CKM and PMNS Mixing Matrices from Discrete Subgroups of SU(2)

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One of the greatest challenges in particle physics is to determine the first principles origin of the quark and lepton mixing matrices CKM and PMNS that relate the flavor states to the mass states. This first principles derivation of both the PMNS and CKM matrices utilizes quaternion generators of the three discrete (i.e., finite) binary rotational subgroups of SU(2) called [3,3,2], [4,3,2], and [5,3,2] for three lepton families in R^3 and four related discrete binary rotational subgroups [3,3,3], [4,3,3], [3,4,3], and [5,3,3] represented by four quark families in R^4. The traditional 3×3 CKM matrix is extracted as a submatrix of the 4×4 CKM4 matrix. The predicted fourth family of quarks has not been discovered yet. If these two additional quarks exist, there is the possibility that the Standard Model lagrangian may apply all the way down to the Planck scale.

1 Introduction

The very successful Standard Model (SM) local gauge group $SU(2)_L \times U(1)_Y \times SU(3)_C$ defines an electroweak (EW) interaction part and a color interaction part. Experiments have determined that the left-handed EW isospin flavor states are linear superpositions of mass eigenstates. One of the greatest challenges in particle physics is to determine the first principles origin of the quark and lepton mixing matrices CKM and PMNS that relate the flavor states to the mass states.

In a recent article [1] I derived the lepton PMNS mixing matrix by using the quaternion (i.e., spinor) generators of three specific discrete (i.e., finite) binary rotational subgroups of the EW gauge group $SU(2)_L \times U(1)_Y$, one group for each lepton family, while remaining within the realm of the SM lagrangian. All the derived PMNS matrix element values are within the 1σ range of the empirically determined absolute values.

The three lepton family groups, binary rotational groups called [3,3,2], [4,3,2], and [5,3,2], (or 2T, 2O, and 2I), have discrete rotational symmetries in R^3. Each group has two degenerate basis states which must be taken in linear superposition to form the two orthogonal fermion flavor states in each family, i.e., (ν_e, e), (ν_μ, μ), and (ν_τ, τ).

In order to have a consistent geometrical approach toward understanding the SM, I have proposed in a series of articles [2–4] over several years that the quark flavor states represent discrete binary rotational groups also. However, one must move up one spatial dimension from R^3 to R^4 and use the related four discrete binary rotational subgroups [3,3,3], [4,3,3], [3,4,3], and [5,3,3], (or 5-cell, 16-cell, 24-cell, and 600-cell), for the quarks, thereby dictating four quark families. Recall that both R^3 and R^4 are subspaces of the unitary space C^2.

Therefore, following up the success I had deriving the neutrino PMNS matrix, the CKM mixing matrix should be derivable by using the same geometrical method, i.e., based upon the quaternion generators of the four groups of specific discrete rotational symmetries. In this quark case, however, first one determines a 4×4 mixing matrix called CKM4 and then extracts the appropriate 3×3 submatrix as the traditional CKM matrix.

These seven closely-related groups representing specific discrete rotational symmetries dictate the three known lepton families in R^3 and four related quark families in R^4, the fourth quark family still to be discovered. That is, neither leptons nor quarks are to be considered as point objects at the fundamental Planck scale of about 10^{-35} meters. If this geometrical derivation of both the PMNS and CKM mixing matrices is based upon the correct reason for the mixing of flavor states to make the mass states, then one must reconcile the empirical data with the prediction of a fourth quark family.

My proposal that leptons are 3-D entities and that quarks are 4-D entities has several advantages. There is a clear distinction between leptons and quarks determined by inherent geometrical properties such as explaining that leptons do not experience the color interaction via $SU(3)_C$ because gluons and quarks would involve 4-D rotations associated with the three color charges defined in R^4. Also, one now has a geometrical reason for there being more than one family of leptons and of quarks. In addition, the mass ratios of the fundamental fermions are determined by the group relationships to the j-invariant of the Monster Group. These physical properties and many other physical consequences are discussed in my previous papers.

2 Review of the PMNS matrix derivation

This section reviews the mathematical procedure used in my 2013 derivation [1] of the PMNS matrix from first principles. One constructs the three SU(2) generators, the U_1 = j, U_2 = k, and the U_3 = i, (i.e., the Pauli matrices in quaternion form), from the three quaternion generators from each of the discrete subgroups [3,3,2], [4,3,2], and [5,3,2]. As you know, the three Pauli matrices, i.e., the quaternions i, j, and k, can generate all rotations in R^3 about a chosen axis or, equivalently, all rotations in the plane perpendicular to this axis. For example,

Table 1: Lepton Family Quaternion Generators U_2

Fam.	Grp.	Generator	Factor	Angle°
ν_e,e	332	$-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{\sqrt{2}}k$	-0.2645	105.337
ν_μ,μ	432	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}k + \frac{1}{2}j$	0.8012	36.755
ν_τ,τ	532	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k$	-0.5367	122.459

the quaternion k is a binary rotation by 180° in the i-j plane.

