Dilatation–Distortion Decomposition of the Ricci Tensor

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We apply a natural decomposition of tensor fields, in terms of dilatations and distortions, to the Ricci tensor. We show that this results in a separation of the field equations of General Relativity into a dilatation relation and a distortion relation. We evaluate these equations in the weak field approximation to show that the longitudinal dilatation mass relation leads to Poisson's equation for a newtonian gravitational potential, and that the transverse distortion wave relation leads to the linearized field equation of gravity in the Transverse Traceless gauge. The results obtained are in agreement with the Elastodynamics of the Spacetime Continuum.

1 Introduction

In a previous paper [1], we proposed a natural decomposition of spacetime continuum tensor fields, based on the continuum mechanical decomposition of tensors in terms of dilatations and distortions. In this paper, we apply this natural decomposition to the Ricci tensor $R^{\mu\nu}$ of General Relativity within the framework of the Elastodynamics of the Spacetime Continuum (*STCED*) [2].

2 Decomposition of the Ricci tensor

As shown in [1], the stress tensor $T^{\mu\nu}$ of General Relativity can be separated into a stress deviation tensor $t^{\mu\nu}$ and a scalar t_s according to

$$T^{\mu\nu} = t^{\mu\nu} + t_s\, g^{\mu\nu} \tag{1}$$

where

$$t^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - t_s\, \delta^{\mu}{}_{\nu} \tag{2}$$

$$t_s = \frac{1}{4}\, T^{\alpha}{}_{\alpha} = \frac{1}{4}\, T. \tag{3}$$

The Ricci curvature tensor $R^{\mu\nu}$ can also be separated into a curvature deviation tensor $r^{\mu\nu}$ (corresponding to a distortion) and a scalar r_s (corresponding to a dilatation) according to

$$R^{\mu\nu} = r^{\mu\nu} + r_s\, g^{\mu\nu} \tag{4}$$

where similarly

$$r^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - r_s\, \delta^{\mu}{}_{\nu} \tag{5}$$

$$r_s = \frac{1}{4}\, R^{\alpha}{}_{\alpha} = \frac{1}{4}\, R \tag{6}$$

where R is the contracted Ricci curvature tensor.

Using (1) to (6) into the field equations of General Relativity [3, see p. 72],

$$R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R = -\kappa T^{\mu\nu} \tag{7}$$

where $\kappa = 8\pi G/c^4$ and G is the gravitational constant, we obtain a separation of the field equations of General Relativity into dilatation and distortion relations respectively:

$$\begin{aligned} \text{dilatation} &: r_s = -\kappa t_s \\ \text{distortion} &: r^{\mu\nu} = \kappa t^{\mu\nu}. \end{aligned} \tag{8}$$

The dilatation relation of (8) can also be expressed as

$$R = -\kappa T. \tag{9}$$

The distortion-dilatation separation of tensor fields is thus also applicable to the field equations of General Relativity, resulting in separated dilatation and distortion relations. This result follows from the geometry of the spacetime continuum (*STC*) used in General Relativity being generated by the combination of all deformations present in the *STC* [2].

3 Weak field approximation

We evaluate these separated field equations (8) in the weak field approximation to show that these relations satisfy the massive longitudinal dilatation and massless transverse distortion results of *STCED* [2].

In the weak field approximation [4, see pp. 435–441], the metric tensor $g_{\mu\nu}$ is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the flat spacetime diagonal metric with signature $(-\ +\ +\ +)$ and $|h_{\mu\nu}| \ll 1$. The connection coefficients are then given by

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) \tag{10}$$

or, after raising the indices,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}(h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu}). \tag{11}$$

The Ricci tensor is also linearized to give

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} \tag{12}$$

which becomes

$$R_{\mu\nu} = \frac{1}{2}(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}). \tag{13}$$

The contracted Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} \simeq \eta^{\mu\nu} R_{\mu\nu} \tag{14}$$

then becomes

$$R = \frac{1}{2}\eta^{\mu\nu}(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}) \tag{15}$$

which, after raising the indices and re-arranging the dummy indices, simplifies to

$$R = h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\alpha,\beta}{}^{\beta}. \tag{16}$$

4 Dilatation (mass) relation

Making use of (16) and (6) into the dilatation relation (9), we obtain the *longitudinal dilatation mass relation*

$$h^{\alpha}{}_{\alpha,\beta}{}^{\beta} - h^{\alpha\beta}{}_{,\alpha\beta} = \kappa T \tag{17}$$

and, substituting for κ from (7) and $T = \rho c^2$ from (30) of [2],

$$\nabla^2 h^{\alpha}{}_{\alpha} - \partial_\alpha \partial_\beta h^{\alpha\beta} = \frac{8\pi G}{c^2}\rho \tag{18}$$

where ρ is the rest-mass density. This equation is shown to lead to Poisson's equation for a newtonian gravitational potential in the next section.

The second term of (18) would typically be set equal to zero using a gauge condition analogous to the Lorentz gauge [4, see p. 438]. However, the second term is a divergence term, and it should not be set equal to zero in the general case where sources may be present.

4.1 Static newtonian gravitational field

We consider the metric perturbation [4, see pp. 412–416]

$$\begin{aligned} h_{00} &= -2\Phi/c^2 \\ h_{ii} &= 0, \quad \text{for } i = 1, 2, 3 \end{aligned} \tag{19}$$

where Φ is a static (i.e. time independent) newtonian gravitational field. Then the term

$$h^{\alpha\beta}{}_{,\alpha\beta} = h^{00}{}_{,00} = 0 \tag{20}$$

and (17) becomes

$$\nabla^2 h^0{}_0 = \kappa T. \tag{21}$$

Using h_{00} from (19) and κ from (7), (21) becomes

$$\nabla^2 \Phi = \frac{4\pi G}{c^2} T. \tag{22}$$

Substituting for $T = \rho c^2$ from (30) of [2], we obtain

$$\nabla^2 \Phi = 4\pi G \rho \tag{23}$$

where ρ is the mass density. This equation is Poisson's equation for a newtonian gravitational potential.

5 Distortion (wave) relation

Combining (13) and (16) with (5) and (6) into the distortion relation of (8), we obtain the *transverse distortion wave relation*

$$\begin{aligned} &\frac{1}{2}\left(h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right) - \\ &-\frac{1}{4}\,\eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\alpha,\beta}{}^{\beta}) = \kappa t_{\mu\nu} \end{aligned} \tag{24}$$

where $t_{\mu\nu}$ is obtained from (2) and (3). This equation can be shown to be equivalent to the equation derived by Misner *et al* [4, see their Eq.(18.5)] from which they derive their linearized field equation and transverse wave equation in the Transverse Traceless gauge [4, see pp. 946–950]. This shows that this equation of the linearized theory of gravity corresponds to a transverse wave equation.

This result highlights the importance of carefully selecting the gauge transformation used to simplify calculations. For example, the use of the Transverse Traceless gauge eliminates massive solutions which, as shown above and in [2], are longitudinal in nature, while yielding only non-massive (transverse) solutions for which the trace equals zero.

6 Discussion and conclusion

In this paper, we have applied a natural decomposition of tensor fields, in terms of dilatations and distortions, to the Ricci tensor. We have shown that this results in a separation of the field equations of General Relativity into a dilatation relation and a distortion relation. We have evaluated these equations in the weak field approximation to show that the longitudinal dilatation mass relation leads to Poisson's equation for a newtonian gravitational potential, and that the transverse distortion wave relation leads to the linearized field equation of gravity in the Transverse Traceless gauge. The results obtained are thus found to be in accord with the Elastodynamics of the Spacetime Continuum.

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