LETTERS TO PROGRESS IN PHYSICS

An Essay on Numerology of the Proton to Electron Mass Ratio

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There are few mathematical expressions for calculation proton to electron mass ratio presented. Some of them are new and some are not. They have been analysed in terms of their simplicity, numerical significance and precision. Expressions are listed in the structured manner with comments. The close attention should be paid to a comparison of the formula similarity via their precision. A brief review of the different attempts in similar search is given.

1 Introduction

The founding of the analytical expression for fundamental dimensionless constant was a dream of a physical science for many years. There are many papers in literature trying to derive or explain fine structure constant from pure numerical theories. Such hypothetical theories can be divided into two types. The first one proposes that the dimensionless constants of the Nature are not actually constant and suggests using some close numbers which deviate from the original ones. This type of the theories requires further experimental research because deviations of the dimensionless constants are still unknown with good precision. For example G. Gamov following Eddington's belief explained the fine structure constant suggesting that it is equal to exactly 137 but it differs from exact number because of some small quantum perturbations similar to those in the case of the Lamb-Rutherford effect [1]. The second type of the theories is less common, it suggests exact relation for the dimensionless constants which is close to current experimental value. Usually such hypotheses derive huge and unnatural formulas that lack of elegance and explain-ability. Moreover physical justification for such expressions doesn't have enough arguments or the physical model is absent. However some of such recent theories may look interesting and promising in the view of the the presented material [2–4].

The part of the physics which involves dimensionless constants is very prone to invasion of numerology. However such cooperation has not been shown to be efficient yet. Though it is worth to notice that numerology itself stays very close to algebra and number theory of mathematics. Numerology itself can be considered as ancient prototype of the modern algebra (as well as alchemy was a base for a modern chemistry) and as it was said by I. J. Good: "At one time numerology meant divination by numbers, but during the last few decades it has been used in a sense that has nothing to do with the occult and is more fully called physical numerology" [5]. At this perspective, physical numerology seems to be a way through back-door which researches also try to enter and finding a key by trying to pickup right numbers. Such attempts should not be ignored as they may provide not only new clues for the researchers, but also in case of null-result they might be an evidence for another consistent principle which can be explored further.

2 Background

The search for mathematical expression for this dimensionless number motivated many serious scientists. A sufficient theory on particle masses and their ratios is not yet ready. The mass ratio of proton to electron ($\mu = m_p/m_e$) — two known stable particles which belong to two different types (leptons and hadrons) — still remains the mystery among other dimensionless numbers.

In 1929 Reinhold Fürth hypothesized that μ can be derived from the quadratic equation involving the fine structure constant [6]. Later on in 1935, A. Eddington who accepted some of Fürth's ideas presented the equation for proton to electron mass ratio calculation ($10\mu^2 - 136\mu + 1 = 0$) which appeared in his book "New Pathways in Science" [17]. However both approaches can not be used nowadays as they give very high deviation from the currently known experimental value of μ, so they are not reviewed in present work. Later on in 1951, it was Lenz [7] (but not Richard P. Feynman!) who noted that μ can be approximated by $6\pi^5$. In 1990, I.J. Good, a British mathematician assembled eight conjectures of numerology for the ratio of the rest masses of the proton and the electron.

Nowadays proton to electron mass ratio is known with much greater precision: $\mu = m_p/m_e = 1836.15267245(75)$, with uncertainty of 4.1×10^{-10} (CODATA 2010, [4]). Recently the professional approach to mathematically decode m_p/m_e ratio was done by Simon Plouffe [8]. He used a large database of mathematical constants and specialized program to directly find an expression. Alone with his main remarkable result for the expression for μ via Fibonacci and Lucas numbers and golden ratio he also noted that expression for μ using π can be improved as $6\pi^5 + 328/\pi^8$, but he concluded that this expression: "hardly can be explained in terms of primes and composites".

Expression	Value	Ref.
$\mu = \left(\frac{7}{2}\right)^6$	**183**8.2656 (1×10^{-3})	1.
$\mu = \sin\left(\frac{\pi}{5}\right)\cdot 5^5$	**1836**.8289 (4×10^{-4})	2.
$\mu = \frac{17}{4}432$	**1836**.0000 (8×10^{-5})	3.
$\mu = 150^{\frac{3}{2}} - 1$	**1836.1**173 (2×10^{-5})	4.
$\mu = 6\pi^5$	**1836.1**181 (2×10^{-5})	5.
$\mu = \frac{200^{300}}{7^{103}}$	**1836.1**179 (2×10^{-5})	6.
$\mu = \frac{22}{(5\cdot3\cdot\alpha)^2}$	**1836.15**56 (2×10^{-6})	7.
$\mu = \frac{5\cdot7^3}{6\cdot67}137\pi$	**1836.15**14 (6×10^{-7})	8.
$\mu = \frac{2^4 3^5}{5\alpha^{-1}}103\pi$	**1836.152**20 (3×10^{-7})	9.
$\mu = \frac{e^8-10}{\phi}$	**1836.15**301 (2×10^{-7})	10.
$\mu = \frac{40}{3}\alpha^{-1}+\frac{800}{9\pi^2}$	**1836.152**98 (2×10^{-7})	11.
$\mu = \frac{86^4}{31^3}$	**1836.152**39 (2×10^{-7})	12.
$\mu = \frac{2267^2}{5\cdot7\cdot11\cdot\alpha^{-1}}6\pi$	**1836.152**5639 $(6*10^{-8})$	13.
$\mu = \frac{11^2 5^{\frac{4}{5}} 7^{\frac{2}{5}} e^3}{6\cdot 2^{\frac{4}{5}}}$	**1836.15267**03 (1×10^{-9})	14.
$\mu = \frac{55\cdot5^{\frac{3}{2}}11^{\frac{15}{32}}}{\phi^{\frac{1}{16}}}$	**1836.15267**48 (1×10^{-9})	15.
$\mu = \frac{3^{\frac{15}{4}}5^{\frac{9}{4}}14^{\frac{3}{2}}}{\pi^3 e^{\frac{3}{4}}}$	**1836.15267**19 (1×10^{-10})	16.

