Fundamental Geometrodynamic Justification of Gravitomagnetism (I)

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At a most fundamental level, gravitomagnetism is generally assumed to emerge from the General Theory of Relativity (GTR) as a first order approximation and not as an exact physical phenomenon. This is despite the fact that one can justify its existence from the Law of Conservation of Mass-Energy-Momentum in much the same manner one can justify Maxwell's Theory of Electrodynamics. The major reason for this is that in the widely accepted GTR, Einstein cast gravitation as a geometric phenomenon to be understood from the vantage point of the dynamics of the metric of spacetime. In the literature, nowhere has it been demonstrated that one can harness the Maxwell Equations applicable to the case of gravitation – i.e. equations that describe the gravitational phenomenon as having a magnetic-like component just as happens in Maxwellian Electrodynamics. Herein, we show that – under certain acceptable conditions where Weyl's conformal scalar [1] is assumed to be a new kind of pseudo-scalar and the metric of spacetime is decomposed as $g_{\mu\nu} = \mathcal{A}_\mu \mathcal{A}_\nu$ so that it is a direct product of the components of a four-vector $\mathcal{A}_\mu$ – gravitomagnetism can be given an exact description from within Weyl's beautiful but supposedly failed geometry.

My work always tried to unite the Truth with the Beautiful, but when I had to choose one or the other, I usually chose the Beautiful.

Herman Klaus Hugo Weyl (1885-1955)

1 Introduction

Exactly 102 years ago, the great, brilliant and esoteric German mathematician *cum* mathematical physicist and philosopher – Herman Klaus Hugo Weyl (1885-1955) – astounded the world of Physics with the first ever unified field theory of gravitation and electromagnetism. At the time, gravitation and electromagnetism were the only known forces of Nature, hence, from the viewpoint of the collective wisdom of the day, Weyl's [1] theory was seen as a unified field theory of all the forces of Nature. Since Weyl's [1] maiden efforts, unification of the gravitational phenomenon with the other forces of Nature has remained as one of the greatest – if not the greatest – and most outstanding problem in all of physics today. This endeavour of unification of all the forces of Nature first conducted by Weyl [1], became Albert Einstein's (1879-1955) final quest in the last 30 years of his brilliant and eventful life.

Since it is a widely accepted position, it perhaps is only fair for us to say at this very point, that – overall – while he failed in his titanic 30-year long quest and battle with the problem of an all-encompassing unified field theory of all the forces of Nature, Einstein made serious meaningful contributions to this seemingly elusive grand dream of a *Final Theory* that ties together all the known forces of Nature – the Gravitational force, the Electromagnetic force, the Weak and the Strong force – into one, giant, neat, beautiful, coherent and consistent mathematical framework that has a direct correspondence with physical and natural reality as we know it.

Despite his legendary lifelong failure to attain a unified field theory, Einstein [2, 3] understood very well the need for tensorial affine connections in the construction of a unified field theory. Einstein [2, 3] was not alone in this esoteric pot of wisdom; amongst others, towering figures of history such as Eddington [4] and Schrödinger [5–7] all but made similar noteworthy attempts to attain a unified field theory that made use of tensorial affines.

In the present work, this idea of tensorial affine connections is a fundamental lynchpin in the construction of what we believe is a noteworthy stepping stone to a *Final Unified Field Theory* (FUFT) of the gravitational phenomenon and the other forces of Nature. When we here say Final Unified Field Theory, we mean this in the context of the path (see [8–10]) that we are pursuing in order to arrive at what we believe is the FUFT.

In order for us to give the reader the correct scope of the present work, we must hasten and say that the present work is part and parcel of our upcoming monograph on this grand dream of Einstein. What we present herein is but a portion thereof. We herein demonstrate that gravitomagnetism has a fundamental geometric justification well within the scheme of Weyl's [1] supposed failure. We strongly believe – or are of the innate view – that the much sought for path to a successful *Quantum Geometrodynamic* (QGD) theory will be achieved very soon *via* a recasting of the gravitational phenomenon into a Maxwell-type formalism where the quantization of the gravitational field will prove to be the trivial exercise of quantizing a four-vector field $\mathcal{A}_\mu$ associated with the gravitational field. Through the well known quantization

procedures discovered in the quantization of the electromagnetic four-vector field in *Quantum Electrodynamics* (QED), the gravitational four-vector field can be quantized too in this very same manner.

We must say that our theory is directly inspired by Weyl's geometry [1] – a geometry that for the first time made the great and esoteric stride and endeavour to bring the electromagnetic and gravitational forces together into a fruitful and harmonious union that did not last beyond Einstein's first criticism of it (see e.g. [11]). Unlike what we have done in our previous work (in [8–10]), we shall not anymore bother our reader with the plethora of the exciting and fascinating historic anecdotes associated with the pursuit of a unified field theory that brings the gravitational and quantum phenomenon into one giant, neat, coherent and consistent mathematical framework. We deal here directly with the purest portions and jewels of our effort.

In their noble quest and search for a unified field theory of the quantum and gravitational phenomenon, physicists – and mathematicians alike – have been motivated by various reasons. In our case, our motivation has been, and is solemnly to overcome the obvious great difficulty associated with the General Theory of Relativity (GTR)'s geodesic equation of motion, namely:

$$\frac{d^2 x^\lambda}{ds^2} - \Gamma^\lambda_{\alpha\delta} \frac{dx^\alpha}{ds} \frac{dx^\delta}{ds} = 0 \qquad (1)$$

where $ds = cd\tau$ is the line element, τ is the relativistic proper time, $c = 2.99792458 \times 10^8 \, \mathrm{m \, s^{-1}}$ (CODATA 2018) is the speed of light in *vacuo*, x^μ is the four-position of the particle in spacetime, and $\Gamma^\lambda_{\mu\nu}$ are the usual Christoffel three-symbols [12]*. Because of the non-tensorial nature of the affine connection $\Gamma^\lambda_{\mu\nu}$, this geodesic (1) of motion does not hold fast – in the truest sense – to the depth of the letter and essence of the philosophy deeply espoused and embodied in Einstein's *Principle of Relativity* (PoE) [13], namely that physical laws must require no special set of coordinates where they are to be formulated.

The non-tensorial nature of the affine connection requires that the equation of motion must first be formulated in special kind of coordinate systems known as a *geodesic coordinate system*†, yet the PoE forbids this. This problem has never been adequately addressed in the GTR. As Einstein [2] noted, a permanent way out of this dilemma is to find a geometry whose affine connections are tensors. This is what we do herein. At the end of our quest – for the gravitational

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*These symbols are named after German mathematician and physicist Elwin Bruno Christoffel (1829-1900). Christoffel first introduced these symbols in a paper on differential forms in n variables, published in Crelle's Journal: see [12].

†A geodesic coordinate system is one in which the Christoffel three-symbols $\Gamma^\lambda_{\mu\nu}$ vanish at all points on the given set of coordinates – i.e. $\Gamma^\lambda_{\mu\nu} = 0$. An example is the flat rectangular (x, y, z) system of coordinates. However, when one moves from this (x, y, z) rectangular system of coordinates to say the spherical (r, θ, φ), the resulting affine $\Gamma^{\lambda'}_{\mu'\nu'}$ is not zero – i.e. $\Gamma^{\lambda'}_{\mu'\nu'} \neq 0$.

phenomenon as a whole – we arrive not by design, but rather by serendipity, at a *gravitomagnetic theory* similar to that of Maxwell [14].

In current efforts being made on both the theoretical (in e.g. [15–19]), and observational front (in e.g. [20–24]), gravitatomagnetism is predominately understood in the context of Einstein's [25–28] linearised first order approximation of the GTR. Our approach is different to this predominant approach.

We herein consider gravitomagnetism as an exact theory independent of the GTR in much the same way it was conceived by Maxwell [14] and Heaviside [29, 30] and further championed (in modern times) e.g. by Jefimenko [31] and Behera [32] amongst others. The present gravitomagnetic theory falls within the realm of a more ambitious attempt that we are currently working on, i.e. an attempt at an all-encompassing *Unified Field Theory* (UFT) of all the forces of Nature (see [10, 33]). We shall say nothing about this attempt but direct the interested reader to these works.

In closing this introductory section, we shall give a brief synopsis of the remainder of the paper. In §2, we give a brief exposition of the GTR. In §4, we give an exposition of Weyl's theory [1]. In §3, we give a non-geometric justification of gravitomagnetism. In §5, we present our theory. Thereafter, in §6, in preparation for the presentation of the gravitomagnetic field equations, we express the new affine ($\bar{\Gamma}^\lambda_{\mu\nu}$) and the Riemann tensor ($R_{\mu\nu}$) in terms of the gravitational Maxwell-type field tensor ($\mathfrak{F}_{\mu\nu}$). Therein §6, we also work out the geometrically derived material tensor ($\mathcal{T}_{\mu\nu}$) so that its terms correspond with what we know from the physical world. In §7, we write down the resultant field equations. Lastly, in §8, a general discussion is given.

2 Brief exposition of the GTR

As is well known, Einstein's *Special Theory of Relativity* [34] deals with inertial observers while the GTR deals with non-inertial observers. The problem with non-inertial observers is that gravitation becomes a problem since it is an all pervading *non-vanishing force*. By analysing the motion of a test body in free fall motion in a gravitational field, Einstein [13] was able to overcome this problem of gravitation by noting that if the gravitational (m_g) and inertia mass (m_i) were equal or equivalent, then gravitation and acceleration are equivalent too. This meant that the effect(s) of acceleration and gravitation are the same. One can introduce or get rid of the gravitational field by introducing acceleration into the system. Because of the importance of this, it came to be known as the *Principle of Equivalence*, and Einstein [25] took this as a foundational pillar to be used for the construction of his GTR.

2.1 Principle of Equivalence

The deep rooted meaning of the Principle of Equivalence is that physical laws should remain the same in a local reference system in the presence of a gravitational field as they do in

an inertial reference system in the absence of gravitation. In Einstein's own words [13]:

> **Einstein's Principle of Equivalence (PoE):** We shall therefore assume the complete physical equivalence of a gravitational field and the corresponding acceleration of the reference system. This assumption extends the Principle of Relativity to the case of uniformly accelerated motion of the reference system.

A consequence of this is that no mechanical or optical experiment can locally distinguish between a uniform gravitational field and uniform acceleration. It is here that we would like to point out that the PoE as used in the formulation of the GTR does not demand that the physics must remain invariant. By "the physics" we mean that the description of a physical event ought to remain invariant unlike, for example, in black hole physics – where, depending on the coordinate system employed (and not the reference system – this is important), a particle can be seen to pass or not pass through the Schwarzschild sphere for the same observer supposedly under the same conditions of experience. Also the chronological ordering of events is violated – i.e. the *Law of Causality* is not upheld.

For example, as first pointed out by the great mathematician, logician and philosopher Kant Gödel [35], in a rotating Universe, it is possible to travel back in time, invariably meaning to say it is possible in principle to violate the *Second Law of Thermodynamics*. Though the idea of time travel is very fascinating and appealing to the mind, it is difficult to visualize by means of binary logical reasoning how it can work in the physical world as we know it. From intuition, the laws of Nature must somehow have deeply engraved and embedded in them the non-permissibility of time travel. We believe that such illogical outcomes emerging out from a legitimate application of the laws of Nature can be solved if the geometry of the Universe is built on tensorial affinities.

2.2 Generalized Principle of Equivalence

Therefore, in order to avoid physical absurdities emerging from supposedly well-founded laws of Nature, we must demand of our theories that "the physics" emerging from the theory, that is to say, the physical state and the chronological ordering of events, must remain invariant – i.e. we must extend the Principle of Equivalence to include the physical state or physical description of events and as well the Law of Causality. Because this must be universal and important, let us call the extended Principle of Equivalence, the *Generalized Principle of Relativity*:

> **Generalized Principle of Relativity (GPR):** Physical laws have the same form in all equivalent reference systems independently of the coordinate system used to express them and the complete physical state or physical description of an event emerging from these

laws in the respective reference systems must remain absolutely and independently unaltered – i.e. invariant and congruent – by the transition to a new coordinate system.

This forms the basic guiding principle of the present theory. The deeper meaning of the GPR is that, if one is describing the same physical event in spacetime e.g. a black hole, it should not be permissible to transform away a singularity by employing a different set of coordinates as is common place in the study of the Schwarzschild metric. If the singularity exists, it exists independently of the coordinate system and reference system used – it is intrinsic and permanent, it must exist at all levels of the theory.

Therefore, if we are to have no singularities, the theory itself must be free of these. If a particle is seen not to pass through the event horizon, it will not be seen to pass through the boundary of the event horizon no matter the coordinate system employed and the reference system to which the current situation is transformed into. In order for this, there is need for the affine connections to be tensors and this is what we shall try to achieve in the present – i.e. a geometry endowed with tensorial affine connections. For completeness, self-containment and latter instructive purposes, in the next subsection, we will take a look at the non-tensor affine connections of Riemann geometry.

