Nuclear Shape Transition Using Interacting Boson Model with the Intrinsic Coherent State

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The values of the potential energy surface (PES) for the even-even isotopic chains of Nd/Sm/Gd/Dy are studied systematically using the simplified form of interacting boson model (IBM) with intrinsic coherent state. The critical points have been determined for each isotope chain. The phase diagrams exhibits first-order shape phase transition from spherical U(5) to deformed axial symmetric prolate SU(3) when moving from light isotopes to heavy ones.

1 Introduction

We note that in the interacting boson model-1 (IBM-1) [1, 2] one describes an even-even nucleus as a system of N bosons able to occupy two levels, one with angular momentum restricted to zero (s boson) and one with angular momentum 2 (d boson).

The bosons are assumed to interact via a two-body residual interaction. Denoting by b_i (i=1,...,6) the creation (annihilation) operators for bosons ($b_1 = s, b_{2,...,6} = d$) it is easy to see that the 36 operators $G_{ii^-} = b_i^\dagger b_{i^-}$ close under the Lie algebra of U(6). This simple model allows the utilization of algebraic symmetric for approaching different type of nuclear spectra, known as dynamical symmetries and corresponding to un-harmonic vibrator (U(5) Symmetry) [3], rigid deformations (SU(3) Symmetry) [4] and γ-instability (O(6) Symmetry) [5]. In these special cases it is possible to find analytical solutions of the boson Hamiltonian and deal with small deviations from these symmetries using different perturbation methods.

However, real nuclei may deviate considerably from the simple dynamical limits. This is represented in the Casten triangle [1–6] with vertices corresponding to the standard dynamical symmetries and the sides of the triangle represent direct transition between the limiting cases, whereas all complex transition regions are contained in the area. Phase transitions between these shapes were studied, and it is known that the phase transition from U(5) to O(6) is second order, while any other transitions within the Casten triangle from a spherical to deformed shape is first order [7–23].

Now, there is a class of symmetries that are formulated in terms of the Bohr Hamiltonian and that can be applied to critical point situation [24–26]. In particular, at the critical point from spherical to γ-unstable shapes, called E(5) [24], at the critical point from spherical to axially deformed shapes, called X(5) [25] and the critical point from axially deformed shapes to triaxial shapes, called Y(5) [26]. Since the introduction of these limits many theoretical [27–32] and experimental [33–39] studies have been presented in order to look for nuclei that exhibit the properties of critically and to classify the corresponding phase transitions. Many studies have extended these original models to more complex situations [40–44].

The relation between the Bohr-Mottelson collective model [45] and the IBM was established [46, 47] on the basis of an intrinsic (or coherent) state for the IBM. Via this coherent state formalism, a potential energy surface (PES) E(β, γ) in the quadruple deformation variables β and γ can be derived for any IBM Hamiltonian and the equilibrium deformation parameters β_0 and γ_0 are then found by minimizing E(β, γ). The deformation parameter β measures the axial deviation from sphericity, while the angle variable γ controls the departure from axial symmetry.

In the present work, we investigate shape phase transition within the IBM-1 using coherent state formalism for various rare earth isotopic chains. The paper is organized as follows. First the IBM and the symmetry triangle used in the present work is briefly described in section 2. In this variation of the IBM, the coherent state approach is treated to produce PES's in section 3. The location of the critical point in the shape transition is identified in section 4. We review the concept of dynamical symmetry in section 5. In section 6 a systematic study of isotopic chains on Nd/Sm/Gd/Dy related to the U(5)-SU(3) shape transition is given and main conclusions arising from the present results are discussed.

2 The IBM-1 Hamiltonian and Coherent State

Denoting by $C_n[G]$ the nth-order Casmir operator of the Lie group G, the general sd-IBM Hamiltonian with up to two-body interactions can be written in the following form:

$$\begin{aligned} H &= \epsilon C_1[U(5)] + k_1 C_2[U(5)] \\ &+ k_2 C_2[O(5)] + k_3 C_2[O(3)] \\ &+ k_4 C_2[SU(3)] + k_5[O(6)] \end{aligned} \tag{1}$$

