Approach to the Schwarzschild Metric with SL(2,R) Group Decomposition

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The paper analyzes the two-step coordinate transformations, known as the simple (or "heuristic") approach to the Schwarzschild metric [3, 5, 22]. The main finding of the analysis is that such transformations are *unique* as they correspond to the Iwasawa decomposition for the special linear group $SL(2,\mathbb{R})$ with the subgroup of rotation $SO(1,1)^+$. It is noted that all original transformations utilize de facto determinant of unity. However, as shown, this property is related to the action invariance under diffeomorphism for gravity. The noted group symmetry of the coordinate transformations may shed light on the "paradox" of the original approach for obtaining the Schwarzschild metric based on the Equivalence Principle only and enable its further study. The path to generalization in $SL(4,\mathbb{R})$ is suggested.

1 Introduction

In work "What is wrong with the Schwarzschild coordinates" [5], J. Czerniawski demonstrated the two-step coordinate transformations from the Minkowski to the Schwarzschild metric. Recently, Christillin and Morchio [3] slightly updated the approach by clarifying the step of the transformation from the Gullstrand-Painlevé (G-P) to the Schwarzschild metric. Without this, the original path would not be consistent. Even if the approach of obtaining the Schwarzschild metric via the "heuristic" to be considered with certain cautiousness, the original work was over-cited, bringing the substantial interest in this topic [1,3,7,14,22]. The approach recently was called the "inherent paradox of GR" [3], and the original question has not been answered. This paper aims to walk through the approach with maximum formality to present the correspondence and possible path to the generalization.

2 Preliminaries and Notation

Diffeomorphism of a manifold $\mathcal{M}$ by definition is a smooth invertible map $\phi : \mathcal{M} \to \mathcal{M}$ such as the inverse map ϕ^{-1} be smooth as well. General diffeomorphism can be thought as the deformation that does not preserve the metric on $\mathcal{M}$. The map $\phi : M \to M$ of the transformation from affine η to curvilinear g coordinates may be considered as a vector-valued function of n-variables. By retaining the requirements of smoothness, the transformation may be defined in terms of partial derivatives in the form of the Jacobian matrices that constitute second rank tensors

$$J_{\mu a} = \frac{\partial x^a}{\partial \overline{x}^\mu} \quad J^{\mu a} = \frac{\partial \overline{x}^a}{\partial x^\mu}\,. \tag{1}$$

The barred symbols denote the curvilinear coordinates, and unbarred are for flat coordinates*. The metric tensor is

$$g_{\mu\nu} = J_{\mu a} J_{\nu b}\, \eta_{ab} \quad g^{\mu\nu} = J^{a\mu} J^{b\nu} \eta^{ab} \tag{2}$$

where indexes are $(0, 1, 2, 3)$ and η has the signature $(-\,+\,+\,+)$. The transformation is non-singular $\mathrm{J} \neq 0$, the matrix is bijective, and the inverse transform represents the simple inverse matrix $\overline{\mathrm{J}} = \mathrm{J}^{-1}$. If the order of indexes as per (1), the equation can be written in the matrix notation (for both covariant and contravariant forms) as

$$\mathrm{g} = \mathrm{J} \cdot \eta \cdot \mathrm{J}^{\mathrm{T}}\,. \tag{3}$$

The capital letters are used for matrices excluding the metric tensor g, and Minkowski η. In matrix notation, the form (covariant or contravariant) will be specified in the text. For the spherical symmetry case, the Jacobian matrices are 4×4

$$\begin{bmatrix} \mathrm{J} & 0 \\ 0 & \mathrm{I}_2 \end{bmatrix}.$$

Therefore, J can be written as 2×2 for the temporal and radial coordinates only, dropping the symmetric angular and tangential terms that are not affected by transformations. The spherical symmetry provides the unique case to consider the transformations as being "two-dimensional" with certain limitations. Though later in Section 7, the four-dimensional form is reviewed. Natural units ($c = 1$) are employed throughout. As a matter of choice, the common hyperbolic notation is used for the radial escape velocity for shortness

$$v = \mathrm{th}(\beta) = \sqrt{\frac{r_g}{r}}$$

$$\sinh(\beta) = \frac{v}{\sqrt{1 - v^2}}$$

$$\gamma = \cosh(\beta) = \frac{1}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{1 - \frac{r_g}{r}}}$$