2.3 Affine connection

Now, back to the main vein: the Principle of Equivalence is, in the context of Riemann geometry, mathematically embodied in the equation:

$$ g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^\delta_{\alpha\mu} g_{\delta\nu} - \Gamma^\delta_{\alpha\nu} g_{\mu\delta} = 0 \qquad (2) $$

where $g_{\mu\nu}$ is the metric tensor describing the geometry of spacetime and $\Gamma^\lambda_{\mu\nu}$ is the affine connection. This affine connection is obtained as follows (e.g. [36, pp. 59–60]): first we write down two equations obtained by way of right-cyclically permuting the $\mu\nu\sigma$-indices in (2) for the term $g_{\mu\nu,\sigma}$, i.e.:

$$ g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^\delta_{\alpha\mu} g_{\delta\nu} - \Gamma^\delta_{\alpha\nu} g_{\mu\delta} = 0 \,, \qquad (3) $$

$$ g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^\delta_{\alpha\mu} g_{\delta\nu} - \Gamma^\delta_{\alpha\nu} g_{\mu\delta} = 0 \,. \qquad (4) $$

Second, we now subtract from (2) the sum of (3) and (4), and use the symmetry of the connection ($\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$) and as well of the metric ($g_{\mu\nu} = g_{\nu\mu}$) to obtain: $\left(g_{\mu\nu,\alpha} - g_{\alpha\mu,\nu} - g_{\nu\alpha,\mu} \right) + 2g_{\alpha\delta}\Gamma^\delta_{\mu\nu} = 0$, hence:

$$ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\delta\lambda} \left(g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta} \right) . \qquad (5) $$

The affine connections play an important role in that they relate tensors between different reference systems and coordinate systems. Its drawback insofar as physical laws are con-

cerned is that it is not a tensor. It transforms as follows:

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^{\delta}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\delta}_{\mu\nu} + \underbrace{\frac{\partial x^{\lambda'}}{\partial x^{\delta}} \frac{\partial^2 x^{\delta}}{\partial x^{\mu'} \partial x^{\nu'}}}_{\textbf{extra term}} . \qquad (6)$$

The extra term on the right makes it a non-tensor and without it, the Christoffel symbol would be a tensor. Most of the problems facing the GTR can be traced back to the non-tensorial nature of the affine connections. Some of the problems will be highlighted in the succeeding section. Due to the nature of these affinities, the real problem is that in its bare form, Riemann geometry does not provide a way to determine permissible and non-permissible coordinate and reference system transformations. The new hybrid geometry on which the UFT being championed is built, does have a way to determine permissible and non-permissible coordinate and reference system transformations.

2.4 Line element

Now, both the invariance and covariance of physical laws under a change of the coordinate system and/or reference system transformation is, in Riemann geometry, encoded and/or expressed through the invariance of the line element: $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. The line element is a measure of the distance between points in spacetime and remains invariant under any kind of transformation of the reference system and/or the coordinate system. This is the essence of the GTR. From this, Einstein was able to deduce that gravitation is and/or can be described by the metric tensor $g_{\mu\nu}$ thus, according to the Einstein doctrine of gravitation, it (gravitation) manifests itself as the curvature of spacetime. Through his (Einstein) own intuition and imagination, he was able to deduce that the curvature of spacetime ought to be proportional to the amount of matter-energy present in spacetime — a fact that has since been verified by numerous experiments.

2.5 Einstein's field equations

The resulting gravitational law emerging from Einstein's thesis stated above – namely that the curvature of spacetime should be proportional to the amount of matter-energy present in spacetime – is:

$$\overbrace{\underbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}}_{\textit{beautiful and splendour}}}^{\textbf{MARBLE}} = \overbrace{\underbrace{\kappa_E T_{\mu\nu}}_{\textit{ugly and loathsome}}}^{\textbf{WOOD}} \qquad (7)$$

where $\kappa_E = 8\pi G/c^4$ is Einstein's constant of gravitation, $G = 6.67430(15) \times 10^{-11}$ kg^{-1} m^3 s^{-2} (CODATA 2018) is Newton's universal constant of gravitation, $R_{\mu\nu}$ is the contracted Riemann curvature tensor, R is the Ricci scalar, and $T_{\mu\nu} = \varrho_g v_{\mu} v_{\nu} + p g_{\mu\nu}$ is the stress and energy tensor where ϱ_g is the density of matter, p is the pressure, v_{μ} the four-velocity, and Λ

is the controversial ad hoc *Cosmological Constant* term added by Einstein [37] so as to stop the Universe from expanding. Einstein [37] was motivated to include the cosmological constant because of the strong influence from the astronomical wisdom of his day that the Universe appeared to be static and thus was assumed to be so.

In the later years of his scientific life while in hot pursuit of a unified field theory – according to his official scientific biographer – Abraham Pais [38], Einstein would look at his equation (7) and compare the left-hand side with marble and the right-hand side with wood, and he would admire the marble side calling it *beautiful and splendour* and, on looking at the right-hand side, he would be filled with sadness whereby he would moan calling it *ugly and loathsome*. His prime and hence immediate goal therefore (see e.g. [39]) was to turn the *ugly wood* into *beautiful marble*.

All Einstein hoped for and wanted in his quest, was that all the fields including the material field $T_{\mu\nu}$, be derived from pure geometry, rather than "glue" the two seemingly independent parts (i.e. the curvature $R_{\mu\nu} - R g_{\mu\nu}/2$ and material tensor $T_{\mu\nu}$) *via* some mere constant κ_E. Einstein was extremely dissatisfied with this state of affairs [38] and thus hoped that a theory would be found in the future where the material tensor is derived directly from the geometry as a direct consequence of the geometry itself. We must say, that, if our ideas prove themselves worthy, it appears we have just managed to derive the material fields from the *Resultant World Geometry*.

3 Present justification of gravitomagnetism

For example, take *Maxwell's Five Equations of Electrodynamics* [14] – i.e. the typical four equations that we are used to involving the reciprocal E and B-fields plus the *Law of Conversation of Electric Charge and Current*. Certainly, to a foremost theoretical physicist such as Paul Dirac (see e.g. [40–42]), these equations are without doubt beautiful in every aspect of the word beauty; and to seal the matter, their foundations are well verified and anchored in experience. But asking what is the fundamental basis for their existence led José Heras [43] to the tangibly solid mathematical fact that Maxwell's equations [14] are nothing more than a consequence of the conservation of electronic charge. That is to say, what you need for the existence of Maxwell's equations [14] is just the conservation of electric charge and current; nothing more and nothing less. Surely – to say that only the conservation of electronic charge and current is all that is needed for Maxwell's Equations to exist – this is certainly deep, isn't it?

Given that the gravitational mass – which is responsible for gravitation – follows a similar law of conversation in the form of the conservation of mass-energy and momentum, rather trivially, one can easily extend this to the gravitational phenomenon and justify the need for gravitomagnetism. Heras [43] did not make this trivial and obvious extrapolation. In addition to this, we must say that we have not

seen in the most recent literature any attempt to use Heras' [43] existence theorem to justify gravitomagnetism. However, by way of analogy with the equations of electrodynamics given the similarity between Newton and Coulomb's inverse square laws, Maxwell [14] and Heaviside [29, 30] already had introduced gravitomagnetism. Sadly, because of lack of experimental backing, gravitomagnetism derived in this way has largely been treated as an endeavour belonging to the realm of *pseudo-science*, rather than science. Many scientists that have followed in an effort to try and investigate this gravitomagnetic phenomenon have struggled to shrug-off the pseudo-science tag hanging at the nimbus of gravitomagnetism.

In the present section, we are going to give a brief exposition of Heras [43] and Behera's [32] existence theorems. These theorems are enough to convince sceptics that like electricity and magnetism, the gravitational phenomenon aught to be described by a four-vector potential. In addition to Heras [43] and Behera's [32] existence theorems, this paper will add a purely geometric justification and this geometric justification follows the same geometric path as the GTR wherein the gravitational phenomenon is described by the metric. Because this demonstration – that we are going to give of the geometric justification of gravitomagnetism – uses the modern description of gravitation as a metric phenomenon, it certainly is not far-off in its outlook, vision and conception with the modern idea of a metric description of gravity. Surely, this aspect of the present ideas must – somehow – make the ideas propagated herein appeal to the reader. In the next subsection, we shall give an exposition of Heras' theorem [43].

3.1 Heras's (2007) existence theorem

In a nutshell, Heras [43] formulated – what in our view is – a very important *Existence Theorem* that states that, given any space and time-dependent localized scalar and vector sources satisfying the continuity equation – as is the case with electromagnetism – there exists in general, two retarded vector fields (X, Y) that satisfy a set of four field equations that are similar in nature and form to Maxwell's equations. By applying the theorem to the usual electrical charge and current densities, the two retarded fields are identified with the reciprocal electric (E) and magnetic (B) fields and the associated field equations with Maxwell's equations [14], i.e.: $X := E, Y := B$.

In brief, what Heras [43] proved is that if ϱ_e is the charge density and $\vec{J}$ is the associated current corresponding to this charge, i.e.:

$$\frac{\partial \varrho_e}{\partial t} = -\vec{\nabla} \cdot \vec{J}, \tag{8}$$

then, there must exist two corresponding fields, X and Y, that

satisfy the following set of equations:

$$
\begin{aligned}
\vec{\nabla} \cdot X &= \alpha \varrho_e & \text{(a)}\\
\vec{\nabla} \cdot Y &= 0 & \text{(b)}\\
\vec{\nabla} \times X + \gamma \frac{\partial Y}{\partial t} &= 0 & \text{(c)}\\
\vec{\nabla} \times Y - \frac{\beta}{\alpha}\frac{\partial X}{\partial t} &= \beta \vec{J} & \text{(d)}
\end{aligned}
\tag{9}
$$

where α, β, γ are arbitrary positive constants and are related to the speed of light c by the equation $\alpha = \beta\gamma c^2$. In the case of electricity and magnetism, if X and Y are identified with the electric and magnetic fields respectively, then we will have Maxwell's classical equations [14] for electrodynamics – in which case $\alpha = 1/\varepsilon$, $\beta = \mu$, and, $\gamma = 1$. Clearly, this axiomatic and fundamental approach of deriving Maxwell's field equations [14] strongly suggests that electric charge and current conservation – and nothing else – can be considered to be the most fundamental assumption underlying Maxwell's equations [14] of electrodynamics. Next, we give an exposition of Behera's [32] theorem.

3.2 Behera's (2006) theorem

Using the Law of Conservation 0f Mass-Energy-Momentum and the Poisson-Laplace equation (10), the endeavour of the present section is to demonstrate – as Behera [32] did – that much the same as the Coulomb electrical potential, the Newtonian gravitational potential ϱ_g has an associated vector field. We shall denote this vector field by the symbol A_g and we shall call it the gravitomagnetic vector potential and in short we shall call it the *g-magnetic* vector potential. This fact that we can associate ϱ_g with A_g has been known for a considerable amount of time now. That is, for more than a century (≥ 120 years), it has been known (since Heaviside [29, 30]) that the inclusion of a magnetic-like vector field in Newtonian gravitational theory can be justified from two immutable facts (see e.g. Behera [32]), i.e. from the Poisson-Laplace equation for gravitation, namely:

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \varrho_g \tag{10}$$

where $\vec{g}$ is the gravitational field intensity at a given point in the gravitational field, ϱ_g is the gravitational potential, and from the equation of conservation of mass-energy and momentum, namely: $\partial \varrho_g / \partial t = -\vec{\nabla} \cdot \vec{\mathfrak{J}}$, where $\vec{\mathfrak{J}} = \varrho_g v$, is the momentum density with v being the velocity of the material whose density is ϱ_g.

In order to see this, from (10) we know very well that: $\dot{\varrho}_g = -(1/4\pi G)(\vec{\nabla} \cdot \dot{\vec{g}})$. Let us set: $\widetilde{\mu} = 1/4\pi G$, so that: $\dot{\varrho}_g = (1/4\pi G)(\vec{\nabla} \cdot \dot{\vec{g}})$ can now be written as: $\dot{\varrho}_g = -\widetilde{\mu}\vec{\nabla} \cdot \dot{\vec{g}}$. From this, it follows that:

$$\frac{\partial \varrho_g}{\partial t} = -\vec{\nabla} \cdot \vec{\mathfrak{J}} = -\vec{\nabla} \cdot \left(\widetilde{\varepsilon} \frac{\partial \vec{g}}{\partial t} \right), \tag{11}$$

hence:

$$\vec{\nabla} \cdot \left[\widetilde{\varepsilon} \frac{\partial \vec{\mathfrak{g}}}{\partial t} + \vec{\mathfrak{J}} \right] = 0. \qquad (12)$$

Now, it is a *bona fide* mathematical fact that for any general vector say $\vec{\mathfrak{B}} = \vec{\mathfrak{B}}(x)$, the following holds always:

$$\vec{\nabla} \cdot \left(\frac{\vec{\nabla} \times \vec{\mathfrak{B}}}{\widetilde{\mu}} \right) \equiv 0. \qquad (13)$$

where $\widetilde{\mu}$ is a constant – which, akin to the electromagnetic permeability (μ_0) and permittivity (ϵ_0) of free space, we shall define this constant $\widetilde{\mu}$ is such that: $\widetilde{c} = 1/\sqrt{\widetilde{\mu}\widetilde{\varepsilon}}$, where $\widetilde{c}$ is the speed of gravity in free space. By comparing (12) and (13), it follows that:

$$\frac{\vec{\nabla} \times \vec{\mathfrak{B}}}{\widetilde{\mu}} = -\vec{\mathfrak{J}} + \widetilde{\varepsilon} \frac{\partial \vec{\mathfrak{g}}}{\partial t}. \qquad (14)$$

What this really means is that the gravitational field $\vec{\mathfrak{g}}$ has an associated magnetic-like field $\vec{\mathfrak{B}}$. Hence, one can make the very bold conclusion that the very laws of Nature (10) and $\partial \varrho_{\mathrm{g}}/\partial t = -\vec{\nabla} \cdot \vec{\mathfrak{J}}$ invariably imply an associated magnetic-like field for the gravitational field. Following tradition, we shall call this magnetic-like field the gravitomagnetic field and for short, we shall call it the *g-magnetic* field.

