partial\Omega} \left(\xi u_2 - \gamma \frac{\partial u_2}{\partial \mathbf{n}} \right) \tag{15}$$

and

$$-\int_{\Omega} \nabla^2 \beta u_1 = -\int_{\Omega} \left(\nabla^2 \xi u_2 - \nabla \cdot \gamma \frac{\partial u_2}{\partial \mathbf{n}} \right). \tag{16}$$

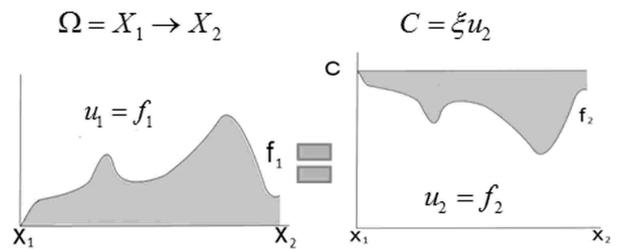Equations 15 and 16 are easily understood graphically as taking the second derivatives of the plots in Fig. 4.

Fig. 4: Equivalent Areas From Gauss' Theorem

3 Conclusion

Although we can assume that some function $\mathbf{g}$ is causal to the appearance of a vector, does the vector appear from nothing or is it result of a change in what is already at that point? If αu exists, what does it physically represent? Calling any field "attractive" or "repulsive" is nothing more than a metaphysical convention, i.e. does the load function cause a change in ϕ resulting in an attraction or a reduced repulsion, as in Fig. 5? From this, we can conclude that although we may possess measurements ∇u and $\nabla^2 u$, we cannot determine the nature of the scalar field u simply from the Poisson equation or Gauss' Theorem.
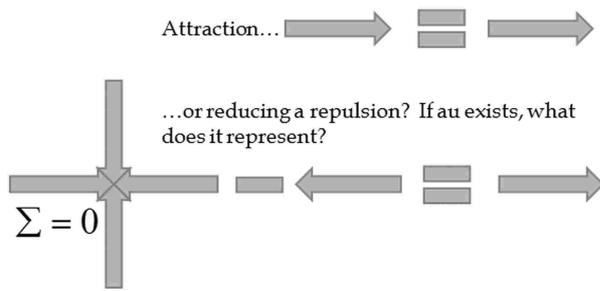
Fig. 5: Attraction or Reduced Repulsion?

4 Motivation: Cosmological Constant and General Relativity

Why is the previous figure important? Although there is a great deal of literature concerning Λ, in order to start a new perspective and to utilize the previous section, we re-examine the first known published physical meaning of the constant. In Einstein's 1917 paper *Cosmological Considerations On The General Theory of Relativity* [2] the first equation Einstein presents is the Poisson equation version of Newton's Law of Gravity

$$\nabla^2 \phi = 4\pi\kappa\rho. \tag{17}$$

Citing Newtonian concerns over the limiting value of ϕ at "spatial infinity" he proposes a modification of the equation to

$$\nabla^2 \phi - \lambda\phi = 4\pi\kappa\rho. \tag{18}$$

This was from an early difficulty in that the derivation required $R_{\mu\nu} = 0$ when matter or energy was not present. Due to cosmological observations though, and despite the rigor of the derivation, this requirement was eventually relaxed [4, see for relation to $G_{\mu\nu} = 0$, p. 410] allowing the introduction of a cosmological constant, even if it is not physically understood.

Setting the Poisson equation aside for the moment, it is also known that one of the interpretations of Λ or λ in Riemannian geometry is as a four dimensional constant of integration, through what is referred to as Unimodular Gravity [9]. This interpretation restricts allowable diffeomorphisms to only those preserving the four volume, but to date this has been treated as but a curious equivalent to General Relativity.

5 Introducing the Lorentz Tensor

Let us take a constant multiple of the metric $g_{\mu\nu}$ and refer to it as Ω. We do not utilize Λ or λ so as not to cause confusion and to allow us to more easily retain a difference in our understanding. Let us enforce $R_{\mu\nu} = 0$ such that

$$\Omega g_{\mu\nu} = G_{\mu\nu} + L_{\mu\nu} \tag{19}$$

where $G_{\mu\nu}$ is the Einstein tensor and $L_{\mu\nu}$ is a tensor we propose to call the "Lorentz" tensor. We shall expand on our

reasoning for calling it this in subsequent papers. We can readily see that

$$G_{\mu\nu} = \Omega g_{\mu\nu} - L_{\mu\nu} \tag{20}$$

and that if $\Omega = 0$ then the Lorentz tensor is simply the negative of the Einstein tensor,

$$G_{\mu\nu} = -L_{\mu\nu}, \tag{21}$$

and should have the same important properties, i.e.

$$G_{\mu\nu;\mu} = -L_{\mu\nu;\mu}. \tag{22}$$

This of course results in

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = G_{\mu\nu} = \Omega g_{\mu\nu} - L_{\mu\nu}. \tag{23}$$

Note that for now cosmological models that rely on only a multiple of the metric remaining with no matter present, such as deSitter space, are not possible since $R_{\mu\nu} = 0$.

Although there are physical arguments for equating the Einstein tensor to the energy momentum tensor ($G_{\mu\nu} = \kappa T_{\mu\nu}$), and thus into analogues for Newton's Law of Gravity, we note simply in this paper that Eq. 17 is ultimately arrived at through $G_{\mu\nu}$. By the symmetry present in Eq. 23 and our arguments concerning the Poisson equation and Gauss' Theorem, our future objective is to use our understanding of Fig. 6 to obtain a rigorous derivation of Fig. 7.
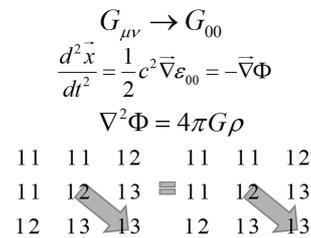

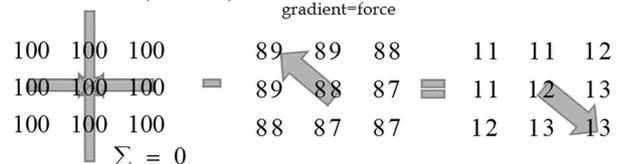Fig. 6: Einstein Tensor to Poisson

Fig. 7: Alternate EFE to Reduced Repulsive Poisson

We do this also in order to ask, should matter subject to the force represented by the vector present in Fig. 7 become

zero after traveling a certain radius from a massive body, what occurs at radii larger than this? It is our motivation to determine whether this is a plausible explanation for phenomena attributed to positive accelerating expansion.

Acknowledgements

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