3 Variability

During the last decade a subject of variability of μ appeared under heavy debate and serious experimental verifications. The main experimental task is to distinguish cosmological red-shift of spectral lines from the shift caused by possible variation of μ. There is also proposed method to observe absorption spectra in the laboratory using the high precision atomic clocks.

Reinhold et al. [9] using the analysis of the molecular hydrogen absorption spectra of quasars Q0405-443 and Q0347-373 concluded that μ could have decreased in the past 12 Gyr and $\Delta\mu/\mu = (2.4 \pm 0.6)\times 10^{-5}$. This corresponds to entry value of μ= 1836.19674. King et al. [9] re-analysed the spectral data of Reinhold et al. and collected new data on another quasar, Q0528-250. They estimated that $\Delta\mu/\mu = (2.6 \pm 3.0)\times10^{-6}$, different from the estimates of Reinhold et al. (2006). So the corresponding value for maximal deviated μ to be something around 1836.1574. The later results from Murphy et al. [15] and Bagdonaite et al. [2] gave a stringent limit $\Delta\mu/\mu < 1.8\times10^{-6}$ and $\Delta\mu/\mu = (0.0 \pm 1.0)\times10^{-7}$ respectively. However these deviations could be valid only for the half of the Universe's current age or to the past of 7 Gyr which may not be enough for full understanding of the evolution of such variation. The results obtained by Planck gave $\Delta\alpha/\alpha = (3.6\pm3.7)\times10^{-3}$ and $\Delta m_e/m_e = (4\pm11)\times10^{-3}$ at the 68% confidence level [13] which provided not so strong limit comparing to found in [9] and [10].

At first sight the variation, if confirmed, may seem to make the numerical search for the mathematical expression meaningless. However possible variability of the μ should not prevent such search further, because the variation means one has to find a mean value of its oscillation or the beginning value from where it has started to change. And such variation would give a wider space for the further numerical sophistication because such value can not be verified immediately as we currently lack experimental verification of the amount of such change. If the fundamental constants are floating and the Nature is fine-tuned by slight the ratio changes from time to time, even so, there should be middle value as the best balance for such fluctuations. In this sense numerologists are free to use more relaxed conditions for their search, and current the precision for μ with uncertainty of 2×10^{-6} (as discussed above) may suffice for their numerical experiments. The formulas listed after number 7 in the table below do fall into this range.

4 Comments to the table

1. This expression is not very precise and given for its simple form. Also the number (7/2) definitely has certain numerological significance. The result actually better fits to the value of the m_n/m_e ratio (relative uncertainty is 2×10^{-4}). It is not trivial task to improve the formula accuracy, but why not, for example: $\mu = \left(\frac{7}{2}\right)^8 \frac{9\cdot13}{10\pi\cdot\alpha^{-1}}$ (relative error: 10^{-6}).
2. It is well known [8] that m_p/m_n ratio can be well approximated as $\cos\left(\frac{\pi}{60}\right)$ with relative uncertainty of 6×10^{-6}. So this is an attempt to build the formula for m_p/m_e ratio of similar form. Next more precise formula of the same form would be: $\mu = \frac{17^{43}}{19^{37}}\sin\left(\frac{\pi}{674}\right) =$

1836.1526661 (relative error is 3×10^{-9}). In the table it would be placed between number 13 and 14.