3 Step one: from Minkowski to Gulfstrand-Painlevé

The first coordinate transformation as given in [5, 22] is

$$dx^1 = d\overline{x}^1 - v d\overline{x}^0 \qquad dx^0 = d\overline{x}^0 \tag{4}$$

*Since the order of indexes for J in the notation is arbitrary, it is chosen such as the covariant form coincides with the "vierbein" or tetrad. So, one can treat them as the same objects.

where v is the radial escape velocity of the gravitational field or the river velocity [7]. The equations have the differential form; therefore, the term "Galilean transformations" can be used with certain cautiousness. Despite the similarity in the look, the latter is defined as the affine transformations of the coordinates*. According to [3,5], this transformation embodies the Equivalence Principle (EP) and therefore plays the central role in the approach.

The Jacobian matrix for the first transformation as per definition (1) is then

$$J^{(1)}_{\mu a} = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{th}(\beta) \\ 0 & 1 \end{bmatrix}$$
$$J^{\mu a}_{(1)} = \begin{bmatrix} 1 & 0 \\ v & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{th}(\beta) & 0 \end{bmatrix} \tag{5}$$

where v can be taken with an arbitrary ± sign as not affecting the final transform [5, 14]. Such transformation can be classified as the spacetime shear deformation. It obviously represents shear mapping transformation on the hyperbolic plane. The value of "shear" is given by the relativistic velocity $v = \mathrm{th}(\beta)$ and the (imaginary) shear angle is β or rapidity. Further, the term shear is used for this transformation for the current purposes leaving aside its physical significance and the relation to the EP. It leads to the G-P coordinates with the metric tensor which has following covariant form

$$\mathrm{g}_1 = \mathrm{J}^{(1)} \cdot \eta \cdot \mathrm{J}^{(1)\mathrm{T}} = \begin{bmatrix} -\left(1 - v^2\right) & v \\ v & 1 \end{bmatrix} = \begin{bmatrix} -\left(1 - \frac{r_g}{r}\right) & \sqrt{\frac{r_g}{r}} \\ \sqrt{\frac{r_g}{r}} & 1 \end{bmatrix}. \tag{6}$$

4 Step two: to the Schwarzschild metric

The second coordinate transformation J_2 is pull-back from the comoving G-P frame to the coordinate frame of reference redefining time coordinate. The covariant form is

$$\mathrm{J}^{(2)} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \tag{7}$$

where b is the *arbitrary* parameter †. The total coordinate transformation is the product of both transforms

$$\mathrm{J} = \mathrm{J}^{(2)} \cdot \mathrm{J}^{(1)} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & v \\ b & vb + 1 \end{bmatrix} \tag{8}$$

that leads to the metric tensor

$$\mathrm{g} = \mathrm{J} \cdot \eta \cdot \mathrm{J}^{\mathrm{T}} = \begin{bmatrix} -(1 - v^2) & (v^2 - 1)\, b + v \\ (v^2 - 1)\, b + v & (vb + 1)^2 - b^2 \end{bmatrix}. \tag{9}$$

Choosing b in the way to eliminate the off-diagonal terms one obtains the Schwarzschild metric

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - v^2\right) & 0 \\ 0 & \left(1 - v^2\right)^{-1} \end{bmatrix} = \begin{bmatrix} -\cosh^{-2}(\beta) & 0 \\ 0 & \cosh(\beta) \end{bmatrix}. \tag{10}$$