Now, (10) and (14) have a seductive and irresistible resemblance with Maxwell's source-coupled equations so much so that for the brave that have set their mind on this, they have proceeded without detouring to make a complete formal analogue with Maxwell's equations [14], in which process, the phenomenon known as *gravitomagnetism* found its original birth certificate. Therefore, as a complete set, the *Field Equations of Gravitomagnetism*, are:

$$\vec{\nabla} \cdot \vec{\mathfrak{g}} = -\varrho_{\mathrm{g}}/\widetilde{\varepsilon} \qquad \text{(a)}$$

$$\vec{\nabla} \times \vec{\mathfrak{g}} = -\frac{1}{\widetilde{c}} \frac{\partial \vec{\mathfrak{B}}}{\partial t} \qquad \text{(b)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (15)$$

$$\vec{\nabla} \cdot \vec{\mathfrak{B}} = 0 \qquad \text{(c)}$$

$$\vec{\nabla} \times \vec{\mathfrak{B}} = -\widetilde{\mu}\vec{\mathfrak{J}} + \frac{1}{\widetilde{c}^2} \frac{\partial \vec{\mathfrak{g}}}{\partial t}. \qquad \text{(d)}$$

This completes our exposition of the non-geometric justification of gravitomagnetism. In the next section, we shall for self-containment and latter instructive purposes, present a brief exposition of Weyl's theory [1] and in the penultimate thereof, we present our partial modification of it.

4 Weyl geometry

In §4.1, we give a brief exposition of Weyl's geometry [1] and thereafter in §4.2, we present the *New Weyl Geometry* (NWG) upon which the proposed gravitomagnetic theory is based.

4.1 Original Weyl geometry

By way of addition of a conformal factor $e^{2\phi}$ to the metric $\mathrm{g}_{\mu\nu} \mapsto e^{2\phi}\mathrm{g}_{\mu\nu}$, Weyl [1] built his geometry by supplementing the Christoffel affine connection $\Gamma^\lambda_{\mu\nu}$ of Riemann geometry with a tensorial affine $\mathcal{W}^\lambda_{\mu\nu}$:

$$\mathcal{W}^\lambda_{\mu\nu} = \mathrm{g}^\lambda{}_\mu \mathcal{A}_\nu + \mathrm{g}^\lambda{}_\nu \mathcal{A}_\mu - \mathrm{g}_{\mu\nu} \mathcal{A}^\lambda, \qquad (16)$$

where $\mathcal{A}_\mu$ is a four-vector that Weyl [1] had to define as the electromagnetic four-vector appearing in Maxwell's theory of electrodynamics [14].

In Weyl's geometry [1] where the length of vector changes from point to the next (see e.g. [33]), the new affine connection $\overline{\Gamma}^\lambda_{\mu\nu}$ (or Christoffel-Weyl connection) is given by:

$$\overline{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \mathcal{W}^\lambda_{\mu\nu}. \qquad (17)$$

The transformational properties of the new Christoffel-Weyl affine connection $\overline{\Gamma}^\lambda_{\mu\nu}$ are identical to those of the original Christoffel three-symbol $\Gamma^\lambda_{\mu\nu}$. So, from a "transformational properties" (topological) standpoint, Weyl's [1] addition is justified.

The versatile and agile Weyl [1] was quick to note that this new Christoffel-Weyl affine (17) is invariant under the following rescaling of the metric $\mathrm{g}_{\mu\nu}$ and the four-vector $\mathcal{A}_\mu$:

$$\begin{aligned} \mathrm{g}_{\mu\nu} &\longmapsto e^{2\chi}\mathrm{g}_{\mu\nu} \\ \mathcal{A}_\mu &\longmapsto \mathcal{A}_\mu + \kappa_0^{-1}\partial_\mu\chi \end{aligned}, \qquad (18)$$

where $\chi = \chi(\boldsymbol{r}, t)$ is a well behaved, arbitrary, smooth, differentiable, integrable and uniform continuous scalar function, and κ_0 is a constant with the dimensions of inverse length. This constant κ_0 has been introduced for the purposes of dimensional consistency, since we here assume that the four-vector $\mathcal{A}_\mu$ and the true scalar χ are dimensionless physical quantities.

Now, because Maxwell's electromagnetic theory [14] is invariant under the same gauge transformation which the four-vector $\mathcal{A}_\mu$ has been subjected to in (18), the great mind of Weyl seized the golden moment and identified this four-vector $\mathcal{A}_\mu$ with the electromagnetic four-vector potential. Weyl went on to assume that the resulting theory was a unified field theory of gravitation and Maxwellian electrodynamics. Weyl's hopes where dashed – first, starting with Einstein's lethal critique of the theory. Later, others joined Einstein in their merciless critique and dismissal of Weyl's theory [1], where they argued that despite its irresistible grandeur and exquisite beauty, Weyl's theory [1] cannot possibly describe the measured reality of our present world.

4.2 New Weyl geometry

Despite the many ingenious attempts (starting with e.g. Weyl [44, 45]) to rework and revive it over the course of time since
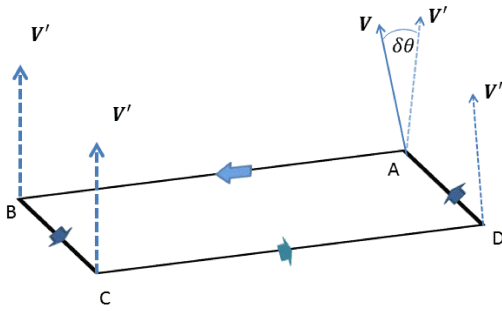
Fig. 1: **Parallel Transport:** The vector V is parallel transported in a closed circuit. Upon arrival at its original position, the vector is not equal to the original vector and this is a result of the curvature of the space in question.

its inception, and its apparent refusal to go away as evidenced by the continued interest* in this beautiful geometry of Weyl [1], it is a generally accepted view that as a basis for a physical theory, Weyl's [1] arcanely beautiful geometry exists beyond redemption. This geometry is the geometry on which [1] made his attempt – the first such – on a UFT of the gravitational and electromagnetic fields. Against this deeply entrenched belief in the non-redeemability of the Weyl [1] geometry into something with a bearing and correspondence with physical and natural reality, we made in [33] the endeavour of calling forth this theory out of the tomb where it was resting. In the present, we go further to give it a perdurable fresh breath of life.

As pointed out by e.g. Schrödinger and Einstein [3, 5–7] and is well known, is that – tensorial affine connections preserve both the length and direction of a vector upon parallel transport. The Christoffel symbols of Riemann geometry preserve only the length and the angle changes from one point the next and this is where the issue with Einstein's GTR [55] lies. Preservation of both the length and angle of a vector upon parallel transport has always been known to be a fundamental key to the attainment of a truly generalized Theory of Relativity [56, 57].

The proposed RWS is a spacetime which preserves both the length and direction of a vector upon parallel transport. As shown in Fig. 1, say the vector V is transported in a closed circuit such that it returns to its original position and V' is the resulting vector after parallel transport; in normal Riemann geometry, while $|V| = |V'|$, the angle $\delta\theta$ between these two vectors, while it can in some cases equal zero, is not necessarily zero i.e. $V \cdot V' \neq 0$. However, on the RWS, we have for all points of space and time on this spacetime the constraints $|V| = |V'|$ and $V \cdot V' = 0$: i.e. both the length and direction of a vector are preserved upon parallel transport of any vector.

*See e.g. [46–54].

The preservation of both the length and angle on the RWS is attained by requiring that the affine connections of this spacetime be tensors. As far as we can tell from our wide ranging searches across the length, breath and depth of the available literature on unified theories (cf. [56, 57]), with the failure to obtain tangible results on this front, the idea of tensorial affinities as key to the attainment of a unified field theory seems to have naturally fallen on the wayside with very few – if any – researchers taking it up. As one will be able to judge for themselves and by themselves, the novelty of our approach lies in our treatment of the unit vectors.

As pointed out in the instance of (18), to attain the desired tensorial affinities, we noted that Weyl [1] had built his very beautiful but failed unified field theory of gravitation and electromagnetism on a pseudo-Riemann spacetime that is invariant under the re-gauging of the metric from $g_{\mu\nu}$ to $e^{2\chi} g_{\mu\nu}$: i.e. after the transformation $g_{\mu\nu} \longmapsto e^{2\chi} g_{\mu\nu}$, the field equations of the resulting geometry or theory thereof remain unaltered provided the four-vector of his theory $\mathcal{A}_\mu$ also underwent the following gauge transformation: $\mathcal{A}_\mu \longmapsto \mathcal{A}_\mu + \kappa_0^{-1} \partial_\mu \chi$. The mathematical structure of the resulting Weyl unified field theory, insofar as the properties of the affine connections is concerned, this theory – despite its elegant introduction of a four-vector field – has the same topological deformations as the original Riemann spacetime.

4.2.1 Riemann-Weyl metric

As already pointed out in §4.1, Weyl added a tensor $\mathcal{W}_{\mu\nu}^\lambda$ to the Christoffel three-symbol $\Gamma_{\mu\nu}^\lambda$, that is to say, if $\widetilde{\Gamma}_{\mu\nu}^\lambda$ is the new Christoffel symbol for the Weyl space, then:

$$\widetilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \mathcal{W}_{\mu\nu}^\lambda. \tag{19}$$

Because $\mathcal{W}_{\mu\nu}^\lambda$ is a tensor, the fundamental transformational properties of the new Christoffel three-symbol $\widetilde{\Gamma}_{\mu\nu}^\lambda$ are the same as the old Christoffel three-symbol $\Gamma_{\mu\nu}^\lambda$; therefore, the Weyl space inherits the same topological and structural defects and problems of the Riemann spacetime – that is, problems to do with non-tensorial affinities.

In [33], for the metric of the RWS $\bar{g}_{\mu\nu}$, instead of making it conformal only at the instance of a gauge transformation, we chose that it ($\bar{g}_{\mu\nu}$) be intrinsically and inherently conformal. That is to say, the fundamental metric $\bar{g}_{\mu\nu}$ of the RWS be such that $\bar{g}_{\mu\nu} = \varrho \, g_{\mu\nu}$, where $g_{\mu\nu}$ remains as the metric of the usual Riemann spacetime and this metric is what is used on the RWS to raise and lower the Greek indices ($\mu\nu \dots$) just as happens in normal Riemann spacetime. In Weyl's theory [1], the function ϱ is a scalar. However in [33], this function takes a decisive new role: . . . it (the scalar χ) must – for better or for worse, yield in the favour of our desideratum – i.e. it must yield for us nothing but tensorial affinities. This is our quest, desire and uncompromising demand.

Thus, in recasting Weyl's theory [1] so that it overcomes once and for all-time Einstein's criticism, we will not take the traditional route that was taken by Weyl [1] because in so doing, we will fall into the same trap which the great Weyl fell victim to. At our point of departure, we wave goodbye to Riemann geometry and efferently prepare to embrace a totally new geometry, a hybrid Riemann geometry which has the same feature as Weyl [1], less of course the change of length of vectors under transformations or translations. The route that we are about to take is equivalent and the reason for changing the sails is that the present route allows us to demonstrate later how Weyl would have overcome Einstein's critique that gave the theory a still birth. Actually, this route allows us to pin down exactly where Weyl's theory [1] makes an unphysical assumption.

4.2.2 Pseudo-scalar and affine vector

In mathematics – linear algebra in particular – a pseudo-scalar is a function which upon a transformation of the coordinate system behaves like a true scalar – albeit – upon a parity transformation, it changes sign (see e.g. [58, 59]). A true scalar does not do this, it remains invariant. As has already been made clear in the exposition of Weyl's theory [1] is the fact that one of the most powerful ideas in physics is that physical laws do not change when one changes the coordinate system used to describe these physical laws. The fact that a pseudo-scalar reverses its sign when the coordinate axes are inverted clearly suggests that these objects are not the best objects to describe a physical quantity, as this could percolate to the physical laws themselves.