The complete mathematical description [5] for the generators operating on the unit vector x in R^3 extending from the origin to the surface of the unit sphere S^2 is given by $R_s = i\,x\,U_s$ where s = 1, 2, 3 and

$$U_1 = j, \;\; U_2 = -icos\frac{\pi}{q} - jcos\frac{\pi}{p} + ksin\frac{\pi}{h}, \;\; U_3 = i, \qquad (1)$$

with h = 4, 6, 10 for the three lepton flavor groups [p,q,2], respectively. Their U_2 generators are listed in Table 1.

My three lepton family binary rotational groups, [3,3,2], [4,3,2], and [5,3,2], all have generators U_1 = j and U_3 = i, but each U_2 is a different quaternion generator operating in R^3. One obtains the correct neutrino PMNS mixing angles from the linear superposition of their U_2's by making the total U_2 = k, agreeing with SU(2). This particular combination of three discrete angle rotations is now equivalent to a rotation in the i-j plane by the quaternion k.

The sum of all three U_2 generators should be k, so there are three equations for the three unknown factors, which are determined to be: -5.537, 16.773, and -11.236. Let the quantity $\phi = (\sqrt{5}+1)/2$, the golden ratio. The resulting angles in Table 1 are the arccosines of these factors (normalized), i.e., their projections to the k-axis, but they are twice the rotation angles required in R^3, a property of quaternion rotations.

Using one-half of these angles produces

$$\theta_1 = 52.67°, \; \theta_2 = 18.38°, \; \theta_3 = 61.23°, \qquad (2)$$

resulting in mixing angles

$$\theta_{12} = 34.29°, \; \theta_{13} = -8.56°, \; \theta_{23} = -42.85°. \qquad (3)$$

The absolute values of these mixing angles are all within the 1σ range of their values for the normal mass hierarchy [6–11] as determined from several experiments:

$$\theta_{12} = \pm 34.47°, \; \theta_{13} = \pm 8.73°, \; \theta_{23} = \pm(38.39° - 45.81°). \quad (4)$$

The experimental 1σ uncertainty in θ_{12} is about 6%, in θ_{13} about 14%, and θ_{23} has the range given. The ± signs arise from the squares of the sines of the angles determined by the experiments.

For three lepton families, one has the neutrino flavor states ν_e, ν_μ, ν_τ, and the mass states ν_1, ν_2, ν_3, related by the PMNS matrix V_{ij}

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}.$$

The PMNS entries are the products of the sines and cosines of the derived angles (3) using the standard parametrization of the matrix, producing:

$$\begin{bmatrix} 0.817 & 0.557 & -0.149e^{-i\delta} \\ -0.413 - 0.084e^{i\delta} & 0.605 - 0.057e^{i\delta} & -0.673 \\ -0.383 + 0.090e^{i\delta} & 0.562 + 0.061e^{i\delta} & 0.725 \end{bmatrix}.$$

For direct comparison, the empirically estimated PMNS matrix for the *normal hierarchy* of neutrino masses is

$$\begin{bmatrix} 0.822 & 0.547 & -0.150 + 0.038i \\ -0.356 + 0.0198i & 0.704 + 0.0131i & 0.614 \\ 0.442 + 0.0248i & -0.452 + 0.0166i & 0.774 \end{bmatrix}$$

Comparing the V_{e3} elements from each, the phase angle δ is confined to be $0° \leq \delta \leq \pm 14.8°$, an angle in agreement with the T2K collaboration value of $\delta \approx 0$ but quite different from other proposed $\delta \approx \pi$ values.

3 The CKM4 matrix derivation

The success of the above geometrical procedure for deriving the lepton PMNS matrix by using the quaternion generators from the 3 discrete binary rotation groups demands that the same approach should work for the quark families in R^4 using the 4 discrete binary rotation groups [3,3,3], [4,3,3], [3,4,3], and [5,3,3]. If this procedure succeeds in deriving the CKM matrix elements as a 3×3 submatrix of CKM4, then a fourth sequential quark family, call its quark states b' and t', exists in Nature.