After b has been defined, the second transformation becomes

$$\mathrm{J}^{(2)}_{\mu \mathrm{a}} = \begin{bmatrix} 1 & 0 \\ \sinh(\beta)\cosh(\beta) & 1 \end{bmatrix}$$
$$\mathrm{J}^{\mu \mathrm{a}}_{(2)} = \begin{bmatrix} 1 & \sinh(\beta)\cosh(\beta) \\ 0 & 1 \end{bmatrix}. \tag{11}$$

As a result, the parameter $\pm b = v\gamma^2 = \sinh(\beta)\cosh$ corresponds to the proper velocity of free-falling observer in the Schwarzschild metric. It stands in the well-known expression for the time coordinate transformation between the G-P and the Schwarzschild metrics.

5 $SL(2, \mathbb{R})$ with the Lorentz signature

The remarkable property of all Jacobian matrices is that they all have the unity determinant‡. In order to classify them as elements of a group, one may note that matrices are defined on the Minkowski basis (space-time or the hyperbolic plane). In fact, the Jacobian matrices can be expressed using an imaginary value for the time coordinate as

$$J_{\mu a} = \frac{\partial x^a}{\partial \overline{x}^\mu} = \begin{bmatrix} \frac{\partial x_0}{\partial \overline{x}_0} & \frac{1}{i}\frac{\partial x_1}{\partial \overline{x}_0} \\ i\,\frac{\partial x_0}{\partial \overline{x}_1} & \frac{\partial x_1}{\partial \overline{x}_1} \end{bmatrix}. \tag{12}$$

In such a way, the Jacobian matrices constitute the subgroup of $\mathrm{SL}(2, \mathbb{C})$ with only two imaginary off-diagonal elements in the matrices. Let's denote this group as $\mathrm{SL}(2, \mathbb{C})^* \in \mathrm{SL}(2, \mathbb{C})$. Then, considering only the real parts, there is one-to-one mapping of $Z' \in \mathrm{SL}(2, \mathbb{C})^*$ to $Z \in \mathrm{SL}(2, \mathbb{R})$ as follows

$$Z' = \begin{bmatrix} a & -i\,b \\ i\,c & d \end{bmatrix} \to Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{13}$$

Ignoring the imaginary unit, in the way as it is done for the Minkowski time coordinate, allows one to use the real values in the matrix as per the defined mapping to $\mathrm{SL}(2, \mathbb{R})$. Introduced in such a way, the group $\mathrm{SL}(2, \mathbb{C})^*$ is isomorphic to $\mathrm{SL}(2, \mathbb{R})$. This mapping is multiplicative and a bijection. Hence, all operations in $\mathrm{SL}(2, \mathbb{R})$ can be translated to $\mathrm{SL}(2, \mathbb{C})^*$ and vice versa using this isomorphism. Such mapping allows one to utilize $\mathrm{SL}(2, \mathbb{R})$ on the Lorentz/Minkowski basis $\mathbb{H}^{1(2)}$, instead of its default, the Euclidean basis $\mathbb{R}^2$.

*The differential form of the Lorentz transformations has the same form and obviously $\Lambda\, \eta\, \Lambda^T = \eta$ is valid for the differential form. For more on differential transformation see [8].

†As suggested in [3] "the requirement to eliminate the off-diagonal term of the P-G metric is generally accomplished just by redefining time in an ad hoc way".

‡To be consistent, the fact is taken *a priori* "knowing" that the resulting metric has $|g| = |\eta| = -1$. Section 8 reviews a physical ground for this.

6 The group decomposition

The Iwasawa decomposition is the factorization of a semisimple Lie group to the product of three closed subgroups as K × A × N ("compact, Abelian and nilpotent") [9, 13]. In the application to $SL(2,\mathbb{R})$ it is well studied [4, 12], and in terms of the matrices is even obvious. Importantly, it implies the *uniqueness* of the factorization of the element of the group to the product of three subgroups, those elements are N is upper triangular, A is diagonal, and K is orthogonal matrices, the spatial rotations $K \in SO(2)$.