The Casmir operators are defined by the following equations

$$C_1[U(5)] = \hat{n}_d \tag{2}$$

$$C_2[U(5)] = \hat{n}_d(\hat{n}_d + 4) \tag{3}$$

$$C_2[O(5)] = 4\left[\frac{1}{10}(\hat{L}.\hat{L} + \hat{T}_3.\hat{T}_3)\right] \tag{4}$$

$$C_2[O(3)] = 2(\hat{L}.\hat{L}) \tag{5}$$

$$C_2[SU(3)] = \frac{2}{3}\left[2(\hat{Q}.\hat{Q}) + \frac{3}{4}(\hat{L}.\hat{L})\right] \tag{6}$$

$$C_2[O(6)] = 2\left[N(N+4) - 4(\hat{P}.\hat{P})\right] \tag{7}$$

where $\hat{n}_d$, $\hat{P}$, $\hat{L}$, $\hat{Q}$ and $\hat{T}_3$ are the boson number, pairing, angular momentum, quadruple and octuple operators defined as

$$\hat{n}_d = (d^\dagger \tilde{d})^{(0)} \tag{8}$$

$$\hat{P} = \frac{1}{2}(\tilde{d}\tilde{d}) - \frac{1}{2}(\tilde{s}\tilde{s}) \tag{9}$$

$$\hat{Q}[\chi] = [d^\dagger \tilde{s} + s^\dagger \tilde{d}]^2 + \chi[d^\dagger x\tilde{d}]^{(2)} \tag{10}$$

$$\hat{L} = \sqrt{10}[d^\dagger x\tilde{d}]^{(1)} \tag{11}$$

$$\hat{T}_3 = [d^\dagger x\tilde{d}]^{(3)} \tag{12}$$

where $s^\dagger(s)$ and $d^\dagger(d)$ are monopole and quadruple boson creation (annihilation) operators respectively. The study of shape phase transition in even-even nuclei can be well done from the simple two parameter IBM Hamiltonian, the well known consistent-Q Hamiltonian

$$H = \varepsilon\hat{n}_d - k\hat{Q}(\chi)\cdot\hat{Q}(\chi). \tag{13}$$

The symbol $(\cdot)$ represents the scalar product and the scaler product of two operators with angular momentum L is defined as $\hat{T}_L.\hat{T}_L = \Sigma_M(-1)^M\hat{T}_{LM}\hat{T}_{L-M}$ where $\hat{T}_{LM}$ corresponds to the M component of the operator $\hat{T}_L$.

The Hamiltonian of equation (13) describes the main features of collective nuclei, it contains the dynamical symmetries of the IBM for spherical choices of the coefficients ε, k and χ, and allows to describe the transitional regions between any of symmetry limits as well. In discussing phase transitions, it is convenient to introduce the control parameter η, such as:

$$\frac{\eta}{1-\eta} = \frac{1}{N}\frac{\varepsilon}{k} \tag{14}$$

where N is the total number of boson. Hamiltonian(1) can be written in the second form

$$H = C\left[\eta\hat{n}_d - \frac{1-\eta}{N}\hat{Q}(\chi).\hat{Q}(\chi)\right]. \tag{15}$$

With

$$C = \varepsilon + Nk, \qquad \eta = \frac{\varepsilon/k}{N + \varepsilon/k}. \tag{16}$$

The second form equation (15) avoids the infinities inherent in the use of the ratio of ε/k as η varies from 0 to 1. The factor C in equation (15) is only a scale factor and η and χ are therefore the two parameters that determine the structure. The values of the control parameter η ranges from 0 to 1 and χ is located in the interval of $-\sqrt{7}/2$ (-1.32) to $\sqrt{7}/2$ (+1.32).

Let us consider the Hamiltonian of equation (5) and the effects of its two parameters η and χ. Clearly, one of the most important features of the IBM is the existence of three distinct dynamical symmetries (DS), each representing a well defined phase of nuclear collective motion. The three DS are: the U(5) symmetry for spherical vibrational nuclei (η=1), the SU(3) symmetry for prolate deformed nuclei (η=0, $\chi = -\sqrt{7}/2$) and the O(6) symmetry for γ-unstable deformed nuclei (η=0, χ=0), the SU(3) symmetry for oblate deformed nuclei corresponding to (η=0, $\chi = +\sqrt{7}/2$). For intermediate values of the control parameters η and χ, the potential energy surface (PES) function will describe a certain point on the IBM symmetry triangle located between the three limits.

Comparing the simplified Hamiltonian equation (15) with equation (1) we see that only two terms of the general form are considered. Rewriting equation (15) in the form of equation (1), we get:

$$\begin{aligned} H \;=\; & \left[\eta + \frac{2}{7N}(1-\eta)\chi\left(\chi + \frac{\sqrt{7}}{2}\right)\right]C_1[U(5)] \\ & + \frac{2}{7N}(1-\eta)\chi\left(\chi + \frac{\sqrt{7}}{2}\right)C_2[U(5)] \\ & + \frac{1}{N}(\eta - 1)\left(1 + \frac{3}{\sqrt{7}}\chi + \frac{2}{7}\chi^2\right)C_2[O(5)] \\ & + \frac{1}{14N}(1-\eta)\chi\left(\chi + 2\sqrt{7}\right)C_2[0(3)] \\ & + \frac{1}{\sqrt{7}N}(\eta - 1)\chi C_2[SU(3)] \\ & + \frac{1}{N}(1-\eta)\left(1 + \frac{2}{\sqrt{7}\chi}\right)C_2[O(6)]. \end{aligned} \tag{17}$$

