New Approach to Measurement in Quantum Tomography

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In this article we propose a new approach to quantum measurement in reference to the stroboscopic tomography. Generally, in the stroboscopic approach it is assumed that the information about the quantum system is encoded in the mean values of certain hermitian operators $Q_1, ..., Q_r$ and each of them can be measured more than once. The main goal of the stroboscopic tomography is to determine under which conditions one is able to reconstruct the initial density matrix $\rho(0)$ on the basis of the measurement results $\langle Q_i \rangle_{t_j}$. In this paper we propose to treat every complex matrix as a measurable operator. This generalized approach to quantum measurement may bring some improvement into the models of stroboscopic tomography.

1   Introduction

In this paper by $\mathcal{H}$ we shall denote the Hilbert space and we shall assume that $dim \mathcal{H} = n < \infty$. By $B(\mathcal{H})$ we shall denote the complex vector space of all bounded linear operators in $\mathcal{H}$. The space $B(\mathcal{H})$ is isomorphic with the space of all complex matrices that shall be represented by $\mathbb{M}_n(\mathbb{C})$. Finally, $B_*(\mathcal{H})$ shall refer to the real vector space of all hermitian (self-adjoint) operators on $\mathcal{H}$. The elements of $B_*(\mathcal{H})$ shall be called observables.

The term *quantum state tomography* refers to a wide variety of methods and approaches which aim to reconstruct the accurate representation of a quantum system by performing a series a measurements. Among many different approaches to quantum tomography, one can especially mention the so-called static model of tomography, which requires $n^2 - 1$ measurements of different observables taken at time instant $t = 0$ (see more in [1–3]). A paper published in 2011 initiated another approach to quantum tomography which is based on weak measurement. The paper revealed that the wave function of a pure state can be measured in a direct way [4]. Further papers proved that this approach can be generalized also for mixed state identification [5].

In this paper we follow yet another approach to quantum tomography – the so-called stroboscopic tomography which originated in 1983 in the article [6]. Subsequently, the approach was developed in other papers, such as [7–9]. The assumption that lies at the very foundation of this method claims that the evolution of an open quantum system can be expressed by a master equation of the form

$$\dot{\rho}(t) = \mathbb{L}[\rho(t)], \tag{1}$$

where the operator $\mathbb{L}$ is called the generator of evolution and its most general form have been introduced in [10]. In order to determine the initial density matrix $\rho(0)$ one assumes to have a set of identically prepared quantum systems which evolve according to the master equation with the generator $\mathbb{L}$. Each system can be measured only once, because any measurement, generally, influences the state.

The other underlying assumption connected with the stroboscopic approach is that the knowledge about the quantum system is provided by mean values of certain observables $\{Q_1, ..., Q_r\}$ (obviously $Q_i^* = Q_i$) such that $r < n^2 - 1$. These mean values are mathematically expressed as

$$\langle Q_i \rangle_t = \text{Tr}(Q_i\, \rho(t)) \tag{2}$$

and are assumed to be achievable from an experiment. If we additionally assume that the knowledge about the evolution enables us to perform measurements at different time instants $t_1, ..., t_g$, we get from an experiment a matrix of data $[\langle Q_i \rangle_{t_j}]$, where $i = 1, ..., r$ and $j = 1, ..., g$. The fundamental question of the stroboscopic tomography that one asks is: whether the matrix of experimental data is sufficient to reconstruct the initial density matrix $\rho(0)$. Other problems relate to the minimal number of observables and time instants, the properties of the observables and the choice of time instants. In general the conditions under which it is possible to reconstruct the initial state have been determined and can be found in [6–8].

Compared with the static model of tomography, the stroboscopic approach makes it possible to decrease significantly the number of different observables that are necessary to perform quantum state tomography. From the experimental point of view, it means that in the static model one needs to prepare $n^2 - 1$ different experimental systems (e.g. for $dim\,\mathcal{H} = 4$ one would need to measure 15 different quantities), which seems rather unrealistic. Therefore, the stroboscopic approach appears to have an advantage over the static model as it aims to reduce the number of distinct observables.