One may see that using the mapping (13), the elements of these three groups become the matrices of the following form

$$N = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix} \\ K = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \tag{14}$$

with $k > 0$. Since the mapping results in the complex conjugation of the angle of rotation ($\beta \to i\beta$), the foremost notable distinction from the decomposition of $SL(2,\mathbb{R})$ is that K becomes the group of hyperbolic rotations $SO(1,1)^+$, that is the pure Lorentz boost.

The covariant form of $J^{(1)}$, and contarvariant $J_{(2)}$ are $\in$ N (upper triangular matrices). Therefore, the decomposition can be applied to contravariant $J_{(1)}$ and to contarvariant $J^{(2)}$ which are lower triangular. In fact, they are explicitly the Iwasawa decomposition $J^{(2)} = A \cdot K \cdot N$ (covariant form) and $J_{(1)} = N \cdot A \cdot K$ (contravariant form). The latter is as follows

$$J^{\mu a}_{(1)} = \begin{bmatrix} 1 & 0 \\ \text{th}(\beta) & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\sinh(\beta)\cosh(\beta) \\ 0 & 1 \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} \cosh(\beta) & 0 \\ 0 & \cosh^{-1}(\beta) \end{bmatrix} \cdot \begin{bmatrix} \cosh(\beta) & \sinh(\beta) \\ \sinh(\beta) & \cosh(\beta) \end{bmatrix}. \tag{15}$$

Notably, that N in the factorization becomes already known matrix $N = J^{-1}_{(2)}$ (11). The resulting transformation is

$$J = J_{(2)}\, J_{(1)} = A \cdot K \tag{16}$$

where $J_{(2)}\, J_{(1)}$ has the form of the product of two upper and lower triangular matrices $N_1 \cdot \overline{N}_2$. And since $K \equiv \Lambda$ is the Lorentz boost, that leaves the original metric invariant $\eta = \Lambda \cdot \eta \cdot \Lambda^T$, then K drops being at the right side of (16). Therefore the resulting Schwarzschild metric

$$g = J \cdot \eta \cdot J^T = A \cdot \eta \cdot A^T \tag{17}$$

is obviously defined by the diagonal matrix A*

$$A^{\nu b} = \begin{bmatrix} \cosh(\beta) & 0 \\ 0 & \cosh^{-1}(\beta) \end{bmatrix}$$

$$A_{\nu b} = \begin{bmatrix} \cosh^{-1}(\beta) & 0 \\ 0 & \cosh(\beta) \end{bmatrix}. \tag{18}$$

Therefore, all approach can be represented as just the diagonalization of the first shear transformation matrix.

Proposition: If J_1 is the shear transformation in the contravariant form with the shear value v, then its Iwasawa decomposition with the mapping (13) provides the diagonal matrix A that uniquely represents the Jacobian matrix J that maps the Minkowski to the Schwarzschild metric. The process is that A normalizes N, or A is the unique diagonal form of the original shear transformation†.

7 The generalization to the Cartesian coordinates

The suggested approach can be generalized to four-dimensional spacetime in the Cartesian coordinates. The hyperbolic shear parameter v is non-Lorentz invariant four-vector $v = (1, v_x, v_y, v_z)$, and its norm is $\|v\| = \cosh(\beta)^{-1}$. It shall constitute the column of contravariant shear transformation in the Cartesian coordinates‡

$$J^{\nu a}_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ v_z & 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

