On the Quantification of Relativistic Trajectories

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Solving the geodesic equation on a relativistic manifold is possible numerically step by step. This process can be transposed into a quantisation. We study here the effect of this quantisation on the Schwarzschild spacetime, more precisely in the Kruskal-Szekeres map.

1 From digitization to quantification

The geodesics are obtained using the Euler-Lagrange variational method, with the Lagrangian $L = g_{\mu\nu} x'^{\mu} x'^{\nu}$ which leads to the well-known equation [1, 8.26]

$$x''^{\alpha} + \Gamma^{\alpha}_{\mu\nu} x'^{\mu} x'^{\nu} = 0\,.$$

The goal is to obtain the extremal solutions for

$$\tau = \int_{\lambda_0}^{\lambda_1} \sqrt{L}\, d\lambda$$

which happens to be the proper time for a test particle subjected to the field g. Except for the mass of the particle, which is in fact an energy, this proper time is an action.

Finding solutions digitally is extremely simple. Given a digitisation step $\delta\lambda$ and an initial state (x, x') of the mobile, the position x is incremented by $x'\delta\lambda$. The geodesic equation gives $x'' = -\Gamma^{\alpha}_{\mu\nu} x'^{\mu} x'^{\nu}$ and the velocity x' is incremented by $x''\delta\lambda$. The process is then iterated.

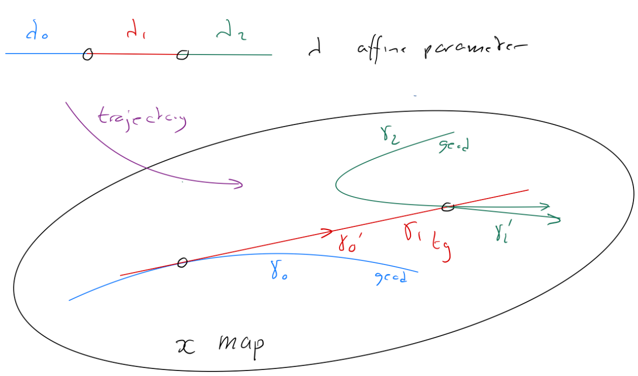

Fig. 1: A test-particle moves from a geodesic $\gamma_0(\lambda_0)$ to a geodesic $\gamma_2(\lambda_2)$ by a trajectory element $\gamma_1(\lambda_1)$ on an interval $\delta\lambda$. This is a straight line in the tangent space. We use the fact that the tangent spaces $T_x\mathbb{R}^n$ are in fact canonically included in $\mathbb{R}^n$.

The choice of the affine step will be made here by keeping the time step constant $\eth\tau$ which gives

$$\delta\lambda = \eth\tau / \sqrt{L}\,.$$

This time step can be physically equated with the quantum of action in the following interpretation.

At each step, the mobile requests a quantum according to the chosen coordinate system x. It uses this quantum to continue its trajectory in its local context, which is the tangent space to the space-time manifold at the current point. Then the new state is considered as such in global space-time. An observer placed on the particle moves during the quantum of time according to a trajectory linearised by the choice of its map.

Some remarkable facts emerge.

First, the coordinate system selected by the observer is essential. The linearisation of the trajectory during $\eth\tau$ depends on the map x and makes the interaction between space-time and the observer contextual. There is an effect of the observation on the trajectory.

Second, it cannot be excluded that the quantisation step involves speeds higher than those of light. This phenomenon can be related to certain quantum effects, such as the possibility for a particle to tunnel through a potential barrier, or to violate the conservation of energy law for a time short enough to be allowed by Heisenberg's uncertainty relations.

Third, in the particular case of the Schwarzschild model with a radius r_S it becomes possible to be in the forbidden zone beyond the naked singularity described below.

2 Reminder on Schwarzschild, Kruskal and Szekeres

Karl Schwarzschild was one of the first to find a solution to the gravitational equations of Einstein's general relativity in 1916. This solution, which describes the field created by a point mass, is expressed by the following metric in polar coordinates, with a speed of light $c = 1$ and a Schwarzschild radius r_S:

$$d\tau^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Two peculiar radiuses were observed immediately. The first one, $r = r_S$, gives the horizon beyond which a particle cannot escape, giving the name of a black hole to this zone. The second one, $r = 0$, is a singularity of the metric, known as naked, where any particle entering the black hole ends its trajectory in a finite time.

The Kruskal-Szekeres coordinate transformation leads to a formulation in terms of the variables (T, X, θ, ϕ) [2]:

$$d\tau^2 = \frac{4r_s^3}{r} \exp\left(-\frac{r}{r_S}\right)\left(dT^2 - dX^2\right) - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

The parameter $r = r_S \left(\mathcal{W}_0 \left(\frac{1}{e} \left(X^2 - T^2 \right) \right) + 1 \right)$ is given by the branch 0 of the Lambert function $\mathcal{W}$.