Now, because we want to introduce a new kind of pseudo-scalar that will help us in our endeavours to obtain tensorial affinities, in order to distinguish this new and soon to be defined pseudo-scalar from the above described pseudo-scalar, we shall call the above described pseudo-scalar a *pseudo-scalar of the first kind*, and the new pseudo-scalar to be defined shortly, a *pseudo-scalar of the second kind*. To that end, we shall hereafter start off by defining a "new" mathematical object, $\mathcal{V}_\mu$, that we shall call an *affine vector*. This quantity, $\mathcal{V}_\mu$, is the derivative of the dot-product of an arbitrary four-vector $\mathcal{B}_\lambda$ and the (non-arbitrary) four-position x^λ i.e.:

$$\mathcal{V}_\mu = \frac{\partial_\mu \mathcal{S}}{\mathcal{S}} = \partial_\mu \ln \mathcal{S} \tag{20}$$

where:

$$\mathcal{S} = \mathcal{B}_\delta x^\delta . \tag{21}$$

From (20) and (21), it follows that:

$$\mathcal{V}_\mu = \mathcal{B}_\mu + \frac{x^\delta \partial_\mu \mathcal{B}_\delta}{\mathcal{S}} . \tag{22}$$

Clearly, upon a coordinate and/or transformation of the reference system, the vector-like quantity $\mathcal{V}_{\mu'} = \partial_{\mu'} \mathcal{S}'/\mathcal{S}'$ is related to $\mathcal{V}_\mu$ as follows:

$$\mathcal{V}_{\mu'} = \frac{\partial x^\delta}{\partial x^{\mu'}} \mathcal{V}_\delta + \frac{\partial^2 x^\delta}{\partial x^{\mu'} \partial x^{\Omega'}} . \tag{23}$$

From (23), we see that the quantity $\mathcal{V}_\mu$ transforms like the affine tensor hence our calling it – affine vector. The scalar $\mathcal{S}$ in (21) is what we shall define as a *pseudo-scalar of the second kind*. Such a scalar has the property that its four-position derivative is not a four-vector as one would expect in the case of a true scalar. In the next section, we shall now consider the Riemann-Weyl covariant derivative in the light of the new mathematical object that we have just defined – i.e. the pseudo-scalar of the second kind.

4.2.3 Riemann-Weyl covariant derivative

Taking into account the above defined pseudo-scalar of the second kind, as we consider the Riemann-Weyl covariant derivative, we will begin with the usual Riemann covariant derivative $g_{\mu\nu;\sigma} = 0$ of Riemann geometry. As already alluded, the condition $g_{\mu\nu;\sigma} = 0$ is the foundation stone of Riemann geometry. We will uphold this covariant derivative condition under the Weyl conformal transformation $g_{\mu\nu} \longmapsto \overline{g}_{\mu\nu} = \varrho g_{\mu\nu}$ of the metric i.e. $\overline{g}_{\mu\nu;\sigma} = 0$. Likewise, the condition $\overline{g}_{\mu\nu;\sigma} = 0$ is to be taken as the foundation stone of the new hybrid Riemann-Weyl geometry. Written in full, the equation $\overline{g}_{\mu\nu;\sigma} = 0$ is given by:

$$\overline{g}_{\mu\nu;\sigma} = \varrho \left[g_{\mu\nu,\sigma} + g_{\mu\nu}\left(\frac{\partial_\sigma \varrho}{\varrho}\right) - g_{\mu\delta}\overline{\Gamma}^\delta_{\nu\sigma} - g_{\delta\nu}\overline{\Gamma}^\delta_{\mu\sigma} \right] = 0 \tag{24}$$

where the "bar" on $\overline{\Gamma}^\lambda_{\mu\nu}$ has been put so that it is made clear that this affine is neither the Christoffel symbol nor the usual Weyl connection, but is the new hybrid Riemann-Weyl connection. In conformity with the definition of a pseudo-scalar of the second kind given in (21), we shall at this point set or define the ϱ-quantity as:

$$\varrho = -2\mathcal{I}_\delta x^\delta \tag{25}$$

where $\mathcal{I}_\sigma$ is the (gravitational) four-current density. With this definition for ϱ, it follows that (24) will reduce to:

$$g_{\mu\nu,\sigma} - g_{\mu\delta}\overline{\Gamma}^\delta_{\nu\sigma} - g_{\delta\nu}\overline{\Gamma}^\delta_{\mu\sigma} = 2\left(\mathcal{I}_\sigma + Q_\sigma\right)g_{\mu\nu} \tag{26}$$

where $Q_\sigma = x^\delta \partial_\sigma \mathcal{I}_\delta / \varrho = x^\delta \partial_\sigma \mathcal{I}_\delta / \mathcal{I}_\delta x^\delta$. As is the case with Weyl's original geometry [1], the covariant derivative $g_{\mu\nu;\sigma}$ does not vanish since $g_{\mu\nu;\sigma} \neq 0$.

4.2.4 Calculation of the Riemann-Weyl affine

Now – we have to calculate the resulting affine connections and for this, we have to write down the three expressions that

result from an anti-clockwise cyclic permutation of the indices μ, ν and σ in $g_{\mu\nu,\sigma}$, i.e.:

$$g_{\mu\nu,\sigma} - g_{\mu\delta}\overline{\Gamma}^\delta_{\nu\sigma} - g_{\delta\nu}\overline{\Gamma}^\delta_{\mu\sigma} \quad = \quad 2\left(\mathcal{I}_\sigma + Q_\sigma\right)g_{\mu\nu} \quad \text{(a)}$$

$$g_{\sigma\mu,\nu} - g_{\sigma\delta}\overline{\Gamma}^\delta_{\mu\nu} - g_{\delta\nu}\overline{\Gamma}^\delta_{\sigma\mu} \quad = \quad 2\left(\mathcal{I}_\nu + Q_\nu\right)g_{\sigma\mu} \quad \text{(b)} \qquad (27)$$

$$g_{\nu\sigma,\mu} - g_{\nu\delta}\overline{\Gamma}^\delta_{\sigma\mu} - g_{\delta\mu}\overline{\Gamma}^\delta_{\nu\sigma} \quad = \quad 2\left(\mathcal{I}_\mu + Q_\mu\right)g_{\nu\sigma} \quad \text{(c)}$$

As usual, subtracting from (27) (a) the sum of (27) (b) and (c), and making use of the symmetries of $g_{\mu\nu}$ and $\overline{\Gamma}^\lambda_{\mu\nu}$ (i.e. $g_{\mu\nu} = g_{\nu\mu}$ and $\overline{\Gamma}^\lambda_{\mu\nu} = \overline{\Gamma}^\lambda_{\nu\mu}$), one obtains:

$$g_{\mu\nu,\sigma} - g_{\sigma\mu,\nu} - g_{\nu\sigma,\mu} + g_{\sigma\delta}\left[\overline{\Gamma}^\delta_{\mu\nu} + g_{\delta\nu}\overline{\Gamma}^\delta_{\sigma\mu}\right] =$$
$$= 2\left[\left(\mathcal{I}_\sigma + Q_\sigma\right)g_{\mu\nu} - \left(\mathcal{I}_\nu + Q_\nu\right)g_{\sigma\mu} - \left(\mathcal{I}_\mu + Q_\mu\right)g_{\nu\sigma}\right]. \qquad (28)$$

Contracting the σ-index in (28) by multiplying (28) throughout by $g^{\sigma\lambda}$ and thereafter resetting this σ-index to δ, we obtain:

$$-g^{\delta\lambda}\left[g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right] + 2\overline{\Gamma}^\lambda_{\mu\nu} =$$
$$-2g^{\delta\lambda}\left[\left(\mathcal{I}_\nu + Q_\nu\right)g_{\delta\mu} + \left(\mathcal{I}_\mu + Q_\mu\right)g_{\nu\delta} - \left(\mathcal{I}_\delta + Q_\delta\right)g_{\mu\nu}\right], \qquad (29)$$

hence:

$$\overline{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \mathcal{W}^\lambda_{\mu\nu} - Q^\lambda_{\mu\nu}, \qquad (30)$$

where $\Gamma^\lambda_{\mu\nu}$ is the usual Christoffel three-symbol given in (5), and

$$\mathcal{W}^\lambda_{\mu\nu} = g^\lambda_{\ \mu}\mathcal{I}_\nu + g^\lambda_{\ \nu}\mathcal{I}_\mu - g_{\mu\nu}\mathcal{I}^\lambda \qquad (31)$$

is the (redefined) Weyl tensor, and the new non-tensorial object:

$$Q^\lambda_{\mu\nu} = g^\lambda_{\ \mu}Q_\nu + g^\lambda_{\ \nu}Q_\mu - g_{\mu\nu}Q^\lambda \qquad (32)$$

is a new affine connection that is defined thereon the hybrid Riemann-Weyl space and its purpose is to allow the hybrid Riemann-Weyl affine $\overline{\Gamma}^\lambda_{\mu\nu}$ to be a tensor. Let us call this affine the Q-affine connection or simply the Q-affine. The geometry that we have just described is what we shall call the *New Weyl Geometry* (NWG).

4.2.5 Transformation of the Riemann-Weyl affine

Now from (6), we know that the old Christoffel three-symbol $\Gamma^\lambda_{\mu\nu}$ transforms as follows:

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\delta}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma^\delta_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial x^{\mu'}\partial x^{\nu'}} \qquad (33)$$

and that $\mathcal{W}^\lambda_{\mu\nu}$ is a tensor, hence, it transforms like a tensor. Verily, if the Q-affine $Q^{\lambda'}_{\mu'\nu'}$ were to transform just as the Christoffel three-symbol $\Gamma^{\lambda'}_{\mu'\nu'}$, as follows:

$$Q^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\delta}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}Q^\delta_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial x^{\mu'}\partial x^{\nu'}}, \qquad (34)$$

then it follows and goes without saying that the object $\overline{\Gamma}^\lambda_{\mu\nu}$ will clearly be a tensor. Because Q_μ is an affine vector, the Q-tensor will transform as desired, that is, as given in (34), hence the object $\overline{\Gamma}^\lambda_{\mu\nu}$ will be a tensor as desired. What this all means is that Q is a pseudo-scalar and not a pure scalar. This is exactly what we did in [33]. Therein [33], we achieved this by forcing Q_μ to yield in the favour of our desires and transform as an affine vector as defined in §4.2.2. The resulting theory that one can build from this NWG has been presented in [33] and, in the present paper, it is this same theory that we are now improving on.

As one can verify for themselves, this theory of [33] produces field equations that we are already familiar with – i.e. the Maxwell equations [14]. At this stage of the development of the theory – whether or not the resulting theory is correct – what is important for the reader to appreciate – as has just been here demonstrated thus far – is that a tensorial affine theory can be attained. The problem suffered by Weyl's theory [1] does not apply to the NWG.

5 Theory

We here lay down our theory. What makes the present work different from the preceding works in [8–10] is that the present work incorporates the new results from various research that we have conducted. Because we shall at five instances (i.e. (37), (44), (79a), (79b), and (79c)) need to do some gauge fixing, we shall start off by addressing this issue of gauge fixing, i.e. within the context of the present work.

5.1 Gauge fixing

In the physics of *Gauge Theories*, gauge fixing (also called choosing a gauge) denotes a mathematical procedure for coping with redundant degrees of freedom in the field variables. The introduction of a gauge effectively reduces the number of degrees of freedom of the theory. In the present expedition, we shall need the fixing of the gauge and this fixing shall be done in such a manner that one seeks to obtain equations that are congruent with reality. That is, equations that we are already used to know. We shall identify two types of gauges, i.e.:

1. **Natural Gauge:** A *natural gauge* shall here be understood as an exogenous constraint the theory must satisfy in order to meet a global physical requirement. For example, in the present pursuit, we seek a theory based on a spacetime which is such that the magnitude and direction of a vector (tensor) upon parallel transport remains unaltered by the act or procedure of parallel transport of the vector on this spacetime. So, the gauge fixing that will lead us to the attainment of this global symmetry, we shall call a natural gauge or – alternatively – an *exogauge constraint*.

2. **Gauge Constraint:** A *gauge constraint* shall here be understood as an endogenous constraint the theory must satisfy in order to yield equations that are congruent with reality as we are used to know. For example, in the present pursuit, we

seek a theory that will at least yield field equations that are similar to Maxwell's equations [14]. So, the gauge fixing that will lead us to the attainment of such equations, we shall call a gauge constraint or – alternatively – an *endogauge constraint*.

Each time we encounter a natural gauge (exogauge constraint) or a gauge constraint (endogauge constraint), we shall make a clear indication of this.

5.2 Hybrid Riemann-Weyl tensor

From Fig. 1, if say we have a (four-) vector v^λ and we parallel transport it along a closed circuit ABCD in the order (A $\longmapsto$ B) then (B $\longmapsto$ C) then (C $\longmapsto$ D) and then finally (D $\longmapsto$ A), if the space in question has a non-zero curvature, upon arrival at its original location, while the length of this vector may be equal to the length of the original vector, its direction will at the very least be different. The infinitesimal changes of this vector's direction and length along these paths (see e.g. [10, for details of the derivation]), are:

$$dv^\lambda = \overline{R}^\lambda_{\mu\sigma\nu} v^\mu da^\nu db^\sigma , \qquad (35)$$

where:

$$\overline{R}^\lambda_{\mu\sigma\nu} = \overbrace{\overline{\Gamma}^\lambda_{\mu\nu,\sigma} - \overline{\Gamma}^\lambda_{\mu\sigma,\nu}}^{\text{linear terms}} + \underbrace{\overline{\Gamma}^\lambda_{\delta\sigma}\overline{\Gamma}^\delta_{\mu\nu} - \overline{\Gamma}^\lambda_{\delta\nu}\overline{\Gamma}^\delta_{\mu\sigma}}_{\text{non-linear terms}} . \qquad (36)$$

is the *Hybrid Riemann-Weyl Tensor*.