In IBM-1, the intrinsic coherent normalized state of a nucleus with N valence bosons outside the doubly-closed shell state is given by:

$$|N\beta\gamma\rangle = \frac{1}{\sqrt{N!}}(\Gamma_C^\dagger)^N|0\rangle \tag{18}$$

where $|0\rangle$ denotes the boson vacuum, and

$$\Gamma_C^\dagger = \frac{1}{\sqrt{1+\beta^2}}\left[s^\dagger + \beta\cos\gamma d_0^\dagger + \frac{1}{\sqrt{2}}\beta sin\gamma(d_2^\dagger + d_{-2}^\dagger)\right]. \tag{19}$$

Here $\beta \geq 0$ and $0 \leq \gamma \leq \pi/3$ are intrinsic shape parameters. We get the PES by calculating the expectation value of Hamiltonian (17) on the boson condensate equation (18). The corresponding PES as a function of the deformations β and γ is given by:

$$
\begin{aligned}
E(N,\eta,\chi,\beta,\gamma) &= \\
&= -5(1-\eta) + \frac{1}{(1+\beta^2)^2} \\
&\quad \Big\{ \left[N\eta - (1-\eta)\left(4N+\chi^2-8\right)\right]\beta^2 \\
&\quad + \left[N\eta - (1-\eta)\left(\frac{(2N+5)}{7}\chi^2 - 4\right)\right]\beta^4 \\
&\quad + 4N(1-\eta)\sqrt{\tfrac{2}{7}}\chi\beta^3\cos 3\gamma \Big\}
\end{aligned} \tag{20}
$$

3 Location of the Critical Symmetries

Minimization of the PES equation (20) with respect to β for given values of the control parameters η and χ, gives the equilibrium value β_e. The phase transition is signaled by the condition at $\beta = 0$

$$\frac{d^2E}{d\beta^2} = 0, \tag{21}$$

which fixes the critical value of the control parameter η. The critical point in the above equation (20) is given by the value of η where the coefficient at β^2 vanishes, *i.e.*

$$\eta_{critical} = \frac{4N+\chi^2-8}{5N+\chi^2-8}. \tag{22}$$

At this value, the second β derivative for $\beta = 0$ changes its sign, which means that $\beta = 0$ maximum becomes a local minimum. Note that the critical point (22) depends on χ, it changes between: $\eta(-\sqrt{7}/2) = (16N-25)/(20N-25)$ at U(S)-SU(3) side if the symmetry triangle, and $\eta(0) = (16N-32)/(20N-32)$ at the U(5)-O(6) side, condition (12) gives in the case of large-N limit the value 4/5.

If we ignore the contribution of one-body term of the quadruple-quadruple interaction and in large N limit (N-1≃N) and $\gamma = 0$, equation (20) takes the form

$$
\begin{aligned}
E(N,\eta,\chi,\beta) = \; & \frac{N\beta^2}{(1+\beta^2)^2}\left[5\eta - 4 + 4\sqrt{\frac{2}{7}}\beta\chi(1-\eta)\right. \\
& \left. +\beta^2\left(\eta - \frac{2}{7}\chi^2(1-\eta)\right)\right].
\end{aligned} \tag{23}
$$

The deformation parameter $\beta = 0$ is always a stationary point. For $\eta < 4/5$, $\beta = 0$ is a maximum, while for $\eta > 4/5$, it becomes a minimum. In the case of $\eta = 4/5$, $\beta = 0$ is an inflection point. The $\eta = 4/5$ is the point at which a minimum at $\beta = 0$ starts to develop and defines the antispinodal line. For $\chi \neq 0$, there exists a region, where two minima, one spherical and one deformed, coexist. This region is defined by the point at which the $\beta = 0$ minimum appears (antispinodal point) and the point at which the $\beta \neq 0$ minimum appears (spinodal point). For $\eta = 1$, the system is in the symmetry phase since the PES has a unique minimum at $\beta = 0$. When η decreases, one reaches the spinodal point $\eta = 0.820361$ for $\chi = -\sqrt{7}/2$ as illustrated in Fig. (1) for boson number N=10.

