# Gravitational Waves from a Sinusoidially Varying Spherical Distribution of Mass

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A theory is developed for the study of spherical gravitational waves by constructing a Generalized Gravitational Field Equation from Newton's gravitational field equation. The Euclidean Laplacian $\nabla^2$ is replaced with the Riemannian Laplacian $\nabla_R^2$. A general gravitational field equation is obtained which resolves the incompleteness in Newton's gravitational field equation. The general gravitational field equation reduces to the pure Newtonian gravitational field equation in the limit of $c^0$ as required by the Principle of Equivalence of Physics. It also contains post Newton correction terms of orders of $c^{-2}$ and all degrees of nonlinearity in the gravitational scalar potential and its derivatives. Considering a sinusoidally varying homogeneous spherical distribution of mass in the frame work of the obtained general gravitational field equation, gravitational waves are predicted with phase velocity equivalent to the speed of light in vacuo.

## 1 Introduction

According to General Relativity Theory, gravitational waves are oscillations of spacetime or small distortions of spacetime geometry, or ripples of spacetime curvature which propagate in the time through space as waves. Gravitational waves are produced mainly by extremely massive binary stellar objects, such as binary neutron stars or binary black holes. Though gravitational waves can be produced by all mass interactions, the amplitude of these waves is far too small to be detected. Normal solar systems produce gravitational waves when their planets orbit their primary, but again, these are incredibly tiny ripples. Even a binary black hole — which produces the most powerful gravitational waves we can imagine — requires measurements of distances of about 1/1000 of the diameter of a proton [1].

The search for gravitational waves has been the centre of current research in Astronomy and Cosmology. Higher precision and more sensitive detectors have been developed over the years. Experiments on gravitational waves started with Weber's experiments on gravitational antennae; in which he registered weak signals [2]. He concluded that some processes at the centre of the Galaxy were the origin of the detected signals. Other attempts were made in detecting gravitational waves such as [3-6]. The most recent experimental attempt by Abbott *et al.* in 2015 [7] claims that two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal.

Much theoretical work has also been done to either proof or disproof the existence of gravitational waves. In a nutshell, theoretical studies of gravitational waves can be classified into three main groups [2]:

- Research targeted at giving an invariant definition for gravitational waves. These include Pirani [8], Bondi [9], and others.
- Searching for solutions to Einstein gravitational field equations by proceeding from physical considerations to describe gravitational radiations. These include studies by Einstein and Rosen [10], Petrov [11], Chifu and Taura [1] and others.
- Studying gravitational inertial waves, covariant with respect of transformations of spatial coordinates and also invariant with respect of transformations of time [12].

This research article falls in the second group. The so-called "Great Metric Tensor" [13-14] is used to deduce a general gravitational wave equation; which is later applied to a sinusoidally varying mass for a homogeneous spherical distribution of mass.

## 2 The general spherical gravitational field equation

Newton's gravitational field equation is given by

$$\nabla^2 f(\underline{r}, t) = 4\pi G\rho_0(\underline{r}, t) \tag{1}$$

where, $\rho_0$ is the density of proper mass in a distribution or system, $\nabla^2$ is the pure Euclidean Laplacian, $G$ is the universal gravitational constant and $f$ is the gravitational scalar potential.

The incompleteness of equation (1) are as follows:

1. The density of proper mass (source of gravitational field) in equation (1) can vary with coordinate time and the Euclidean Laplacian cannot account for this possible variation.

2. Time variation of proper mass should result in the radiation of energy possibly in the form of gravitational waves or radiation that can propagate in space-time with or without gravitational field.
3. Newton's gravitational intensity vector $\underline{g}$ is given by

$$\underline{g} = -\underline{\nabla} f \tag{2}$$

where $\underline{\nabla}$ is the Euclidean gradient operator.

The Euclidean operator in equation (2) above has no variation with time and hence will not be sufficient for the complete description of gravitational intensity vector of time dependent gravitational fields.

From the foregoing it becomes necessary to seek a general gravitational field equation which will be sufficient for the description of all gravitational fields. Howusu in 2009 [13] proposed that a general gravitational field equation based on Riemannian coordinate geometry may be obtained by replacing the Euclidean Laplacian with Riemannian Laplacian to obtain

$$\nabla_R^2 f(\underline{r}, t) = 4\pi G \rho_0(\underline{r}, t) \tag{3}$$

where $\nabla_R^2$ is the Riemannian Laplacian based on the great metric tensor for all possible gravitational fields. The gravitational intensity (acceleration due to gravity) for all possible gravitational fields can also be defined in terms of the Riemannian gradient operator $\nabla_R$. The most general form of the Riemannian Laplacian is given as

$$\nabla_R^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) \tag{4}$$

where $g^{\mu\nu}$ is the contravariant metric tensor. Thus, for any function $f(\underline{r}, t)$ we can write

$$\begin{aligned} \nabla_R^2 f(\underline{r}, t) = & \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) f(\underline{r}, t) \\ & + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} \left( \sqrt{g} g^{00} \frac{\partial}{\partial x^0} \right) f(\underline{r}, t). \end{aligned} \tag{5}$$