2   Generalized observables and measurement results

According to one of the most fundamental concepts of quantum mechanics, to every physical quantity we can assign a hermitian operator which is called an observable. Thus, when talking about measurements in the context of the stroboscopic tomography, we consider mean values of certain hermitian operators [6]. In general, any hermitian operator can be de-

composed according to the spectral theorem:

$$Q = \sum_i \lambda_i P_i. \tag{3}$$

where P_i is the projector onto the eigenspace of Q with the eigenvalue λ_i [11]. Physically speaking, the possible results of measurement correspond to the eigenvalues of Q, whereas the probability of getting the result λ_i (upon measuring the state ρ) can be expressed as:

$$p_i = \text{Tr}(P_i \rho). \tag{4}$$

Finally, we can compute the expectation value of Q as:

$$\langle Q \rangle = \sum_i \lambda_i \, p_i = \text{Tr}(Q \rho), \tag{5}$$

which gives the famous formula for the mean value of any observable.

In other words, any observable is associated with a projective measurement, which stems from the spectral theorem. The main goal of this section is to prove that this approach to measurement can be generalized in such a way that any complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be considered a measurable operator.

We shall formulate and employ the following theorem.

Theorem 1. *(Hermitian decomposition of a complex matrix) For any matrix $A \in \mathbb{M}_n(\mathbb{C})$ there exist two matrices $Q, R \in B_*(\mathcal{H})$ such that the matrix A can be decomposed as*

$$A = Q + i R. \tag{6}$$

Proof. Let us first denote $A = [a_{ij}]$ and since in general $a_{ij} \in \mathbb{C}$ we can put

$$a_{ij} = \text{Re} \, a_{ij} + i \, \text{Im} \, a_{ij}. \tag{7}$$

Moreover we can denote $Q = [q_{ij}]$ and $R = [r_{ij}]$. Then we shall define the entries of the matrices Q and R in the way:

$$q_{ij} := \frac{\text{Re} \, a_{ij} + \text{Re} \, a_{ji}}{2} + i \, \frac{\text{Im} \, a_{ij} - \text{Im} \, a_{ji}}{2}, \tag{8}$$

$$r_{ij} := \frac{\text{Im} \, a_{ij} + \text{Im} \, a_{ji}}{2} + i \, \frac{\text{Re} \, a_{ji} - \text{Re} \, a_{ij}}{2}. \tag{9}$$

One can easily notice that $\overline{q_{ij}} = q_{ji}$ and $\overline{r_{ij}} = r_{ji}$. Therefore $Q, R \in B_*(\mathcal{H})$.

Furthermore, one can check that

$$q_{ij} + ir_{ij} = \frac{\text{Re} \, a_{ij} + \text{Re} \, a_{ji}}{2} + i \, \frac{\text{Im} \, a_{ij} - \text{Im} \, a_{ji}}{2} + $$
$$+ i \, \frac{\text{Im} \, a_{ij} + \text{Im} \, a_{ji}}{2} + \frac{\text{Re} \, a_{ij} - \text{Re} \, a_{ji}}{2} = a_{ij}, \tag{10}$$

which implies that

$$A = Q + i R. \tag{11}$$

$\square$

The above theorem states that every complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be uniquely decomposed into two hermitian matrices. In other words, every complex matrix can be regarded as a pair of observables (hermitian matrices), i.e.

$$A \rightarrow (Q_1, Q_2), \text{ where } Q_1, Q_2 \in B_*(\mathcal{H}). \tag{12}$$

Since in general any observable is considered measurable, therefore, any complex matrix can also be considered a measurable operator.

In this paper it has been proven that for any $A \in \mathbb{M}_n(\mathbb{C})$ there exist two observables $Q_1, Q_2 \in B_*(\mathcal{H})$ such that

$$A = Q_1 + i \, Q_2. \tag{13}$$

If we generalize the idea of quantum measurement, we can define the mean value of any operator $A \in \mathbb{M}_n$ measured upon a quantum system characterized by a density matrix $\rho(t)$. Such a quantity, denoted by $\langle A \rangle_t$, shall be defined in the following way:

$$\langle A \rangle_t := \text{Tr}[A\rho(t)] = \text{Tr}\left[(Q_1 + i \, Q_2)\rho(t)\right]. \tag{14}$$

Taking into account the fact that *trace* is linear, one obtains

$$\langle A \rangle_t = \text{Tr}[Q_1\rho(t)] + i \, \text{Tr}[Q_2\rho(t)], \tag{15}$$

which can be equivalently presented as

$$\langle A \rangle_t = \langle Q_1 \rangle_t + i \, \langle Q_2 \rangle_t. \tag{16}$$

One can observe that if we generalize the idea of quantum measurement in such a way that we treat any complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ as a measurable operator, the mean value of A is a complex number such that its real and imaginary parts are mean values of the observables Q_1, Q_2 which appear in the hermitian decomposition of A. Therefore, the measurement of any complex operator A can be understood as the measurement of two physical quantities that are mathematically represented by the hermitian matrices Q_1, Q_2.