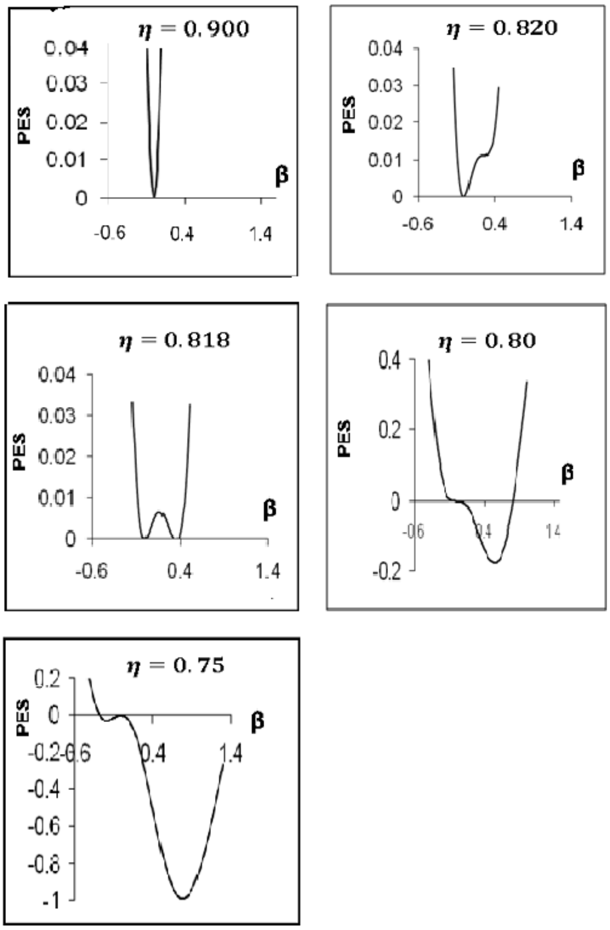

Fig. 1: Potential energy surface (PES) equation (3) for N=10 calculated with IBM without normalization along the axial trajectory $\gamma = 0°, 60°$ as a function of the shape parameter β. The curves describe the first order shape phase transition between spherical to prolate deformed U(5)-SU(3) for control parameter η: $\eta = 0.900$, $\eta = 0.820$ (spinodal), $\eta = 0.818$ (critical point), $\eta = 0.800$ (antispinodal) and $\eta = 0.750$.

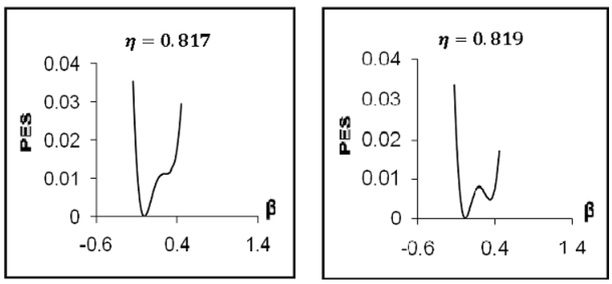

Fig. 2: For two cases in the coexistence region $\eta = 0.817$ and $\eta = 0.819$.

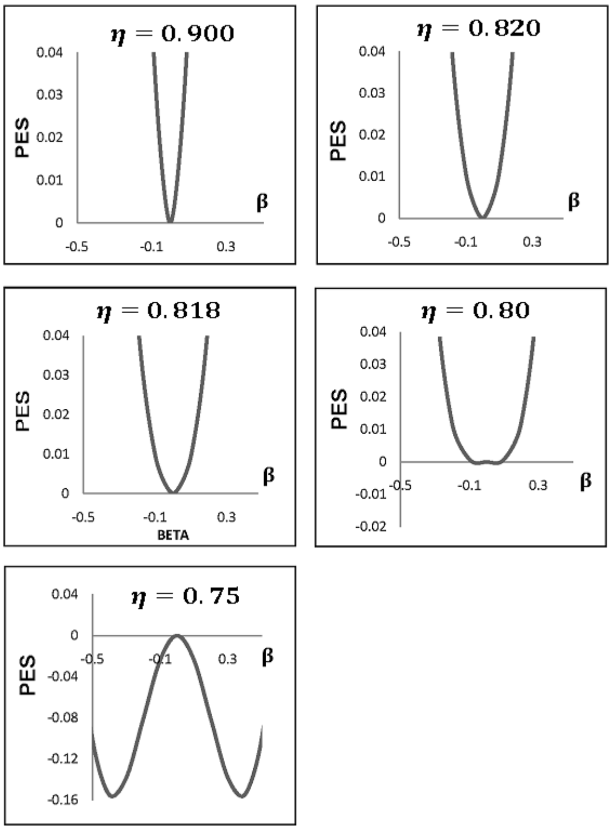

Fig. 3: For $\chi = 0$.

In the coexistence region, the critical point is at the situation in which both minima of spherical point is at the situation in which both minima of spherical and deformed are degenerate. At the critical point, the two degenerated minima are at $\beta_0 = 0$ and $\beta_0 = -\sqrt{7}/4$ and their energy is equal to zero. The critical point line is at $\eta_c = (4 + 2/7\chi^2)/(5 + 2/7\chi^2)$.

The $\chi = -\sqrt{7}/2$ provides $\eta_c = 9/11$ (0.818181). According to the previous analysis, a first order phase transition appears for $\eta \neq 0$, $\chi \neq 0$, while for $\chi = 0$ there is an isolated point of second order phase transition as a function of η. Spinodal, antispinodal and critical point coincide at the critical value $\eta = 4/5$.