5.3 Linear Riemann tensor

Given that we have attained a geometry with tensorial affinities, it goes without saying that – insofar as the beleaguering problems besieging pure Riemann geometry is concerned – now is our time to reap the sweet fruits of our hard labour i.e. it is time to take the fullest advantage of the tensorial nature of the affinities. We now have at our disposal the mathematical and physical prerogative, legitimacy and liberty to choose a spacetime where the non-linear terms do not vanish identically i.e. $\overline{\Gamma}^\lambda_{\mu\nu} \neq 0$, but are bound by the gauge constraint*:

$$\overline{\Gamma}^\delta_{\mu\nu}\overline{\Gamma}^\lambda_{\delta\sigma} = \overline{\Gamma}^\lambda_{\delta\nu}\overline{\Gamma}^\delta_{\mu\sigma} . \qquad \textbf{(gauge constraint)} \qquad (37)$$

Clearly, from this, the resulting Riemann tensor becomes linear, i.e.:

$$\overline{R}^\lambda_{\mu\sigma\nu} = \overline{\Gamma}^\lambda_{\mu\nu,\sigma} - \overline{\Gamma}^\lambda_{\mu\sigma,\nu} . \qquad (38)$$

Just like that, we have thrown the non-linear terms out of our sight once and for all-time.

Clearly and without any doubt, this fact that we have chosen a spacetime that is governed by the gauge constraint (37),

means that we have just rid ourselves of the troublesome non-linear terms in the Riemann tensor (38), because with this beautiful and elegant choice (37), the non-linear terms now vanish identically to become but footnotes of history. The justification for this choice of gauge will become clear later when we derive from this tensor (38), the Maxwell equations [14] that we are used to know – albeit this time, these equations are being derived not for the electrodynamic phenomenon, but for the gravitodynamic phenomenon. In the next subsection, we will redefine the Riemann metric $g_{\mu\nu}$ in terms of the four-vector field $\mathcal{A}_\mu$ *via* the decomposition of the metric.

5.4 Decomposition of the metric tensor

A key feature of the present theory, as well as the previous versions of it as given in [8–10], is that of the decomposition of the metric tensor. The Riemann metric $g_{\mu\nu}$ is a compound rank two tensor field symmetric in the $\mu\nu$-indices and because of this symmetry, it consists of ten independent functions. In the present, the components of the metric tensor $\overline{g}_{\mu\nu}$ are a product of the components of a four-vector field $\mathcal{A}_\mu$, thus – this metric consists of four independent functions instead of ten as is the case in pure Riemann geometry.

The covariant $\mathcal{A}_\mu$ and contravariant $\mathcal{A}^\mu$ four-vectors are here to be defined as follows:

$$\mathcal{A}_\mu = (\mathcal{A}^\mu)^\dagger \qquad (39)$$

where the dagger-operation (†) is the usual *transpose-complex-conjugate operation* applied to the object in question[†], while the covariant $g_{\mu\nu}$, contravariant $g^{\mu\nu}$ and mixed covariant and contravariant metric $g_\mu{}^\nu, g^\mu{}_\nu$ tensors are defined in terms of the covariant $\mathcal{A}_\mu$ and contravariant $\mathcal{A}^\mu$ four-vectors as follows:

$$\begin{aligned} g_{\mu\nu} &= \mathcal{A}_\mu \mathcal{A}_\nu , & g_\mu{}^\nu &= \mathcal{A}_\mu \mathcal{A}^\nu , \\ g^{\mu\nu} &= \mathcal{A}^\mu \mathcal{A}^\nu , & g^\mu{}_\nu &= \mathcal{A}^\mu \mathcal{A}_\nu . \end{aligned} \qquad (40)$$

The mixed covariant and contravariant metric $g_\mu{}^\nu$ and $g^\mu{}_\nu$ tensors are in Riemann defined such in terms of the covariant $g_{\mu\nu}$ and contravariant $g^{\mu\nu}$ as follows:

$$\begin{aligned} g_\mu{}^\nu &= g_{\mu\delta} g^{\delta\nu} = \mathcal{A}_\delta \mathcal{A}^\delta g_\mu{}^\nu = g_\delta{}^\delta g_\mu{}^\nu = \delta_\mu{}^\nu \\ g^\mu{}_\nu &= g^{\mu\delta} g_{\delta\nu} = \mathcal{A}^\delta \mathcal{A}_\delta g^\mu{}_\nu = g^\delta{}_\delta g^\mu{}_\nu = \delta^\mu{}_\nu \end{aligned} \qquad (41)$$

where $\delta_\mu{}^\nu$ and $\delta^\mu{}_\nu$ are the usual Kronecker-Delta functions. From (41), it follows that:

$$g_\delta{}^\delta = \mathcal{A}_\delta \mathcal{A}^\delta = g^\delta{}_\delta = \mathcal{A}^\delta \mathcal{A}_\delta = 4 . \qquad (42)$$

---

*This gauge constraint allows us to obtain linear equations. This constraint is made possible by the fact that the affine connections are tensors.

[†]The four-vector $\mathcal{A}_\mu$ can either be a 4×4 or zero rank object. We are not sure at the moment which is which. If it turns out that $\mathcal{A}_\mu$ is a zero rank object, then the *dagger-operation* simple reduces to a *complex-conjugate operation*.

On this new Riemann-Weyl spacetime, the usual raising and lowering of the indices applicable in Riemann geometry holds, i.e.:

$$V_\mu = g_{\mu\delta} V^\delta = g_\mu{}^\delta V_\delta$$
$$V^\mu = g^{\mu\delta} V_\delta = g^\mu{}_\delta V^\delta$$
(43)

With the metric now having been redefined and its nature regarding the lowering and raising of indices, and that the length of the four-vector $\mathcal{A}_\mu$ is four units throughout all spacetime, we will proceed in the next subsection to deduce the first set of the field equations.

5.5 Field equations

Having set the stage, we shall now proceed to write down the resulting field equations.

5.5.1 Field equations I

If both the length and angles are to remain unaltered upon parallel transport, this can only happen if the curvature tensor $\overline{R}^\lambda_{\mu\sigma\nu}$ vanishes at all points of this spacetime, i.e.:

$$\overline{R}^\lambda_{\mu\sigma\nu} = 0. \quad \textbf{(natural gauge)} \quad (44)$$

Eq. (44) is a natural equation of the geometry; it emanates from the hypothesis of requiring that both the length and angles are to remain unaltered upon parallel transport. In general, the affine $\overline{\Gamma}^\lambda_{\mu\nu}$ is non-vanishing, i.e. $\overline{\Gamma}^\lambda_{\mu\nu} \neq 0$. So, the present *Hybrid Riemann-Weyl Spacetime* (HRWS) is a curvature-less space because vectors maintain or preserve both their length and orientation under parallel transport. Embedded or cojoined in this HRWS curvature tensor $\overline{R}^\lambda_{\mu\sigma\nu}$ are the Riemann curvature tensor $R^\lambda_{\mu\sigma\nu}$ and the geometrically derived material tensor $\mathcal{T}^\lambda_{\mu\sigma\nu}$. Because of the vanishing nature of HRWS curvature tensor $\overline{R}^\lambda_{\mu\sigma\nu}$, together with its linear nature (see §5.3), we will in the next subsection use these facts to unbundle the Riemann curvature tensor and the material tensor, thereby achieve what Einstein desired but failed to achieve – i.e. a material field derived from pure geometry.

5.5.2 Field equations II

Now that we have a theory linear in the curvature tensor – i.e. a theory in which the non-linear terms vanish – we can use this to separate the Weyl terms $\mathcal{W}^\lambda_{\mu\sigma\nu}$ from the Riemann terms $R^\lambda_{\mu\sigma\nu}$ and as well from the Q-tensor $Q^\lambda_{\mu\sigma\nu}$. That is, we can now rewrite the linear Riemann-Weyl curvature tensor $\overline{R}^\lambda_{\mu\sigma\nu}$ as is given in (38) as follows:

$$\overline{R}^\lambda_{\mu\sigma\nu} = R^\lambda_{\mu\sigma\nu} - \underbrace{\left(\mathcal{W}^\lambda_{\mu\sigma\nu} + Q^\lambda_{\mu\sigma\nu} \right)}_{\mathcal{T}^\lambda_{\mu\sigma\nu}} \quad (45)$$

where:

$$\begin{array}{llll} R^\lambda_{\mu\sigma\nu} & = & \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} & \text{(a)} \\[6pt] \mathcal{W}^\lambda_{\mu\sigma\nu} & = & \mathcal{W}^\lambda_{\mu\nu,\sigma} - \mathcal{W}^\lambda_{\mu\sigma,\nu} & \text{(b)} \\[6pt] Q^\lambda_{\mu\sigma\nu} & = & Q^\lambda_{\mu\nu,\sigma} - Q^\lambda_{\mu\sigma,\nu} & \text{(c)} \end{array} \quad (46)$$

are the *linear Riemann curvature tensor* (46a), the *linear Weyl curvature tensor* (46b), and the *linear Q-curvature tensor* (46c) or simply the *Q-tensor*.

An excogitative inspection of the Riemann curvature tensor will clearly reveal that this tensor is a function of the four-vector field $\mathcal{A}_\mu$, i.e. $R^\lambda_{\mu\sigma\nu} = R^\lambda_{\mu\sigma\nu}(\mathcal{A}_\alpha)$, while the Weyl and the Q-tensors are functions of ϱ, i.e. $\mathcal{W}^\lambda_{\mu\sigma\nu} = \mathcal{W}^\lambda_{\mu\sigma\nu}(\mathcal{J}_\alpha)$ and $Q^\lambda_{\mu\sigma\nu} = Q^\lambda_{\mu\sigma\nu}(\varrho)$. The Q-tensor is a direct function of ϱ while the Weyl tensor is not – remember (25) that $\mathcal{J}_\alpha = -\frac{1}{2}\partial_\alpha\varrho$, hence, as said $\mathcal{W}^\lambda_{\mu\sigma\nu} = \mathcal{W}^\lambda_{\mu\sigma\nu}(\varrho)$. Why are we talking of the functional dependence of these tensors?

The reason for excogitating on the functional dependence of these tensors is that we not only want to, but shall identify the Riemann curvature tensor as describing *Einstein's beautiful marble* that, in Einstein's vision and desideratum, is described by the metric tensor $g_{\mu\nu}$; while the Weyl curvature tensor and the Q-curvature tensor describe *Einstein's ugly wood* – albeit – varnished (polished) wood this time around since the field ϱ is later to be identified with the beautiful – albeit – arcane quantum mechanical object, namely the quantum probability amplitude.

After the above deliberations, it therefore makes much sense to house the Weyl curvature tensor and the Q-curvature tensor under one roof since they constitute the material tensor. To that end, let us represent the sum total material curvature tensor using the symbol $\mathcal{T}^\lambda_{\mu\sigma\nu}$ where:

$$\mathcal{T}^\lambda_{\mu\sigma\nu} = \mathcal{W}^\lambda_{\mu\sigma\nu} + Q^\lambda_{\mu\sigma\nu}. \quad (47)$$

With the above definition (47) of the material tensor, it follows that the Riemann-Weyl curvature tensor $\overline{R}^\lambda_{\mu\sigma\nu}$ can now be written as an object comprising two main tensors expressing the fields ($R^\lambda_{\mu\sigma\nu}$) and their corresponding material ($\mathcal{T}^\lambda_{\mu\sigma\nu}$) counterpart:

$$\overline{R}^\lambda_{\mu\sigma\nu} = R^\lambda_{\mu\sigma\nu} - \mathcal{T}^\lambda_{\mu\sigma\nu}. \quad (48)$$

What we have done – from (45) to (48) above – is to indulge and cajole the reader to the idea of envisioning the Riemann-Weyl tensor in Einstein's vision of a *marble* and *wood* component, albeit, with the wood now recast into its quantum mechanical description.

Now, from (44) and (48), it follows that:

$$R^\lambda_{\mu\sigma\nu} = \mathcal{T}^\lambda_{\mu\sigma\nu}. \quad (49)$$

At this point – if it turns out that this theory proves to be a correct description of physical and natural reality as we know it – we have no doubt in our mind that if Einstein were watching from above or from wherever in the interstices of spacetime,

he must be smiling endlessly because his lifelong endeavour was to derive* the material tensor from pure geometry and not to insert it by sleight of mind as he did with his gravitational field (7). In-line with Einstein's deepest quest and longing insofar in attaining a final UFT of all the forces of Nature, we have in the present derived the material tensor from pure geometry.

As we saw previously in §2.5, Einstein's ultimate goal was to turn *wood* into *marble* so to speak, which meant deriving the material field from pure geometry. Einstein wanted to find the final theory; this he pursued to the very end of his life to a point that while on his deathbed on April 18, 1955, instead of worrying about the imminent end of his fruitful life, he asked for a pen and his notes so that he could continue to work on the unified field theory that he was working on at the time. It is sad to say that Einstein never laid a fertile egg on this front – i.e. the front of unification.