The KAN decomposition of this form provides the unique Jacobian matrix for the metric as described in the Proposition. In case if $v_y = v_z = 0$, implying that one direction via coordinate x is considered, then it converges to the reviewed case above. It is known that the Iwasawa decomposition can be also applied to elements of $SL(4,\mathbb{R})$ group [4, 19]. The straightforward approach is to use the Gram–Schmidt process that leads to QR decomposition, from which the KAN form can be obtained [19]. However, the more elegant way is to use the Givens rotations, which are literally spatial rotations of the SO(3) group. Obviously, the shear vector in the Cartesian coordinates can be represented as

$$v = (1, \text{th}(\beta)\sin(\theta)\cos(\phi), \text{th}(\beta)\sin(\theta)\sin(\phi), \text{th}(\beta)\cos(\theta))$$

where θ and ϕ are the angles between vector v and the coordinate axes. Hence, two sequential spatial rotations $R_z(\phi) \in SO(3)$ and $R_y(\frac{\pi}{2} - \theta) \in SO(3)$ reduce the matrix to the case above, eliminating second and third components (v_y and v_z). Treated in such way, a general transformation in four-dimensional spacetime (19) is $\{SL(2,\mathbb{R}), SO(3)\}$.

The details and the analysis of the decomposition of (19) lay out of the scope of this work and can be an interesting topic for future research.

*It coincidences with the Schwarzschildian vierbein or "metric squared".

†NAK, as shown, results in contrvariant form of A, similarly KAN decomposition gives the covariant form of A.

‡Note, the Jacobian's column vectors' signature becomes *opposite* to the metrics signature (η and g) as per definition of $SL(2,C)^*$ above.

8 Discussion

At the critical angle, a possible weak point of the original path should be also noted. When one uses "one coordinate change" transformations (5) and (7), in fact, the additional condition on the determinant $|J| = 1$ is taken "under the hood". During the classical derivation of the Schwarzschild metric in the GR, $|g| = -1$ is the obtained results from the field equations (note: even with $T_{\mu\nu} = 0$). Contrary to that, the reviewed "heuristic" approach uses $|J| = 1$ that explicitly leads to $|g| = -1$ *a priori* knowing the resulting metric.

Once this principle physically has solid ground, then the above parallel can be considered fundamental. Without this, one may still regard this approach as a coincidence. From the prospect of the physics, the value of g_{00} for the Schwarzschild metric can be obtained from the Newtonian gravitation [15] or the equivalence principle and red-shift experiments [20, 21]. If one would a priori know that $|g| = -1$, then the Schwarzschild metric easily follows by defining $g_{rr} = -g_{00}^{-1}$.

From another perspective, the fact is that the spherically symmetric static gravitational field has explicitly $|g| = -1$ cannot be just a coincidence but may potentially signal a hidden symmetry attached to such property.

Consider the action in the Minkowski spacetime $S_1(x) = \int \mathcal{L}(x, \dot{x})\, dV^4$ and in the spacetime with the curvature $S_2(x) = \int \sqrt{-g}\mathcal{L}(x, \dot{x})\, dV^4$ expressed by the Lagrangian density. The diffeomorphism invariance of the action would require that under the map $\phi : S_1 \to S_2 = S_1$ and therefore $|g| = -1$. On the other hand, the action invariance under diffeomorphism implies the equivalence of the conservation of energy, momentum, and the continuity equations for the system.

9 The conclusion

The analyzed approach shows the striking correspondence between coordinate transformation from the Minkowski spacetime to the Schwarzschild metric and $SL(2, \mathbb{R})$ group using the mapping to the Lorentz base. The original "heuristic" approach to the Schwarzschild metric can be considered via the unique group decomposition by obtaining the first coordinate transformation's corresponding diagonal form.

$SL(2, \mathbb{R})$ group has already appeared in the application to the gravitation metric in [10] and in two-dimensional quantum gravity [17]. This review gives a more classical and intuitive outlook on the group's correspondence to the coordinate transformations of the metrics.

The work outlines a critical point of the original approach, though suggesting further prospects for the method generalization and research. The reviewed case brings an additional question on the action invariance under diffeomorphism for the gravity. The group symmetry of the reviewed coordinate transformations may probably shed light on the resolution of the mentioned "inherent paradox of GR".

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