We show in Figures (1,2,3) a sketch at this evolution for the special case $\chi = -\sqrt{7}/2$, the two cases in the coexistence region and for $\chi = 0$. From Figure (3), we observe the evolution from the spherical potential $\eta = 0.9$, whose minima is found at $\beta = 0$ to potentials with well-deformed minima $\eta = 0.75$. For intermediate η values one finds a set of potential energy curves which are practically degenerated along the prolate axis in the interval [0, 0.4]. These curves show two minima, on spherical and a prolate deformed one. In particular, for $\eta = 0.81818$, the spherical and the prolate deformed minima are degenerate and this condition defines precisely the critical point of the first order phase transition where the order parameter is the deformation β.

For $\eta = 1$,the Hamiltonian H of equation (15) reduces to the U(5) limit of the IBM corresponds to a spherical shape with vibration

$$H(U(S)) = \hat{n}_d. \tag{24}$$

The PES of H is given by:

$$E(U(5)) = \frac{N\beta^2}{1+\beta^2}. \tag{25}$$

The equilibrium value of the deformation parameter β is easily obtained by solving $\partial E/\partial\beta = 0$ to give $\beta_e = 0$ which corresponds to a spherical shape.

For $\eta = 0$ and $\chi = \mp\sqrt{7}/2$, the schematic Hamiltonian of equation (15) reproduces the SU(3) Limit corresponds to a shape of ellipsoid with rotation (or axial rotation)

$$H((SU(3)) = -\frac{1}{N}\hat{Q}(\chi).\hat{Q}(\chi). \tag{26}$$

If we eliminate the contributions of the one-body terms of quadruple-quadruple interaction, for this case the PES of H is given by:

$$E((SU(3)) = -\frac{(N-1)}{(1+\beta^2)^2}(4\beta^2 + \frac{1}{2}\beta^4 \pm 2\sqrt{2}\beta^3 cos3\gamma). \tag{27}$$

The equilibrium values are given by solving $\frac{dE}{d\beta} = \frac{\partial E}{\partial\gamma} = 0$ to give $\beta_e = \sqrt{2}$ and $\gamma_e = 0$ for $\chi = -\sqrt{7}/2$ and by $\beta_e = \sqrt{2}$ and $\gamma_e = \pi/3$ for $\chi = \sqrt{7}/2$ corresponding to prolate and oblate deformed shape respectively.

For $\eta = 0$ and $\chi = 0$, one recovers the O(6) limit corresponds to γ-unstable

$$H(O(6)) = -\frac{1}{N}\hat{Q}(\chi = 0).\hat{Q}(\chi = 0). \tag{28}$$

Eliminating the one-body terms, the PES depends only on β

$$E(O(6)) = -\frac{(N-1)}{(1+\beta^2)^2}4\beta^2. \tag{29}$$

The equilibrium value is given by $\beta_e = 1$, corresponding to a γ-unstable deformed shape. For intermediate values of the control parameters η and χ, the PES function will describe a certain point on the IBM symmetry triangle, located between the three limits.

4 First-Order U(5)-SU(3) Phase Transition in Nd/Sm/Gd/Dy Rare Earth Nuclei

In a first order phase transition, the state of the rearrangement happens, which means that there involves an irregularity at the critical point.

The study is carried out considering specific isotopic chains of even-even rare earth nuclei $_{60}$Nd, $_{62}$Sm, $_{64}$Gd and $_{66}$Dy displaying first order phase transition from sphericity to