3. It was Werner Heisenberg in 1935 [14] who suggested to use number $2^4 3^3$ (which is equal to 432) to calculate alpha as $\alpha^{-1} = 432/\pi$, so m_p/m_e ratio can be also obtained approximately via 432. The expression can be rewritten as $1836 = 17 \cdot 108$ (the number 108 was considered to be sacred by ancients). There are other possible representations for the number 1836 which were noticed in the past, for example: $1836 = (136 \cdot 135)/10$ (see review in [5] and [22]).
4. This expression has some certain theoretical base related to original R. Fürth ideas [6], but it won't be discussed here. The precision has the same order as famous $6\pi^5$.
5. This is a Lenz's formula and it remains the favorite among the physicists. Recently Simon Plouffe also suggested yet another adjustment to this formula as following: $\mu = \dfrac{1}{5\cosh(\pi)} + 6\pi^5 + \dfrac{1}{5\sinh(\pi)}$ which looks remarkably symmetric and natural. The relative error is also extremely good: 4×10^{-9}. This formula has not been published before, it definitely has to attract further attention of the researchers.
6. The simplest way to approximate m_p/m_e ratio using powers of 2 and 7. Similar formula: $\mu = \dfrac{3^5 7^{16}}{2^{42}}$.
7. The elegant expression which uses almost 'kabalistic' numbers 22, 5, 3 and fine structure constant. Other possible expression with similar look and with the same precision: $\mu = \dfrac{5^{76}}{2^{127} 3^{25}}$. Being combined together one can derive approximation for fine structure constant as 137.035999761 (with good relative deviation of 5×10^{-9}): $\alpha^{-2} = \dfrac{5^{78}}{11 \cdot 2^{127} 3^{23}}$.
8. Parker-Rhodes in 1981, see [21] and review in [5]. McGoveran D.O. [20] claimed that this formula does not have anything in common with numerology as it was derived entirely from their discrete theory.
9. This elegant expression uses only the fine structure constant α, powers of 2, 3, 5 and the number 103. As J.I. Good said: "the favoured integers seem all to be of the form $2^a 3^b$ " [5].
10. By unknown source. No comment.
11. The expression can be also rewritten in more symmetric form: $\mu = 2\left(\dfrac{20}{3}\alpha^{-1} + \left(\dfrac{20}{3\pi}\right)^2\right)$. It can be noted that the number (20/3) appears in the author previous work [18] in the expression for the gravitational constant G.
12. One of the found expressions by author's specialized program. The search was performed for the expression of the view: $\mu = p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$, where p_i — some prime numbers, n_i — some natural numbers. Also: $\mu = \left(\dfrac{19}{5}\right)^{21} \dfrac{1}{13^8}$.
13. Number 2267 has many interesting properties; it is a prime of the form $(30n - 13)$ and $(13n + 5)$, it is congruent to 7 mod 20. It is father primes of order 4 and 10 etc. In the divisor of this formula there are sequential primes 5, 7, 11. There are other possible expressions of the similar form with such precision (10^{-8}), for example: $\mu = \dfrac{45 * 49 * 53^2}{8 * 29 * \alpha^{-1}} 5\pi$. It is also hard to justify why in expressions 9 and 13 α^{-1} stays opposite to π as by definition they supposed to be on the same side: $\alpha^{-1} = \hbar c/ke^2$ or $(2\pi\alpha^{-1}) = hc/ke^2$. But the author did not succeed in finding similar expressions with α and π on the same side with the same uncertainty. There are some few other nice looking formulas which the use of big prime numbers, for example: $\mu = \sqrt{4^3 \cdot 52679}$ (9×10^{-8}).
14. Another possible expression was found using web based program Wolframalpha [23]. The precision is the same as in next formula.
15. Simon Plouffe's approximation using Fibonacci and Lucas numbers [8] - slightly adjusted from its original look. Another elegant form for this expression is following: $\mu^{32} = \dfrac{11^{47} 5^{80}}{\phi^2}$.
16. This formula has the best precision alone the listed. Though, powers of π and e seem to despoil its possible physical meaning.

5 Conclusions

At the present moment big attention is paid to experimental verification of possible proton-electron mass ratio variation. If experimental data will provide evidence for the ratio constancy then only few expressions (14-16 from the listed) may pretend to express proton-electron mass ratio as they fall closely into current experimental uncertainty range (4.1×10^{-10} as per CODATA 2010). Of course Simon Plouffe's formula (14) seems as a pure winner among them in terms of the balance between it simplicity and precision. However, some future hope for the other formulas remains if the variability of the proton to electron mass ratio is confirmed. Important to note that there could be unlimited numbers of numerical approximations for dimensionless constant. Some of them may look more simple and "natural" than others. It is easy to see that expression simplicity and explain-ability in opposite determines its precision. As all formulas with uncertainty 10^{-8} and better become obviously more complex. And at the end: "What is the chance that seemingly impressive formulae arise purely by chance?" [15].

Remembering mentioning words said by Seth Lloyd [19] "not to follow in Dirac's footsteps and take such numerology too seriously" the author encourages the reader to continue such mathematical experiments and in order to extend the table of the formulas and submit your expressions to the author. Special attention will be brought to simple expressions with relations to: power of two (2^n), prime numbers and properties of Archimedean solids. Besides that it may be interesting mathematical exercise it may also reveal some hidden properties of the numbers. But how complexity of the mathematical expression can be connected to the complexity of the numbers? What is the origin of the Universe complexity? How much we can encode by one mathematical expression?

The mass ratio of proton to electron — two stable particles that define approximately 95% of the visible Universe's mass — can be related to the total value Computational capacity of the Universe (see [19]). So as a pure numbers they supposedly have to be connected to prime numbers, entropy, binary and complexity. So, possibly, their property should be investigated further by looking through the prism of the algorithmic information theory.

Let's hope that presented material can be a ground for someone in his future investigation of this area.

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