These 4 binary rotational groups for the quark family flavors each have rotation subgroups of SO(4) = SO(3) × SO(3), and they also have the double covering SU(2) × SU(2). The SO(4) is the rotation group of the unit hypersphere S^3 in R^4, with every 4-D rotation being simultaneous rotations in two orthogonal planes.

The only finite (i.e., discrete) quaternion groups are [12]

$$2I, \quad 2O, \quad 2T, \quad 2D_{2n}, \quad 2C_n, \quad 1C_n \; (n \; odd) \qquad (5)$$

with the 2 in front meaning binary (double) group, the double cover of the normal 3-D rotation group by SU(2) over SO(3). Mathematically, the 4 discrete binary groups for the quark families each can be identified as $(L/L_K; R/R_K)$ with the homomorphism $L/L_K = R/R_K$. Here L and R are specific discrete groups of quaternions and L_K and R_K are their kernels.

P. DuVal [13] established that one only needs the cyclic groups $2C_n$ and $1C_n$ when considering the four discrete rotational symmetry groups, i.e., the ones I am using for the quark families. Essentially, vertices on the 4-D regular polytope can be projected to be a regular polygon on each of the two orthogonal planes in R^4.

There will be 6 quaternion generators for each of the 4 groups, producing simultaneous rotations in two orthogonal planes. The two sets of Pauli matrices for producing continuous rotations can be identified as i, j, k, and another i, j, k, but they act on the two different S^2 spheres, i.e, in the two orthogonal planes. One can consider this 4-D rotational transformation as the result of a bi-quaternion operation [14], or equivalently, a bi-spinor or Ivanenko-Landau-Kähler spinor or Dirac-Kähler spinor operation.

For three quark families, one has the "down" flavor states d', s', b', and their mass states d, s, b, related by the CKM matrix. This quark mixing matrix for the left-handed components is defined in the standard way as

$$V = U_L D_L^\dagger, \tag{6}$$

but for four quark families the mathematics is a little different, for one must consider the bi-quaternion case in which there will be Bogoliubov mixing [14], producing two subfactors for each component, i.e.,

$$U_L = W^u_{14,23} W^u_{12,34}, \quad D_L = W^d_{14,23} W^d_{12,34} \tag{7}$$

with the W^u and W^d factor on the right mixing the 1st and 2nd generations and, separately, mixing the 3rd and 4th generations. The Bogoliubov mixing in the factor on the left mixes the 1st and 4th generations and, separately, the 2nd and 3rd generations. Therefore, the CKM4 matrix derives from

$$V_{CKM4} = U_L D_L^\dagger = W^u_{14,23} W^u_{12,34} (W^d_{14,23} W^d_{12,34})^\dagger. \tag{8}$$

The product $W^u_{12,34} W^{d\dagger}_{12,34}$ is given by

$$W^u_{12,34} W^{d\dagger}_{12,34} = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ z_1 & w_1 & 0 & 0 \\ 0 & 0 & x_2 & y_2 \\ 0 & 0 & z_2 & w_2 \end{bmatrix}.$$

The upper left block is an SU(2) matrix that mixes generations 1 and 2 while the lower right block is an SU(2) matrix that mixes generations 3 and 4. Each 2x2 block relates the rotation angles and the phases via

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} cos\theta\, e^{i\alpha} & -sin\theta\, e^{i\beta} \\ sin\theta\, e^{i\gamma} & cos\theta\, e^{i\delta} \end{bmatrix}.$$

The 4×4 matrix that achieves the Bogoliubov mixing has four possible forms for the four possible isospin cases obeying SU(2) × SU(2): (0, 0), (1/2, 0), (0, 1/2), and (1/2, 1/2). The (1/2, 1/2) is the one for equal, simultaneous, isospin 1/2 rotations in the two orthogonal planes for CKM4:

$$W^{u,d}_{14,23} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Table 2: Quark Family Discrete Group Assignments for U_2

Fam.	Grp.	Generator	Angle°	Factor	Angle°
u,d	333	exp[2πi/5]	72	1.132	81.504
c,s	433	exp[2πi/8]	45	1.132	50.940
t,b	343	exp[2πi/12]	30	1.132	33.960
t',b'	533	exp[2πi/30]	12	1.132	13.584

Multiplying out these three 4×4 bi-quaternion mixing matrices, one determines that

$$V_{CKM4} = \frac{1}{2} \begin{bmatrix} x_1 + x_2 & y_1 + y_2 & x_1 - x_2 & y_1 - y_2 \\ z_1 + z_2 & w_1 + w_2 & z_1 - z_2 & w_1 - w_2 \\ x_1 - x_2 & y_1 - y_2 & x_1 + x_2 & y_1 + y_2 \\ z_1 - z_2 & w_1 - w_2 & z_1 + z_2 & w_1 + w_2 \end{bmatrix}$$

in which the phases $\alpha, \beta, \gamma, \delta$ have been ignored.