Be that as it may, it is without an iota of doubt that we say that if what is before us proves itself to have a correspondence with physical and natural reality, then we can safely say we have achieved one of Einstein's goals to attaining the "elicit dream of a Final Theory" by deriving the material tensor from pure geometry – wood, one way or the other, has finally been turned into marble! This we are certain has been achieved in the present UFT. The only question is, *Does the theory correspond with physical and natural reality?* This we leave in the able hands of our reader so that they may be their own judge on that very important matter.

5.5.3 Field equations III

First Voss-Bianchi Identities: Further, we shall derive other field equations. We know that the Riemann curvature tensor satisfies the first Voss-Bianchi[†] identity, namely:

$$R^\lambda_{\mu\sigma\nu} + R^\lambda_{\nu\mu\sigma} + R^\lambda_{\sigma\nu\mu} \equiv 0\,. \tag{50}$$

From this first Bianchi identity and as well from (49), it follows that:

$$\mathcal{T}^\lambda_{\mu\sigma\nu} + \mathcal{T}^\lambda_{\nu\mu\sigma} + \mathcal{T}^\lambda_{\sigma\nu\mu} \equiv 0\,. \tag{51}$$

In the next subsection, we present the second Voss-Bianchi identity.

---

*Here, we must hasten to say that we have not exactly derived the material tensor field $\mathcal{T}^\lambda_{\mu\sigma\nu}$, but merely justified its physical existence on the fundamental basis of the need for tensorial affinities. Thus, this material field is not only justifiable on a fundamental physical level, but very much a part and parcel of the whole edifice of the marvellous structure of the spacetime continuum.

†In the wider literature – if not every common text where these identities are considered – they are referred to as the *Bianchi Identities* after the Italian mathematician – Luigi Bianchi (1856-1928) who published them in 1902 [60]. However, the reality to the matter is that these identities were first derived and published by the German mathematician Aurel Voss (1845-1931) in 1880 [61]. Hence, keeping matters in their correct historic record and perspective, and to give due credit and acknowledgement of the work of Aurel Voss, we herein refer to these identities ((50) and (52)) as the *Voss-Bianchi Identities*.

5.5.4 Field equations IV

Second Voss-Bianchi Identities: Furthermore, we are going to derive our last set of field equations. We know that the Riemann curvature tensor satisfies the second Voss-Bianchi identity, namely:

$$R^\lambda_{\upsilon\sigma,\nu} + R^\lambda_{\upsilon\nu\mu,\sigma} + R^\lambda_{\upsilon\sigma\nu,\mu} \equiv 0\,. \tag{52}$$

From this second Bianchi identity and as well from (49), it follows that:

$$\mathcal{T}^\lambda_{\upsilon\sigma,\nu} + \mathcal{T}^\lambda_{\upsilon\nu\mu,\sigma} + \mathcal{T}^\lambda_{\upsilon\sigma\nu,\mu} \equiv 0\,. \tag{53}$$

In the next section, we shall explore (49), (50), (51), (52) and (53), and from these equations, we shall see that one is able to obtain field equations that we are already familiar with. Before we depart this section, we must say that while we have shown that the material tensor $\mathcal{T}^\lambda_{\upsilon\mu\sigma,\nu}$ does satisfy the Voss-Bianchi identities, the subcomponents $(\mathcal{W}^\lambda_{\upsilon\mu\sigma,\nu}; Q^\lambda_{\upsilon\mu\sigma,\nu})$ of this tensor also satisfy the Voss-Bianchi identities, i.e.:

$$
\begin{aligned}
\mathcal{W}^\lambda_{\mu\sigma\nu} + \mathcal{W}^\lambda_{\nu\mu\sigma} + \mathcal{W}^\lambda_{\sigma\nu\mu} &\equiv 0 & \text{(a)} \\
Q^\lambda_{\mu\sigma\nu} + Q^\lambda_{\nu\mu\sigma} + Q^\lambda_{\sigma\nu\mu} &\equiv 0 & \text{(b)} \\
\mathcal{W}^\lambda_{\upsilon\mu\sigma,\nu} + \mathcal{W}^\lambda_{\upsilon\nu\mu,\sigma} + \mathcal{W}^\lambda_{\upsilon\sigma\nu,\mu} &\equiv 0 & \text{(c)} \\
Q^\lambda_{\upsilon\mu\sigma,\nu} + Q^\lambda_{\upsilon\nu\mu,\sigma} + Q^\lambda_{\upsilon\sigma\nu,\mu} &\equiv 0 & \text{(d)}
\end{aligned}
\tag{54}
$$

where in (54a,b) and (54c,d), we have the first and second Voss-Bianchi identities of $\mathcal{W}^\lambda_{\upsilon\mu\sigma,\nu}$ and $Q^\lambda_{\upsilon\mu\sigma,\nu}$ respectively.

6 Affine, Riemann and the material tensor

In the present section, we are going to calculate or express the affine tensor $\Gamma^\lambda_{\mu\nu}$, the Riemann tensor $R_{\mu\nu}$, and the material tensor $\mathcal{T}_{\mu\nu}$ in terms of a Maxwell field tensor $\mathfrak{F}_{\mu\nu}$. This exercise is meant to prepare us for the work to be conducted in §7 where we are going to write down our desired *Maxwell Gravitomagnetic Field Equations*.

6.1 Affine tensor

We already know from (5) that the affine connection $\Gamma^\lambda_{\mu\nu}$ is such that $2\Gamma^\lambda_{\mu\nu} = g^{\delta\lambda}\left(g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right)$, and from the present new findings that the decomposed Riemann metric tensor is such that $g_{\mu\nu} = \mathcal{A}_\mu \mathcal{A}_\nu$. What we want – and will – do here is to substitute the decomposed metric into the affine wherefrom we expect to obtain the usual Maxwell-type field tensor of electromagnetism. To that end, we substitute the metric into the affine and then differentiate this metric as required by the differentials in the affine – doing so, we obtain:

$$2\Gamma^\lambda_{\mu\nu} = g^{\delta\lambda}\Big[\overbrace{\mathcal{A}_\delta \mathcal{A}_{\mu,\nu}}^{\text{Term I}} + \overbrace{\mathcal{A}_\mu \mathcal{A}_{\delta,\nu}}^{\text{Term II}} + \overbrace{\mathcal{A}_\delta \mathcal{A}_{\nu,\mu}}^{\text{Term III}} + \underbrace{\mathcal{A}_\nu \mathcal{A}_{\delta,\mu}}_{\text{Term IV}} - \underbrace{\mathcal{A}_\mu \mathcal{A}_{\nu,\delta}}_{\text{Term V}} - \underbrace{\mathcal{A}_\nu \mathcal{A}_{\mu,\delta}}_{\text{Term VI}} \Big]\,. \tag{55}$$

Now, we shall identify the labelled terms in (55), that is, terms that will yield for us the desired Maxwell-type field tensor of electromagnetism.

1. **Terms II and V:** Combining Term II and Term V, we will have:

$$\mathcal{A}_\mu \mathfrak{F}_{\delta\nu} = \mathcal{A}_\mu \left(\mathcal{A}_{\delta,\nu} - \mathcal{A}_{\nu,\delta} \right) \tag{56}$$

where:

$$\mathfrak{F}_{\delta\nu} = \mathcal{A}_{\delta,\nu} - \mathcal{A}_{\nu,\delta} \tag{57}$$

is the gravitomagnetic field tensor. This tensor (57) is our desired Maxwell-type field tensor of electromagnetism – albeit – this time – as per our desire – it is appearing in the equations of gravitation and not electromagnetism.

2. **Terms IV and VI:** Further, combining Term IV and Term VI, we will have:

$$\mathcal{A}_\nu \mathfrak{F}_{\delta\mu} = \mathcal{A}_\nu \left(\mathcal{A}_{\delta,\mu} - \mathcal{A}_{\mu,\delta} \right) \tag{58}$$

where – as in (57):

$$\mathfrak{F}_{\delta\mu} = \mathcal{A}_{\delta,\mu} - \mathcal{A}_{\mu,\delta} \tag{59}$$

is the same gravitomagnetic field tensor – the only difference is the interchange of the indices.

3. **Terms I and III:** Lastly, combining Term I and Term III, we will have:

$$\mathcal{A}_\delta \Omega_{\mu\nu} = \mathcal{A}_\delta \left(\mathcal{A}_{\mu,\nu} + \mathcal{A}_{\nu,\mu} \right) \tag{60}$$

where – this time:

$$\Omega_{\mu\nu} = \mathcal{A}_{\mu,\nu} + \mathcal{A}_{\nu,\mu} \tag{61}$$

is not a gravitomagnetic field tensor, but some non-tensorial object that will prove to be absolutely essential and necessary in the generation of the source-free Maxwell-type equations for gravitomagnetism.

From the foregoing, it follows from (57), (59) and (61), that:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\delta\lambda} \left[\mathcal{A}_\mu \mathfrak{F}_{\delta\nu} + \mathcal{A}_\nu \mathfrak{F}_{\delta\mu} + \mathcal{A}_\delta \Omega_{\mu\nu} \right]. \tag{62}$$

Now, multiplying the terms in the square bracket by $g^{\delta\lambda}$, the meaning of which is that we have to raise the δ-index in these square brackets and reset it so that it now equals λ, i.e.:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \left[\mathcal{A}_\mu \mathfrak{F}^\lambda{}_\nu + \mathcal{A}_\nu \mathfrak{F}^\lambda{}_\mu + \mathcal{A}^\lambda \Omega_{\mu\nu} \right]. \tag{63}$$

In (63), we most importantly have expressed the Christoffel affine in terms of the Maxwell field tensor $\mathfrak{F}_{\mu\nu}$. In the next section, we shall proceed to express the Riemann tensor in terms of the same Maxwell field tensor $\mathfrak{F}_{\mu\nu}$.

For the purposes of convenience in the coming computations to be made in the subsequent sections, we shall write down the Christoffel affine (i.e. (63)), as follows:

$$\Gamma^\lambda_{\mu\nu} = \check{\Gamma}^\lambda_{\mu\nu} + \Omega^\lambda_{\mu\nu} \tag{64}$$

where:

$$\check{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} \left(\mathcal{A}_\mu \mathfrak{F}^\lambda{}_\nu + \mathcal{A}_\nu \mathfrak{F}^\lambda{}_\mu \right) \tag{65}$$

and:

$$\Omega^\lambda_{\mu\nu} = \frac{1}{2} \mathcal{A}^\lambda \Omega_{\mu\nu}. \tag{66}$$

The object $\check{\Gamma}^\lambda_{\mu\nu}$ is a tensor while $\Omega^\lambda_{\mu\nu}$ is not, for, upon a transformation of the system of coordinates, this affine $\Omega^\lambda_{\mu\nu}$ transforms in the exact same manner as the Christoffel symbols (see (6)), that is, it transforms as follows:

$$\Omega^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\delta} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Omega^\delta_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\delta} \frac{\partial^2 x^\delta}{\partial x^{\mu'} \partial x^{\nu'}}. \tag{67}$$

In the next subsection, as we continue to work toward the writing down of the resultant field equations, we shall express the Riemann tensor in terms of the gravitomagnetic Maxwell-type tensor $\mathfrak{F}_{\mu\nu}$.

6.2 Riemann tensor

We are not only going to express the Riemann tensor in terms of the gravitomagnetic Maxwell-type field tensor $\mathfrak{F}_{\mu\nu}$ but decompose this tensor into three tensors. To that end, we will start-off by substituting the newly re-expressed Christoffel affine in (64) into the linear Riemann tensor (46a); so doing, we obtain:

$$R^\lambda_{\mu\sigma\nu} = \check{\Gamma}^\lambda_{\mu\nu,\sigma} - \check{\Gamma}^\lambda_{\mu\sigma,\nu} + \Omega^\lambda_{\mu\nu,\sigma} - \Omega^\lambda_{\mu\sigma,\nu}$$
$$= \check{R}^\lambda_{\mu\sigma\nu} + \Omega^\lambda_{\mu\sigma\nu} \tag{68}$$

where:

$$\check{R}^\lambda_{\mu\sigma\nu} = \check{\Gamma}^\lambda_{\mu\nu,\sigma} - \check{\Gamma}^\lambda_{\mu\sigma,\nu}$$
$$\Omega^\lambda_{\mu\sigma\nu} = \Omega^\lambda_{\mu\nu,\sigma} - \Omega^\lambda_{\mu\sigma,\nu} \tag{69}$$

are tensors. The reader will need to verify for themselves that – indeed – these objects are tensors.