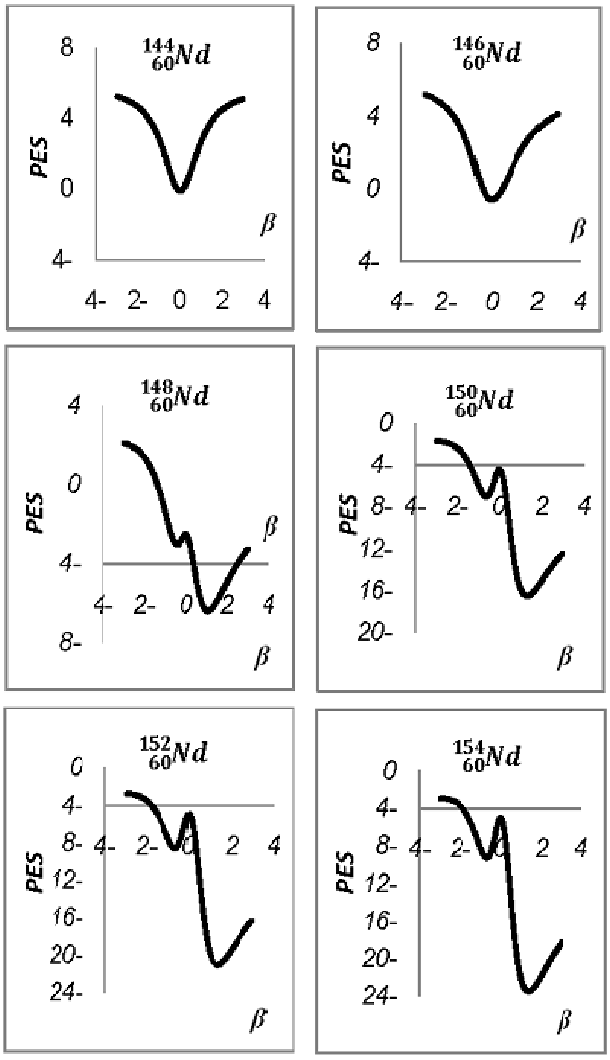

Fig. 4: PES for first order shape phase transition between spherical to prolate deformed U(5)-SU(3) for Neodymium isotope chain $^{144-154}_{60}$Nd (with $N_\pi = 5$ proton bosons and $N_\nu = 1-6$ neutron bosons).

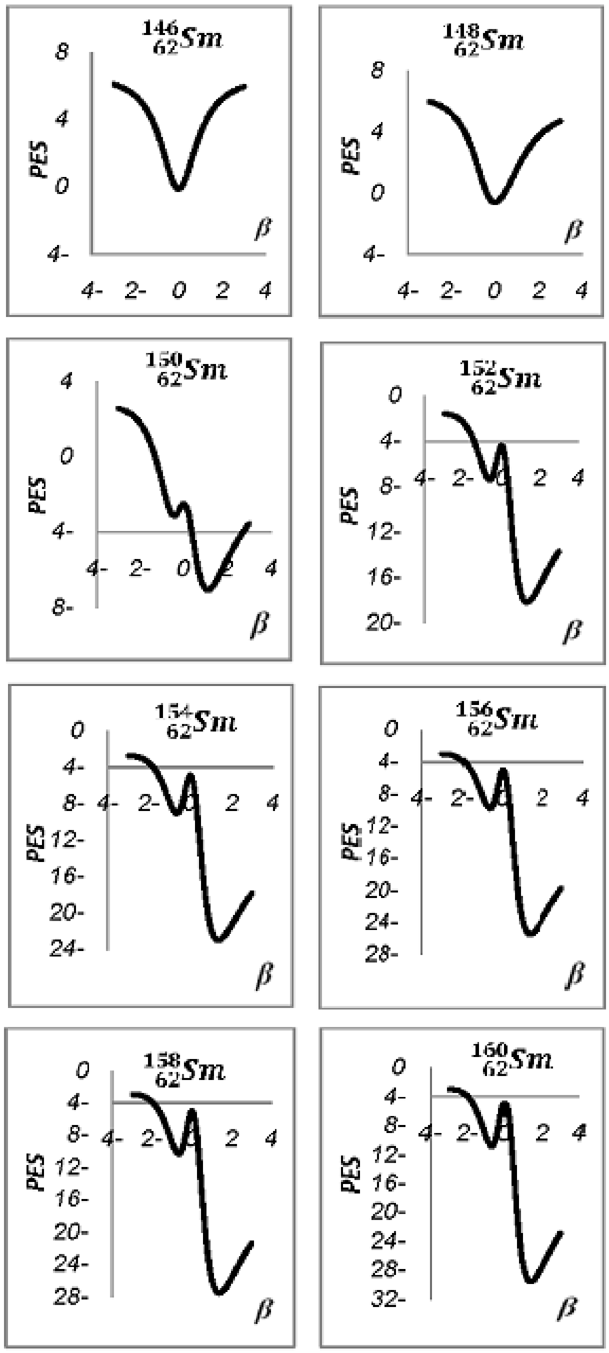

Fig. 5: The same as Fig. (4) but for Samarium isotope chain $^{146-160}_{62}$Sm (with $N_\pi = 6$ proton bosons and $N_\gamma = 1$–8 neutron bosons).

axial symmetric deformed U(S)-SU(3). That is for the nuclei included in this study; all chains begin as vibrational with energy ratio $R_{(4/2)} = E(4^+_1)/E(2^+_1)$ near 2.0 and move towards rotational $R_{(4/2)} = 3.33$ as neutron number is increased. For control parameter $\eta = 1$, we get the U(5) limit and for $\eta = 0$ and $\chi = -\sqrt{7}/2$ the SU(3) limit. For intermediate values of the control parameters η and χ, the PES function will describe a certain point on the IBM symmetry triangle, located between the U(5) and SU(3) limits. To describe a phase transition, one has to establish the values of the control parameter for each nucleus.