One determines the angles θ_1 and θ_2 from the quaternion generators of the 4 discrete binary rotation groups for the quark families. Projections of each of the four discrete symmetry 4-D entities onto the two orthogonal planes produces a regular polygon [5, 13] with the generator iexp[2πj/h], as given in Table 2, where the h values are 5, 8, 12, 30, for the [3,3,3], [4,3,3], [3,4,3], and [5,3,3], respectively.

Again, we need to determine the contribution from each group generator that will make the sum add to 180°, i.e., make their collective action produce the rotation U_2 = k. Expanding out the exponentials in terms of sines and cosines reveals four unknowns but only two equations. Alternately, because the four rotation angles sum to only 159°, we can use the same factor for each group, i.e., the ratio 180°/159° = 1.132.

In the last column of Table 2 are the normalized angles which are twice the angle required. Therefore, taking the appropriate half-angle differences produces the mixing angles

$$\theta_1 = 15.282°, \quad \theta_2 = 10.188°. \tag{9}$$

Substituting the cosines and sines of these two derived angles into the CKM4 matrix form above produces a mixing matrix symmetrical about the diagonal. Remember that I have ignored up to eight possible phases in the 2x2 blocks.

$$V_{CKM4} = \begin{bmatrix} 0.9744 & 0.2203 & 0.0098 & 0.0433 \\ 0.2203 & 0.9744 & 0.0433 & 0.0098 \\ 0.0098 & 0.0433 & 0.9744 & 0.2203 \\ 0.0433 & 0.0098 & 0.2203 & 0.9744 \end{bmatrix}.$$

One can compare the upper left 3×3 submatrix to the most recent *estimated absolute values* [7]

$$V_{CKM} = \begin{bmatrix} 0.9745 & 0.2246 & 0.0036 \\ 0.2244 & 0.9736 & 0.0415 \\ 0.0088 & 0.0407 & 0.9991 \end{bmatrix}.$$

Note that most of these estimated V_{CKM} values are probably good to within a few percent but some could have uncertainties as large as 10% or more.

Of concern are my low values of 0.2203 for V_{us} and V_{cd}. However, according to the Particle Data Group (2013) there are two possible values [7]: 0.2253 and 0.2204, the latter from tau decays. Also, my derived symmetric CKM4 matrix V_{ub} value is high while the V_{td} value is reasonable, i.e., V_{td} at 0.0098 compares well with the estimated value of 0.0088.

The V_{tb} element of CKM4 is 0.9744, quite a bit smaller than the suggested 0.9991 V_{tb} value for the 3×3 CKM matrix. However, if one imposes the unitarity condition on the rows and columns of the extracted CKM, the new value for this V_{tb} matrix element would be 0.999, in agreement.

My final comment is that if one calculates CKM using only the first three quark groups [3,3,3], [4,3,3], and [3,4,3], the resulting 3×3 CKM matrix will disagree significantly with the known CKM matrix. Therefore, one cannot eliminate a fourth quark family when discrete rotational subgroups of SU(2) are considered.

4 Discussion

In the SM the EW symmetry group is the Lie group $SU(2)_L \times U(1)_Y$. This local gauge group operating on the lepton and quark states works extremely well, meaning that all its predictions agree with experiments so far. However, in this context there is no reason for Nature to have more than one fermion family, and certainly no reason for having 3 lepton families and at least 3 quark families. As far as I know, the normal interpretation of the SM provides no answer that dictates the actual number of families, although the upper limit of 3 lepton families with low mass neutrinos is well established via Z^0 decays and via analysis of the CMB background.

My geometrical approach with *discrete* symmetries alters the default reliance upon SU(2) and its continuous symmetry transformations, for I utilize discrete binary rotational subgroups of SU(2) for the fundamental fermion states, a different subgroup for each lepton family and for each quark family. In this scenario one can surmise that the enormous success of the SM occurs because $SU(2)_L \times U(1)_Y$ is acting like a mathematical "cover group" for the actual underlying discrete rotations operating on the lepton states and quark states.