Further, we will express $\check{R}^\lambda_{\mu\sigma\nu}$ in terms of the field tensor $\mathfrak{F}_{\mu\nu}$ by substituting $\check{\Gamma}^\lambda_{\mu\nu}$ as it is given in (65); so doing, one obtains:

$$\check{R}^\lambda_{\mu\sigma\nu} = \frac{1}{2} \left(\mathcal{A}_\mu \mathfrak{F}^\lambda{}_{\nu,\sigma} + \mathcal{A}_\nu \mathfrak{F}^\lambda{}_{\mu,\sigma} \right) -$$
$$- \frac{1}{2} \left(\mathcal{A}_\mu \mathfrak{F}^\lambda{}_{\sigma,\nu} + \mathcal{A}_\sigma \mathfrak{F}^\lambda{}_{\mu,\nu} \right) +$$
$$+ \frac{1}{2} \left(\mathcal{A}_{\mu,\sigma} \mathfrak{F}^\lambda{}_\nu + \mathcal{A}_{\nu,\sigma} \mathfrak{F}^\lambda{}_\mu \right) -$$
$$- \frac{1}{2} \left(\mathcal{A}_{\mu,\nu} \mathfrak{F}^\lambda{}_\sigma + \mathcal{A}_{\sigma,\nu} \mathfrak{F}^\lambda{}_\mu \right)$$
$$= \acute{R}^\lambda_{\mu\sigma\nu} + \grave{R}^\lambda_{\mu\sigma\nu} \tag{70}$$

where:

$$\acute{R}^\lambda_{\mu\sigma\nu} = \frac{1}{2} \left(\mathcal{A}_\mu \mathfrak{F}^\lambda{}_{\nu,\sigma} + \mathcal{A}_\nu \mathfrak{F}^\lambda{}_{\mu,\sigma} \right)$$
$$- \frac{1}{2} \left(\mathcal{A}_\mu \mathfrak{F}^\lambda{}_{\sigma,\nu} + \mathcal{A}_\sigma \mathfrak{F}^\lambda{}_{\mu,\nu} \right) \tag{71}$$

and:

$$\grave{R}^\lambda_{\mu\sigma\nu} = \frac{1}{2} \left(\mathcal{A}_{\mu,\sigma} \mathfrak{F}^\lambda{}_\nu + \mathcal{A}_{\nu,\sigma} \mathfrak{F}^\lambda{}_\mu \right) -$$
$$- \frac{1}{2} \left(\mathcal{A}_{\mu,\nu} \mathfrak{F}^\lambda{}_\sigma + \mathcal{A}_{\sigma,\nu} \mathfrak{F}^\lambda{}_\mu \right) \tag{72}$$

are tensors. Once again, the reader will need to verify for themselves that these objects are indeed tensors. Therefore, from (68) and (70), it follows that:

$$R^\lambda_{\mu\sigma\nu} = \acute{R}^\lambda_{\mu\sigma\nu} + \check{R}^\lambda_{\mu\sigma\nu} + \Omega^\lambda_{\mu\sigma\nu}\,. \tag{73}$$

In (73), we have – as desired – not only re-expressed the Riemann tensor, but decomposed it into three part tensors. Now – in the next subsection, we will conduct the same exercise with the material tensor. All this re-expression and decomposition is all gearing up for the derivation of the result field equation of the theory.

6.3 Material tensor

Just as we have decomposed the Riemann curvature tensor into three parts in (73), we are now going to decompose the material curvature tensor $\mathcal{T}^\lambda_{\mu\sigma\nu}$ into three parts by decomposing into two parts, the linear Weyl curvature tensor $\mathcal{W}^\lambda_{\mu\sigma\nu}$. To that end, decomposing the Weyl part of the material tensor field by differentiating the products $g^\lambda{}_\mu \mathcal{I}_\nu$, we obtain that:

$$\begin{aligned}
\mathcal{T}^\lambda_{\mu\sigma\nu} =\ & \left(g^\lambda{}_\mu \mathcal{I}_{\nu,\sigma} + g_\nu{}^\lambda \mathcal{I}_{\mu,\sigma} - g_{\mu\nu} \mathcal{I}^\lambda{}_{,\sigma}\right) - \\
& - \left(g^\lambda{}_\mu \mathcal{I}_{\sigma,\nu} + g_\sigma{}^\lambda \mathcal{I}_{\mu,\nu} - g_{\mu\sigma} \mathcal{I}^\lambda{}_{,\nu}\right) + \\
& + \left(g^\lambda{}_{\mu,\sigma} \mathcal{I}_\nu + g_{\nu,\sigma}{}^\lambda \mathcal{I}_\mu - g_{\mu\nu,\sigma} \mathcal{I}^\lambda\right) - \\
& - \left(g^\lambda{}_{\mu,\nu} \mathcal{I}_\sigma + g_{\sigma,\nu}{}^\lambda \mathcal{I}_\mu - g_{\mu\sigma,\nu} \mathcal{I}^\lambda\right) + \\
& + Q^\lambda_{\mu\sigma\nu} \\
=\ & \acute{\mathcal{T}}^\lambda_{\mu\sigma\nu} + \grave{\mathcal{T}}^\lambda_{\mu\sigma\nu} + Q^\lambda_{\mu\sigma\nu}
\end{aligned} \tag{74}$$

where the newly introduced tensors $\acute{\mathcal{T}}^\lambda_{\mu\sigma\nu}$ and $\grave{\mathcal{T}}^\lambda_{\mu\sigma\nu}$ are explicitly defined as follows:

$$\begin{aligned}
\acute{\mathcal{T}}^\lambda_{\mu\sigma\nu} =\ & \left(g^\lambda{}_\mu \mathcal{I}_{\nu,\sigma} + g_\nu{}^\lambda \mathcal{I}_{\mu,\sigma} - g_{\mu\nu} \mathcal{I}^\lambda{}_{,\sigma}\right) - \\
& - \left(g^\lambda{}_\mu \mathcal{I}_{\sigma,\nu} + g_\sigma{}^\lambda \mathcal{I}_{\mu,\nu} - g_{\mu\sigma} \mathcal{I}^\lambda{}_{,\nu}\right) \\
=\ & \left[g_\nu{}^\lambda \mathcal{I}_{\mu,\sigma} - g_{\mu\nu} \mathcal{I}^\lambda{}_{,\sigma}\right] - \left[g_\sigma{}^\lambda \mathcal{I}_{\mu,\nu} - g_{\mu\sigma} \mathcal{I}^\lambda{}_{,\nu}\right]
\end{aligned} \tag{75}$$

and:

$$\begin{aligned}
\grave{\mathcal{T}}^\lambda_{\mu\sigma\nu} =\ & \left(g^\lambda{}_{\mu,\sigma} \mathcal{I}_\nu + g_{\nu,\sigma}{}^\lambda \mathcal{I}_\mu - g_{\mu\nu,\sigma} \mathcal{I}^\lambda\right) - \\
& - \left(g^\lambda{}_{\mu,\nu} \mathcal{I}_\sigma + g_{\sigma,\nu}{}^\lambda \mathcal{I}_\mu - g_{\mu\sigma,\nu} \mathcal{I}^\lambda\right)\,.
\end{aligned} \tag{76}$$

Written in a much clearer manner:

$$\mathcal{T}^\lambda_{\mu\sigma\nu} = \acute{\mathcal{T}}^\lambda_{\mu\sigma\nu} + \grave{\mathcal{T}}^\lambda_{\mu\sigma\nu} + Q^\lambda_{\mu\sigma\nu}\,. \tag{77}$$

At this juncture, having now written down the Riemann and the material curvature tensors in the manner that we have written them in (73) and (77), we are now ready to explore the *Resultant Field Equations*.

7 Resultant field equations

Having calculated in (73) and (77), the Riemann and the material curvature tensors into a form that allows us to execute the main business of the day of deriving (deducing) the source-coupled and source-free field equations respectively, we are going to start by writing main field (49) with the decoupled Riemann and the material curvature tensors, i.e.:

$$\underbrace{\overbrace{\underset{\text{L I}}{\underbrace{\acute{R}^\lambda_{\mu\sigma\nu}}} + \underset{\text{L II}}{\underbrace{\check{R}^\lambda_{\mu\sigma\nu}}} + \underset{\text{L III}}{\underbrace{\Omega^\lambda_{\mu\sigma\nu}}}}^{\text{marble}}}_{R^\lambda_{\mu\sigma\nu}(\mathcal{A}_\alpha)} = \underbrace{\overbrace{\underset{\text{R I}}{\underbrace{\acute{\mathcal{T}}^\lambda_{\mu\sigma\nu}}} + \underset{\text{R II}}{\underbrace{\grave{\mathcal{T}}^\lambda_{\mu\sigma\nu}}} + \underset{\text{R III}}{\underbrace{Q^\lambda_{\mu\sigma\nu}}}}^{\text{varnished wood}}}_{\mathcal{T}^\lambda_{\mu\sigma\nu}(\varrho)}\,. \tag{78}$$

Eq. (78) is the single most important equation of our theory and it is out of this equation that we are to derive the rest of the field equations of the theory. The setting up of the said field equations of the theory we shall do by way of introduction of the appropriate gauge constraints. If it were us creating the Universe out of (78), how were we going to proceed to accomplish this monumental task? Our thinking is that a term on the left-hand side in (78) has a corresponding term on the right. Therefore, if our said thinking is reasonable or correct, then our task to finding the sought-for field equations is simply to correctly match the left- and right-hand side terms in (78). If the choice we make turns out to describe our Universe as we know it, then this choice will somehow be the choice that has been made in creating the Universe! This should give us a foothold in seeking answers to some of Einstein's deep philosophical questions about the creation of the Universe.

With regard to the creation of the Universe, Einstein is famously quoted as having said *I want to know the mind of God ... whether or not He had a choice in making the Universe* and on a different occasion, as having said *When I am judging a theory, I ask myself whether, if I were God, I would have arranged the World in such a way.* [62]. These are very deep questions that Einstein was asking about physical and natural reality. Using Einstein's words as a source of inspiration, strength and guidance, we find ourself asking *How are we to construct the resulting field equations from (78)?*

It is with great equanimity that we say that we are of the veritable standpoint that the first term (labelled L I) on the left-hand side of (78) corresponds to the first term on the right-hand side (labelled R I); that, the second term on the left (labelled L II) corresponds to the second term on the right-hand side (labelled R II); and, likewise, that, the L III term corresponds to the R III term, i.e.:

$$\begin{aligned}
\acute{R}^\lambda_{\mu\sigma\nu} &= \acute{\mathcal{T}}^\lambda_{\mu\sigma\nu} & \text{(a)} \\
\check{R}^\lambda_{\mu\sigma\nu} &= \grave{\mathcal{T}}^\lambda_{\mu\sigma\nu} & \text{(b)} \\
\Omega^\lambda_{\mu\sigma\nu} &= Q^\lambda_{\mu\sigma\nu} & \text{(c)}
\end{aligned} \tag{79}$$

(gauge constraints)

Eqs. (79a), (79b) and (79c) are constraints on (78), albeit endogauge constraints of the theory. Shortly in §7.1 and §7.2, we shall show that (79a) and (79c) are the gravitational source-coupled and source-free Maxwell's field equations [14]. Exploration of (79b) is left for a later paper.

7.1 Source-coupled field equations

As claimed above, we shall now proceed to show that (79a) is indeed the gravitomagnetic Maxwell-type source-coupled field equation. To see this, we shall multiply (79a) on both sides by $\mathcal{A}^\alpha$ and thereafter contracting the (α, μ) and (λ, σ)-indices by setting $\alpha = \mu = \beta$ and $\lambda = \sigma = \delta$; so doing, we obtain:

$$\mathcal{A}^\beta \acute{R}^\delta_{\beta\delta\nu} = \mathcal{A}^\beta \acute{T}^\delta_{\beta\delta\nu} \,. \tag{80}$$

On the other hand, for $\mathcal{A}^\beta \acute{R}^\delta_{\beta\delta\nu}$, we have that:

$$\mathcal{A}^\beta \acute{R}^\delta_{\beta\delta\nu} = \mathfrak{F}^\delta_{\ \nu,\delta} \,, \tag{81}$$

and this already looks very familiar – is this not the well known left-hand side of Maxwell's source-coupled field equation [14] – albeit – in the realm of the gravitational phenomenon? It certainly is.

For $\mathcal{A}^\beta \acute{T}^\delta_{\beta\delta\nu}$, we have that:

$$\begin{aligned} \mathcal{A}^\beta \acute{T}^\delta_{\beta\delta\nu} &= -2\mathcal{A}^\delta \mathcal{I}_{\delta,\nu} - \mathcal{I}^\delta_{\ ,\delta} \mathcal{A}_\nu \\ &= -2\mathcal{A}^\delta \partial_\delta \partial_\nu \varrho + (\Box \varrho / 2)\, \mathcal{A}_\nu \\ &= -\widetilde{\mu} \mathfrak{J}_\nu + \kappa^2 \mathcal{A}_\nu \end{aligned} \tag{82}$$

where from our foreknowledge and, by way of inference and inspiration from experience, we have set in (82):

$$2\mathcal{A}^\delta \partial_\delta \partial_\nu \varrho \;=\; \widetilde{\mu} \mathfrak{J}_\nu \,,$$

with $\widetilde{\mu}$ being a coupling constant that restores dimensional consistency and $\mathfrak{J}_\nu$ is the conserved gravitational four-current density (or four-momentum density). Thus from the foregoing, it follows that $\mathfrak{F}^\delta_{\ \nu,\delta} = -\widetilde{\mu} \mathfrak{J}_\nu$. We expect that $\widetilde{\mu}$ should embody (represent) Newton's gravitational constant. For aesthetic reasons, we prefer to write this equation $\mathfrak{F}^\delta_{\ \nu,\delta} = -\widetilde{\mu} \mathfrak{J}_\nu$ in the form:

$$\partial^\mu \mathfrak{F}_{\mu\nu} = -\widetilde{\mu} \mathfrak{J}_\nu + \kappa^2 \mathcal{A}_\nu \,. \tag{83}$$

The above (83) is Maxwell's source-coupled field equations [14], albeit in the present case, these equations are emerging not in the realm and domain of electrodynamics, but pure gravitation. This derivation of (83) completes the first part of the main task of the present paper. In the next section, we tackle the second part where we shall derive the source-free gravitomagnetic field equations.