For our rare- earth nuclei, we keep χ at the fixed value $\chi = -\sqrt{7}/2$, because some Gd isotopes clearly exhibit the character of the SU(3) dynamical symmetry. This assumption is very successful in describing the Sm nuclei which form neighboring nuclei.

The system passes from the U(5) to the SU(3) limit when the number of bosons is increasing from N=6 towards N=17. The values of the control parameter η is adjusted for each nucleus by using a computer simulated search program in order

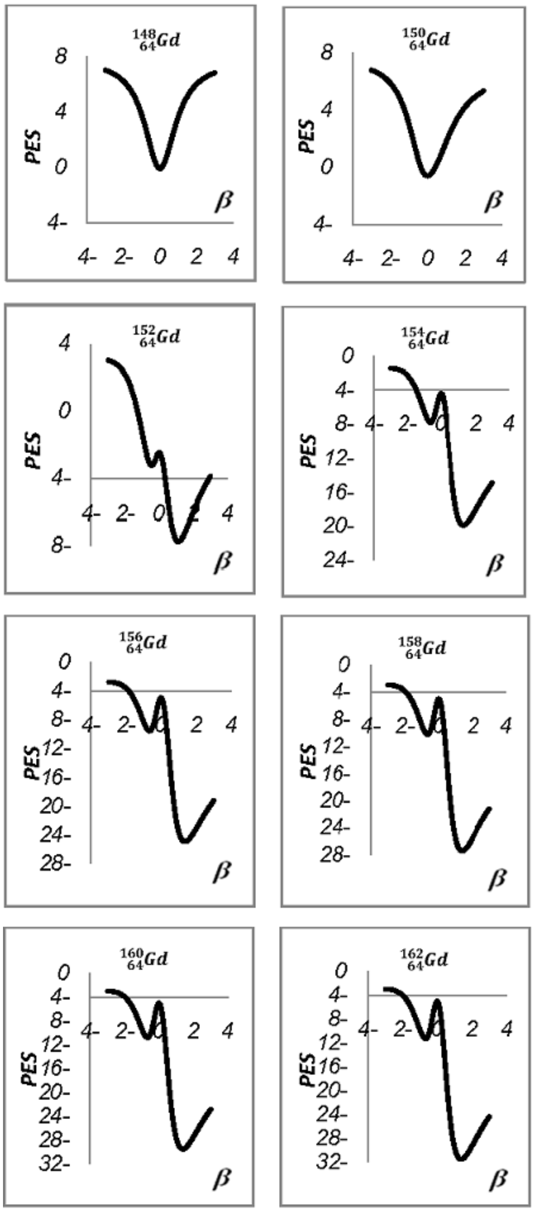

Fig. 6: The same as Fig. (4) but for Gadolinium isotope chain $^{148-162}_{64}$Gd (with $N_\pi = 7$ proton bosons and $N_\gamma = 1$–9 neutron bosons).

to describe the gradual change in the structure as boson number is varied and to reproduce the properties of the selected states of positive parity excitation ($2^+_1, 4^+_1, 6^+_1, 8^+_1, 0^+_2, 2^+_3, 4^+_3, 2^+_2, 3^+_1$ and 4^+_2) and the two neutron separation energies of all isotopes in each isotopic chain. Typically, η decreasing from 1 to 0 as boson number increases and the nuclei evolve from vibrational to rotational as expected. This trend is observed for the studied isotopic chains and illustrated in figures (4-7) by plotting the PES from Hamiltonian (12) as a function of quadruple deformation parameter β for different values of the