Using Einstein's coordinates with $x^0 = ct$, equation (5) can be written explicitly as

$$\begin{aligned} \nabla_R^2 f(\underline{r}, t) = & \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) f(\underline{r}, t) \\ & + \frac{1}{c^2 \sqrt{g}} \frac{\partial}{\partial t} \left( \sqrt{g} g^{00} \frac{\partial}{\partial t} \right) f(\underline{r}, t). \end{aligned} \tag{6}$$

Hence equation (3) can be written more explicitly as

$$\begin{aligned} 4\pi G \rho_0(\underline{r}, t) = & \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) f(\underline{r}, t) \\ & + \frac{1}{c^2 \sqrt{g}} \frac{\partial}{\partial t} \left( \sqrt{g} g^{00} \frac{\partial}{\partial t} \right) f(\underline{r}, t). \end{aligned} \tag{7}$$

Equation (7) is the general field equation which resolves the incompleteness of Newton's gravitational field equation. Remarkably, the general gravitational field equation reduces to the pure Newton's gravitational field equation in the limit of $c^0$ ( as required by the Principle of Equivalence of Physics). It may also be noted that the gravitational field equation contains post Newton correction terms of orders of $c^{-2}$ and all degrees of nonlinearity in the gravitational scalar potential and its derivatives.

The Great Metric Tensor for all spherical gravitational fields in spherical polar coordinates $(r, \theta, \phi, x^0)$ is given as [13-14]:

$$g_{11}(r, \theta, \phi, x^0) = \left( 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right)^{-1}, \tag{8}$$

$$g_{22}(r, \theta, \phi, x^0) = r^2, \tag{9}$$

$$g_{33}(r, \theta, \phi, x^0) = r^2 \sin^2 \theta, \tag{10}$$

$$g_{00}(r, \theta, \phi, x^0) = -\left( 1 + \frac{2}{c^2} f(r, \theta, \phi, x^0) \right) \tag{11}$$

where $f$ is the gravitational scalar potential. From equation (8) to (11) it can be deduced that

$$\sqrt{g} = r^2 \sin \theta. \tag{12}$$

Equation (7) can thus be written as:

$$\begin{aligned} 4\pi G \rho_0(r, t) = & \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \left( 1 + \frac{2}{c^2} f \right) r^2 \right] f \\ & + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) f \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f \\ & - \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \left( 1 + \frac{2}{c^2} f \right)^{-1} \frac{\partial}{\partial t} f \right]. \end{aligned} \tag{13}$$

Equation (13) is the general spherical gravitational field equation interms of the great metric tensor. The following important facts can be drawn from equation (13):

1. It contains the $\left(1 + \frac{2}{c^2} f\right)$ term which is not found in Newton's gravitational field equation. The consequence of this is that it predicts correction terms to the gravitational field of all massive spherical bodies.
2. The time component of this equation predicts the existence of gravitational waves with velocity which is equal to the speed of light in vacuo.

## 3 Special case: sinusoidally varying homogenous spherical distribution of mass

Now, consider a sinusoidally varying homogenous spherical distribution of mass. In this case, the mass varies in such a way that $f$ is independent of the polar angle $\theta$ and the azimuthal angle $\phi$, [15] such that equation (13) reduces to

$$4\pi G\rho_0(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(1+\frac{2}{c^2}f\right)r^2\frac{\partial}{\partial r}\right]f - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\left(1+\frac{2}{c^2}f\right)^{-1}\frac{\partial}{\partial t}f\right]. \tag{14}$$

Linearizing equation (14) we obtain:

$$f'' + \frac{2}{r}f' - \frac{1}{c^2}\ddot{f} = 4\pi G\rho_0. \tag{15}$$

Suppose we have a dipole antenna which consists of two spherical bodies where electrons are driven by an oscillator [1]; then the movement of the electric charges driven by the oscillator is equivalent to an exponential factor. We therefore modify equation (15) in such a way that the proper mass density varies sinusoidally within a homogeneous spherical mass distribution such that:

$$f'' + \frac{2}{r}f' - \frac{1}{c^2}\ddot{f} = 4\pi G\rho_e\, e^{i\omega t}. \tag{16}$$