7.2 Source-free field equations

Having derived the source-coupled field (83), we are now going to deduce (derive) the source-free field equations from

the field (79c) by means of the first Voss-Bianchi identities (in (50)). To that end, we shall achieve this by conducting a cyclic permutation of the $\mu\sigma\nu$-indices in (79c), i.e.:

$$\Omega^\lambda_{[\mu\sigma\nu]} = Q^\lambda_{[\mu\sigma\nu]} \,. \tag{84}$$

The square-brackets in (84) here and after indicate the cyclic permutation of the indices for the particular tensor in question.

Now for $Q^\lambda_{[\mu\sigma\nu]}$, we already know from (54b) that $Q^\lambda_{[\mu\sigma\nu]} \equiv 0$. For $\Omega^\lambda_{[\mu\sigma\nu]}$, a computation of this tensor will yield $\Omega^\lambda_{[\mu\sigma\nu]} = \mathcal{A}^\lambda \mathfrak{F}_{\mu\sigma,\nu} + \mathcal{A}^\lambda \mathfrak{F}_{\nu\mu,\sigma} + \mathcal{A}^\lambda \mathfrak{F}_{\sigma\nu,\mu}$. Therefore, combining this with (54b) and (84), it follows that:

$$\partial_\nu \mathfrak{F}_{\mu\sigma} + \partial_\sigma \mathfrak{F}_{\nu\mu} + \partial_\mu \mathfrak{F}_{\sigma\nu} \equiv 0 \,. \tag{85}$$

If anything, the above (85) is indeed Maxwell's source-free field equations [14] written in terms of the covariant derivative, albeit in the present case, this equation is emerging deep within the full domains of gravitation, i.e. from the pure soils of geometry. The derivation of (85) technically completes the main task of the present paper. We surely have shown that one can derive Maxwell's equations [14] from the viewpoint of a Riemann-Weyl geometry standpoint. This must give a strong leverage and impetus to gravitomagnetism as a legitimate and plausible fundamental phenomenon lying well within the domain and realm of real science that is well worthy of the attention of a knowledge seeking scientific mind.

8 Discussion

For what we wanted to achieve in the present paper, we are of the view that we have succeeded – i.e. succeeded in demonstrating that – a legitimate fundamental geometrodynamic justification of gravitomagnetism can be found from the fertile soils of Weyl's [1] beautiful but now thought to be dead and obsolete theory. We further believe that this justification adds much greater impetus to the justification one obtains from say Heras's [43] insightful and powerful existence theorem, or from Behera's [32] interesting theorem that much like the electromagnetic force, the gravitational force is susceptible to a four-vector description. Furthermore, we are also confident that what we have presented herein is being presented for the first time in the scientific literature, hence, these are new blossoms in the realm of ideas.

In the following subsections (i.e. §8.1 and §8.3), we shall discuss (in §8.2) rather briefly, the gauge conditions arising in the present theory and in §8.3, our thoughts regarding a *Quantum Theory of Gravity*. No tangible conclusion is drawn from this paper as this is left for our able and agile reader to makeup their own mind regarding what has herein been presented. We are of the view that this paper is clear and straight forward enough, so much that it should not be difficult to come to a conclusion as to what this paper really means regarding gravitomagnetism.

8.1 Architecture and design of theory

We have used Weyl's modified theory [1] to give a legal and fundamental basis for the existence of gravitomagnetism, and this gravitomagnetic theory can and will be extended in the next paper to demonstrate a possible unity between gravitation and electricity. Naturally and with justification, one will (or may) ask the interesting question: *What in the present have we now done differently that no one has done in the past to this 102 year old theory that suffered a moumental still-birth under the able hands and agile eyes of Albert Einstein's razor sharp intellect whose criticism made sure that Weyl's theory [1] failed?*

In a nutshell, what we have done in our quest to give a fundamental geometrodynamic justification of gravitomagnetism, is to modify Weyl's [1] supposedly failed geometry whose endeavour was to bring the gravitational and electromagnetic forces into one grand scheme, *via* the subtle addition of a conformal scalar leading to the addition of a tensorial affine connection that is a function of a four-vector field and have turned Weyl's [1] scalar into a pseudo-scalar of the second kind. Succinctly stated – in just nine major steps – this is what we have done:

1. The first insight has been to make the Weyl [1] conformal scalar a pseudo-scalar of the second kind and this allows us to obtain tensorial affinities within the realm of Weyl's theory [1].

2. The second insight is to realize that the Riemann metric tensor $g_{\mu\nu}$ can be decomposed into a product of a four-vector $\mathcal{A}_\mu$ so that, instead of describing the metric using ten potentials, it is now described by only four potentials: $g_{\mu\nu} = \mathcal{A}_\mu \mathcal{A}_\nu$.

3. Third – in a Weyl [1] fashion – *via* the newly introduced pseudo-scalar, we added a new non-tensorial affine connection $Q^\lambda_{\mu\nu}$ (i.e. $\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \mathcal{W}^\lambda_{\mu\nu} - Q^\lambda_{\mu\nu}$) and demanded of it to yield for us a resultant affine connection that is a tensor. Once we have a tensorial affine connections, it means we now have the tool required to obtain Einstein's desired geometry that is such that both the length and direction of a vector under parallel transport are preserved.

4. Fourth, the preservation of both the direction and length of the vector under parallel transport automatically implies that the curvature tensor $\bar{R}^\lambda_{\mu\sigma\nu}$ will vanish identically everywhere, i.e. $\bar{R}^\lambda_{\mu\sigma\nu} \equiv 0$. The equation $\bar{R}^\lambda_{\mu\sigma\nu} \equiv 0$ becomes our theory's first and main field equation.

5. Fifth – because the affine connections are now tensors, it is possible to construct for ourself – by way of choice (gauge constraint) – an effective geometry which is such that the non-linear terms $\bar{\Gamma}^\delta_{\mu\nu}\bar{\Gamma}^\lambda_{\delta\alpha}$ and $\bar{\Gamma}^\lambda_{\delta\nu}\bar{\Gamma}^\delta_{\mu\alpha}$ in the curvature tensor $\bar{R}^\lambda_{\mu\sigma\nu}$ vanish identically. This gauge choice results in three separate linear curvature tensors making up the resultant curvature tensor, namely $R^\lambda_{\mu\sigma\nu}$,

$\mathcal{T}^\lambda_{\mu\sigma\nu}$, and $Q^\lambda_{\mu\sigma\nu}$.

6. Sixth – the main field equation $\bar{R}^\lambda_{\mu\sigma\nu} \equiv 0$ is split into parts as $R^\lambda_{\mu\sigma\nu} = \mathcal{T}^\lambda_{\mu\sigma\nu}$ where $R^\lambda_{\mu\sigma\nu}$ is the Riemann curvature tensor and $\mathcal{T}^\lambda_{\mu\sigma\nu}$ the material curvature tensor.

7. Seventh – a set of gauge conditions (constraints) are then deliberately introduced – i.e. conditions which, when used in conjunction with the source-coupled field equation $R^\lambda_{\mu\sigma\nu} = \mathcal{T}^\lambda_{\mu\sigma\nu}$, yield for us the desired source-coupled Maxwell Geometrodymanic Equations [14].

8. Ante-penultimate – we split each of the curvature tensors $R^\lambda_{\mu\sigma\nu}$ and $\mathcal{T}^\lambda_{\mu\sigma\nu}$ into three parts each of which are also tensors.

9. Penultimate – we deduce the resultant field equations by relating each of the three tensors making up the Riemann curvature tensor $R^\lambda_{\mu\sigma\nu}$ to the three parts making up the material curvature tensor $\mathcal{T}^\lambda_{\mu\sigma\nu}$, wherefrom we obtain the first and second Maxwell's field equations [14], albeit in the realm of gravitomagnetism.

The above nine steps are an executive summary of the road leading to the theory here laid down. There is not much to say any further regarding the construction and architecture of the theory, except that we have given gravitomagnetism a fundamental geometric justification that we hope will lead researchers to reconsider gravitomagnetism as a fundamental phenomenon to be considered separately and independently as a physical phenomenon.

8.2 Gauge conditions

In total, the theory has required five gauge conditions for its architecture and design. These gauge conditions are presented in (37), (44), (79a), (79b), and (79c). Of these gauge conditions, (44) is the only natural gauge condition, while the rest are gauge constraints. The solo natural gauge is necessary in order that on a global level, the theory meets our most sought for requirement – of a geometry whose vectors during parallel transport in spacetime will have both their lengths and angles remain invariant. The gauge constraints (37), (79a), (79b), and (79c) have been instituted (imposed) so that we obtain a theory whose resulting equations have the form that we desire or that we are used to – which in this case, is the Maxwell form [14].

8.3 Quantum theory of gravity

Lastly, as our final word, we will briefly touch on the long sought – albeit elusive and contentious – dream of attaining a *Quantum Theory of Gravity* (QTG). Given the obvious similarities not only in the formulae of Sir Isaac Newton's universal law of gravitation $F_g = -G\mathcal{M}_g m_g/r^2$ and Coulomb's electrostatic law $F_e = Q q/4\pi\epsilon r^2$, but in the two physical phenomenon themselves, we can learn one or two things from QED if we are to one day find a quantum mechanical description of the gravitational field.

For example, if we are to accept the thesis presented herein – this would mean that, like electricity, gravity is represented by a four-vector field. From this deduction, logically and intuitively, it would appear that the same method(s) used to quantize the electrodynamic phenomenon – can (and must) be applied somehow to the much sought for quantization program of the gravitational field. We know very well that QED is built on the fundamental soils of three very beautiful equations, namely the Dirac equation [63, 64] and Maxwell's two equations of electrodynamics [14], i.e.:

$$
\begin{aligned}
\imath\hbar\,\gamma_\mu\partial^\mu\psi &= m_0 c\psi & \text{(a)} \\
\partial^\mu F_{\mu\nu} &= \mu_0 J_\nu & \text{(b)} \\
\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} &= 0 & \text{(c)}
\end{aligned}
\tag{86}
$$

where (86a) is the Dirac equation [63, 64] and (86b & c) are Maxwell's two equations of electrodynamics [14] respectively. In the Dirac equation (86a), γ_μ, m_0, and ψ are the usual four 4×4 Dirac matrices, the rest mass of the particle, and the four-component Dirac wavefunction, respectively.

Thus, in much the same manner, the gravitational field might be quantizable *via* the quantization of the gravitational four-vector field $\mathcal{A}_\mu$, in much the same way the electromagnetic four-vector A_μ has been quantized in QED under the scheme of the three equations given in (86). In order for this, the Dirac equation will have to be replaced by its curved spacetime equivalent. In [65], we did propose such a curved spacetime version of the Dirac equation, namely $\imath\hbar\gamma_\mu^{(a)}\mathcal{A}_\mu\partial^\mu\psi = m_0 c\psi$, and in our search for a QTG, we shall take this equation as the appropriate curved spacetime Dirac equation. Thus, we propose that the three equations to be used in the quantization program are:

$$
\begin{aligned}
\imath\hbar\,\gamma_\mu^{(a)}\mathcal{A}_\mu\partial^\mu\psi &= m_0 c\psi & \text{(a)} \\
\partial^\mu \mathfrak{F}_{\mu\nu} &= -\widetilde{\mu}\mathfrak{J}_\nu & \text{(b)} \\
\partial_\nu \mathfrak{F}_{\mu\sigma} + \partial_\sigma \mathfrak{F}_{\nu\mu} + \partial_\mu \mathfrak{F}_{\sigma\nu} &\equiv 0\,. & \text{(c)}
\end{aligned}
\tag{87}
$$

At the time when the curved spacetime Dirac equation (87a) was proposed, we where not sure how to identify the gravitational four-vector field $\mathcal{A}_\mu$ because we had not conceived of the gravitational field as capable of being described by a four-vector. But after the fundamental work of Behera [32] and Heras [43], and what we have presented herein, we are more than convinced that the gravitational field must submit to a four-vector description as suggested herein and e.g. by Heras [43], Behera [32], Heaviside [29,30] and Maxwell [14].

8.4　In closing

In closing, allow us to say that as already stated a number of times, the purpose of the present paper has been to show that gravitomagnetism can be given a geometric description $\overline{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$ on spacetime in exactly the same manner as Einstein gave gravity a geometric description on spacetime *via*

the metric tensor $g_{\mu\nu}$. For fear of digression and loss of focus, we have avoided going deeper in the many areas that this paper can possibly touch. We shall be making follow-up work which will dwell on these matters. We are very much aware of these many areas and we have not even mentioned some of them but silently passed as though we are not aware of them – this has been done intentionally. Further, for the same reasons, we have not done a serious comparative analysis of the present ideas with similar attempts in the literature. We must say that, the present paper is already an unavoidably lengthy one, so mush so that there really is no need to burden you our reader with more material. This can efficiently be done in separate papers in the future.

Dedications

Acknowledgements

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