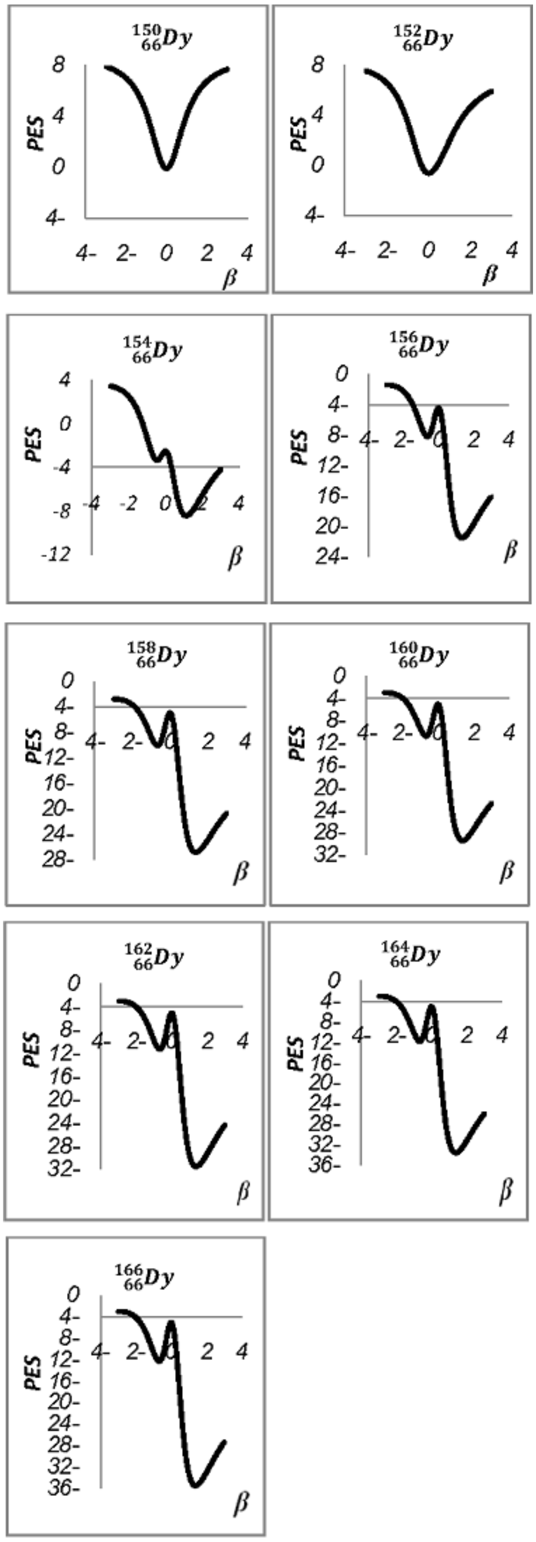

Fig. 7: The same as Fig. (4) but for Dysprosium isotope chain $^{150-166}_{66}$Dy (with $N_\pi = 8$ proton bosons and $N_\gamma = 1$–9 neutron bosons).

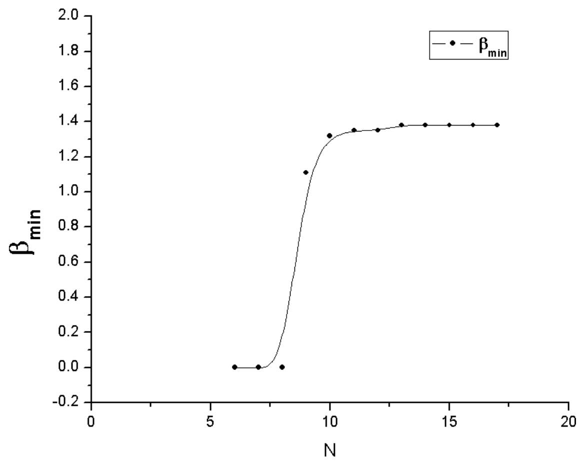

Fig. 8: Position of the absolute minima β_{min} versus the total number of bosons N from N = 6 to N = 17.

Table 1: Neutron Number.

Nucleus	η/N_{crit}			
$_{66}$Dy	0.08183	0.07339	0.04166	0.00993
$_{64}$Gd	0.08183	0.07339	0.04166	0.00993
$_{62}$Sm	0.0982	0.08807	0.5	0.01192
$_{60}$Nd	0.10911	0.09786	0.55555	0.01324
N	84	86	88	90

Nucleus	η/N_{crit}			
$_{66}$Dy	0.00149	0.0002	0.0000	0.0000
$_{64}$Gd	0.00149	0.0002	0.0000	0.0000
$_{62}$Sm	0.00179	0.00024	0.00003	0.0000
$_{60}$Nd	0.00199	0.00027		
N	92	94	96	98

the control parameter η and varying boson number N.

Here, we observe that the transition from spherical to prolate deformed occurs between N=9 and N=12. In the $^{144-154}$Nd, the nuclei $^{146-150}$Nd are transitional isotopes between the spherical nucleus ^{144}Nd and the well prolate deformed nuclei $^{152-154}$Nd. The ^{150}Gd nucleus still shows a vibrational structure while $^{156-162}$Gd are considered as rather good SU(3) example.

The $^{158-162}$Gd are corresponds to $\eta = 0$. One can observe a sudden transition in the Gd isotopes from a vibrational region into the rotational SU(3) limit. The control parameter η for each nucleus is shown in Table (1). The position of the absolute minimum $\beta_{min}(N)$ of the different PES's is illustrated in Figure (8).

Table (1) lists values of the control parameter η/N_{crit} for each Nd/Sm/Gd/Dy isotopic chain as a function of the neutron number.

5 Conclusion

In the present paper we have analyzed systematically the $PES's$ for the even-even Nd/Sm/Gd/Dy isotopes using the simplified form of IBM in its sd-boson interaction. We have analyzed the critical points of the shape phase transitional region U(5)-SU(3) in the space of two control parameters η and χ.

In all isotopic chains one observes a change from spherical U(5) shape to axially symmetric deformed shape SU(3) when moving from the lighter to the heavier isotopes.

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