3 Connection with the stroboscopic tomography

When considering problems in the stroboscopic tomography, one needs to bear in mind the necessary condition that the set of observables $Q_1, Q_2, ..., Q_r$ has to satisfy so that an open quantum system with dynamics given by (1) will be reconstructible.

Theorem 2. *An open quantum system with evolution given by Eq. 1 is $(Q_1, ...Q_r)$-reconstructible if and only if the operators Q_i satisfy the condition [6, 7]*

$$\bigoplus_{i=0}^r K_\mu(\mathbb{L}, Q_i) = B_*(\mathcal{H}), \tag{17}$$

where $\bigoplus$ refers to the Minkowski sum, μ is the degree of the minimal polynomial of $\mathbb{L}$ and $K_\mu(\mathbb{L}, Q_i)$ denotes the Krylov subspace which standard definition reads:

$$K_\mu(\mathbb{L}, Q_i) := \text{Span}\{Q_i, \mathbb{L}^* Q_i, (\mathbb{L}^*)^2 Q_i, ..., (\mathbb{L}^*)^{\mu-1} Q_i\}. \tag{18}$$

In reference to this condition for observability of a quantum system we can propose the following theorem.

Theorem 3. *Assume that the set of hermitian matrices denoted by* $\{\lambda_1, \lambda_2, ..., \lambda_{n^2}\}$ *constitutes a basis in the space of all hermitian operators* $B_*(\mathcal{H})$, *where* $n = dim\,\mathcal{H}$. *Then they also constitute a basis in the space of all linear operators* $\mathbb{M}_n(\mathbb{C})$.

Proof. Taking into account the assumption, one can write:

$$\forall_{Q \in B_*(\mathcal{H})} \quad \exists_{\alpha_1,...,\alpha_{n^2} \in \mathbb{R}} \quad Q = \sum_{k=1}^{n^2} \alpha_k \lambda_k. \tag{19}$$

Then from the theorem on hermitian decomposition of a complex matrix it follows that $\forall_{A \in \mathbb{M}_n(\mathbb{C})} \; \exists_{Q,R \in B_*(\mathcal{H})}$ such that the matrix A can be decomposed as

$$A = Q + i\,R. \tag{20}$$

Assuming that Q has such decomposition as in (19) and taking R in the analogous form:

$$R = \sum_{k=1}^{n^2} \beta_k \lambda_k, \quad \beta_k \in \mathbb{R}, \tag{21}$$

matrix A can be represented as

$$A = \sum_{k=1}^{n^2} \alpha_k \lambda_k + i\left(\sum_{k=1}^{n^2} \beta_k \lambda_k\right), \tag{22}$$

which can be transformed into the form

$$A = \sum_{k=1}^{n^2} (\alpha_k + i\beta_l)\,\lambda_k. \tag{23}$$

Finally, the matrix A can be decomposed as

$$A = \sum_{k=1}^{n^2} z_k \lambda_k, \tag{24}$$

where $z_k \in \mathbb{C}$ and $z_k = \alpha_k + i\beta_k$.

From (24) one can easily draw the conclusion that the set of matrices $\{\lambda_1, \lambda_2, ..., \lambda_{n^2}\}$ is a basis in $\mathbb{M}_N(\mathbb{C})$. □

The link between the above theorem and the stroboscopic tomography is that in (17), which expresses the necessary condition for observability, on the right hand side you can put either $B_*(\mathcal{H})$ or $B(\mathcal{H})$. On the basis of theorem 3 one can conclude that if certain operators span one of these spaces, they also have to span the other.

4 Summary

In this paper it has been proved that any complex matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be uniquely determined by two hermitian matrices

(i.e. observables). In general, mean values of hermitian matrices can be obtained from an experiment (based on projective measurement). Thus, from this observation one can conclude that any complex matrix can be regarded as a measurable operator. The measurement of a complex matrix should be understood as the measurement of the mean values of two observables which determine the complex operator. The measurement result of a complex matrix is then a complex number which real and imaginary parts are obtained from an experiment. Further research is planned to investigate whether the generalized approach to measurable operators can improve the models of the stroboscopic tomography.

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