In order to solve equation (16) we seek a solution such that

$$f(r,t) = R(r)\, e^{i\omega t} \tag{17}$$

where $R$ is the radius of the spherical mass distribution. Equation (15) will thus become

$$R''(r) + \frac{2}{r}R'(r) + \frac{1}{c^2}\omega^2 R(r) = 4\pi G\rho_e. \tag{18}$$

Let

$$R(r) = \frac{1}{r}F(r),$$

then

$$R' = -\frac{1}{r^2}F(r),$$

and

$$R''(r) = \frac{1}{r}F''(r) - \frac{2}{r^2}F'(r) + \frac{2}{r^3}F(r).$$

It therefore follows that equation (18) becomes

$$\frac{1}{r}F''(r) + \frac{\omega^2}{c^2 r}F(r) = 4\pi G\rho_e. \tag{19}$$

Hence, the interior field equation for this distribution of mass is given as

$$\frac{1}{r}F''(r) + \frac{\omega^2}{c^2 r}F(r) = 4\pi G\rho_e\,;\ r < R \tag{20}$$

and the corresponding exterior field equation as:

$$\frac{1}{r}F''(r) + \frac{\omega^2}{c^2 r}F(r) = 0\,;\ r > R. \tag{21}$$

Equation (21) is a simple harmonic function which can have three solutions viz:

$$F(r) = Be^{ikr}, \tag{22}$$

$$F(r) = D\cos(kr), \tag{23}$$

and

$$F(r) = E\sin(kr). \tag{24}$$

Taking the first and second derivatives of equation (22) we have

$$F'(r) = ikB\, e^{ikr}$$

and

$$F''(r) = -k^2 B\, e^{ikr},$$

which can be substituted into (21) to yield

$$-k^2 B\, e^{ikr} + \frac{\omega^2}{c^2}B\, e^{ikr} = 0, \tag{25}$$

hence

$$k = \pm\frac{\omega}{c}. \tag{26}$$

We thus state the complimentary solution as

$$F_c^-(r) = E\sin\left(\frac{\omega}{c}r\right);\ r > R \tag{27}$$

$$F_c^+(r) = D\cos\left(\frac{\omega}{c}r\right);\ r < R. \tag{28}$$

The particular solution for the interior field equation is given by

$$\frac{1}{r}F''(r) + \frac{\omega^2}{c^2 r}F(r) = 4\pi G\rho_e; r < R. \tag{29}$$

Let $F(r) = Ar$, then $F'(r) = A$ and $F''(r) = 0$ and equation(29) yields

$$A = \frac{4\pi G c^2\rho_e}{\omega^2}, \tag{30}$$

and hence

$$F_p^-(r) = \frac{4\pi G c^2\rho_e}{\omega^2}r. \tag{31}$$

Equation (31) is thus the particular solution for the exterior field equation. The general solution for the exterior field is then given as

$$R^+(r) = \frac{D}{r}\cos\left(\frac{\omega}{c}r\right)\frac{4\pi G c^2\rho_e}{\omega^2}. \tag{32}$$

Equation (17) can thus be fully expressed as

$$f^+(r,t) = \frac{D}{r}\cos\left(\frac{\omega}{c}r\right)\cos(\omega t) + \frac{iD}{r}\cos\left(\frac{\omega}{c}r\right)\sin(\omega t) \tag{33}$$

with independent solutions

$$f^+(r,t) = \frac{D}{r}\cos\left(\frac{\omega}{c}r\right)\cos(\omega t) \tag{34}$$

and

$$f^{+}(r,t) = \frac{D}{r}\cos\left(\frac{\omega}{c}r\right)\sin(\omega t). \tag{35}$$

The two solutions (34) and (35) can be combined to yield

$$f^{+}(r,t) = \frac{1}{2}\frac{D}{r}\left[\cos\left(\omega\left(\frac{r}{c}+t\right)\right)+\cos\left(\omega\left(\frac{r}{c}-t\right)\right)\right]. \tag{36}$$

From equation (36) it is clear that the phase of the wave equation $\phi$ is given by

$$\phi = \frac{\omega r}{c} \pm \omega t, \tag{37}$$

hence

$$\frac{dr}{dt} = c. \tag{38}$$

## 4 Concluding remarks

In this paper we have shown [equation (36)] that in the limit of linear terms, the general gravitational field equation predicts gravitational waves with phase velocity which is equal to the speed of light in empty space. These waves will not vary with any angle, hence they will move along radial lines from inside the sphere outwards(radial waves). A sinusoidally varying mass thus radiates spherical gravitational waves. The obtained results gives similarlar predictions as in [1, 16] in the limit $c^{-2}$ though in the limit $c^{0}$ [16] predicts gravitational waves with imaginary phase.

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