Assuming that the above matrix derivations are correct, the important question is: Where is the b' quark of the predicted 4th quark family? In 1992 I predicted a top quark mass of about 160 GeV, a b' quark mass of 65–80 GeV, and a t' quark at a whopping 2600 GeV. These mass predictions were based upon the mass ratios being determined by the j-invariant function of elliptic modular functions and of fractional linear transformations, i.e., Möbius transformations. Note that all seven discrete groups I have for the fermions are related to the j-invariant and Möbius transformations, which have direct connections to numerous areas of fundamental mathematics.

With a predicted b' mass that is much smaller than the top quark mass of 173.3 GeV and even smaller than the W mass at 80.4 GeV, one would have expected some production of the b' at LEP, Fermilab, and the LHC. Yet, no clear indication of the b' quark has appeared.

Perhaps the b' quark has escaped detection at the LHC and lies hidden in the stored data from the runs at 7 TeV and 8 TeV. With a mass value below the W and Z masses, the b' quark must decay via flavor changing neutral current (FCNC) decay channels [16] such as b' $\rightarrow$ b + γ and b' $\rightarrow$ b + gluon. The b' could have an average lifetime too long for the colliders to have detected a reasonable number of its decays within the detector volumes and/or the energy and angle cuts. However, the b' quark and t' quark would affect certain other decays that depend upon the heaviest "top" quark in a box diagram or penguin diagram.

Another possibility is that a long lifetime might allow the formation of the quarkonium bound state b'-anti-b', which has its own specific decay modes, to bb-bar, gg, $\gamma\gamma$, and WW* $\rightarrow \nu\nu\ell\ell$. Depending upon the actual quarkonium bound state, the spin and parity $J^{PC} = 0^{++}$ or 0^{-+}.

And finally, there is an important theoretical problem associated with the mismatch of three lepton families to four quark families, e.g., the famous triangle anomalies do not cancel in the normal manner. Perhaps my fundamental leptons and quarks, being extended particles into 3 and 4 dimensions, respectively, can avoid this problem which occurs for point particles. Someone would need to work on this possibility.

5 The bigger picture!

We know that the SM is an excellent approximation for understanding the behavior of leptons, quarks, and the interaction bosons in the lower energy region when the spatial resolution is less than 10^{-24} meters. At smaller distance scales, perhaps one needs to consider a discrete space-time, for which the discrete binary rotation groups that I have suggested for the fundamental particles would be appropriate. Quite possibly, with this slight change in emphasis to discrete subgroups of the local gauge group, the SM lagrangian will hold true all the way down to the Planck scale.

If indeed the SM applies at the Planck scale, then one can show [2] that the Monster group dictates all of physics! The surprising consequence: The Universe *is* mathematics and is *unique*. Indeed, we humans are mathematics!

This connection to the Monster Group is present already in determining the lepton and quark mass ratios, which are proportional to the j-invariant of elliptic modular functions, the same j-invariant that is the partition function for the Monster Group in a quantum field theory [17].

The mathematics of these discrete groups does even more for us, for there is a direct connection [2] from the lepton groups [3,3,2], [4,3,2], [5,3,2], and the quark groups [3,3,3], [4,3,3], [3,4,3], [5,3,3], in R^3 and R^4, respectively, via special quaternions called icosians to the discrete space R^8. One then brings in another R^8 for relativistic space-time transformations. The two spaces combine into a 10-D discrete space-time obeying the discrete symmetry transformations of "Weyl" SO(9,1) = Weyl $E_8 \times$ Weyl E_8. This proposed *unique* connection to "Weyl" SO(9,1) was a surprise to me because one has two 8-D spaces combining to make a 10-D space-time! Its direct and unique relationship to the SM certainly is a welcome replacement to the 10^{500} ways for M-theory.

Finally, among the advantages to having a fourth family of quarks is a possible explanation of the baryon asymmetry of the Universe (BAU). From the CKM and the PMNS matrices, one learns that the predicted CP violation (CPV) is at least 10 orders of magnitude too small to explain the BAU. That is, the important quantity called the Jarlskog value is much too small. But a 4th quark family resolves this issue [18] because substituting the fourth quark family mass values into the Jarlskog expression increases the CPV value by more than 10^{13}! Voilà. One now has penguin diagrams distinguishing the particle and antiparticle decays with sufficient difference to have the particle Universe we experience.

6 Conclusion

The quark mixing matrix CKM4 has been derived using four quark families. Using quaternion generators from four specific related discrete binary rotational groups [3,3,3], [4,3,3], [3,4,3], and [5,3,3], I have derived the quark CKM4 and its CKM submatrix. However, neither quark of the 4th quark family has been detected at the colliders. Their appearance could mean that the Standard Model lagrangian might be a good approximation to the ultimate lagrangian all the way down to the Planck scale if space-time is discrete.

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