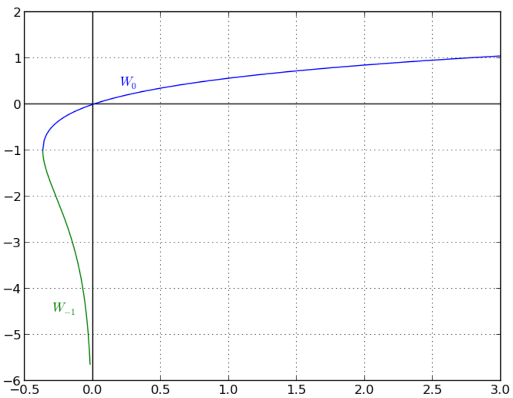

Fig. 2: Real branches of the Lambert function.

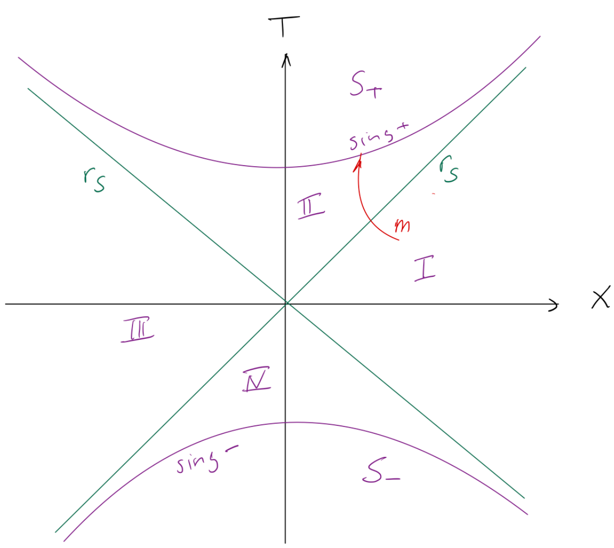

Fig. 3: Kruskal-Szekeres map.

The diagram in Fig. 3 shows the following regions:

I space-time outside the black hole
II black hole
III other component of space-time
IV white hole
S+ inside of the naked singularity
S– other component inside the naked singularity.

This map shows that the Schwarzschild horizon is not a physical singularity, but only an artefact due to the choice of the map.

The diagonal lines represent the Schwarzschild horizon, and the two boundary branches of the **sing+** and **sing–** hyperbola the entrance and exit of the naked singularity.

A particle from region **II** ends its trajectory on **sing+**, without being able to exit. Conversely, a particle in region **IV** cannot do anything else, but exit; hence the name of the white hole. One also finds the expressions *sink* and *source* for these two regions.

S– and **S+** are inaccessible, or forbidden, because they are outside the map domain. These two regions and their boundaries are associated with a single point, the zero of the polar coordinates, and can be considered as collapsed. At least in the hypothesis of a strictly continuous world.

3 Appearance of tachyons

Traditionally, the term *tachyon* has been applied to a hypothetical particle with a speed greater than the speed of light. The exit of the speed of the future light cone is identified by the fact that $L < 0$ and thus an imaginary quantisation step. Here, we propose using a complex proper time:

$$\tau = \tau_r + i\tau_i \in \mathbb{C}\,.$$

This time is measured by two clocks, one real and the other imaginary. The increase in the affine parameter becomes $\delta\lambda = \delta\tau_r / \sqrt{L}$ if $L > 0$ or $\delta\lambda = i\delta\tau_i / \sqrt{-L}$ if $L < 0$. In this way, the trajectory remains real in the map x. For a tachyon, it is the imaginary clock that works, the other one remains fixed, and the opposite is true for a standard particle.

For any coordinate system on space-time, the notions of time and space are found locally by placing an orthonormal basis in the tangent space which diagonalizes the metric. Afterwards, thanks to a possible permutation of the axes and a calibration of the units, we can obtain the diagonal metric of Minkowski Diag$(1, -1, -1, -1)$. The zero coordinate is then time and the others define the space. The base obtained in this manner is generally referred to as a *tetrad*.

The proper speed $\dot{x} = \frac{\delta\lambda}{\delta\tau} x'$ is transformed into a quad-speed $u = \gamma \begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}$ where $\boldsymbol{v}$ is the space velocity of the mobile. Let v be its Euclidean norm and $\boldsymbol{n}_G$ be the unit vector $\boldsymbol{v}/v$, the so-called *slip* vector. We easily obtain $\gamma = (1 - v^2)^{-1/2}$.

If $v < 1$, it is possible to put the mobile at rest with a Lorentz boost $\Lambda(\boldsymbol{v})$ such as

$$\Lambda(\boldsymbol{v})\, \boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If $v > 1$, γ becomes purely imaginary. Nevertheless, it is possible to extend this boost by

$$\Lambda(\boldsymbol{v}) = \Lambda \left(\frac{\boldsymbol{n}_G}{v} \right) R \left(\boldsymbol{n}_G, \frac{\pi}{2} \right)$$

where $R(\boldsymbol{n}, \theta)$ is the rotation of angle θ and axis $\boldsymbol{n}$. It can be seen that $\Lambda(\boldsymbol{v})\, \boldsymbol{u} = \begin{pmatrix} 0 \\ i\boldsymbol{n}_G \end{pmatrix}$. The "putting at rest" with this extended boost makes a particle appear in the direction $\boldsymbol{n}_G$ with a proper time marked by its imaginary clock. As $1/v <$

1, this transformation is physically feasible for an external observer, and the tachyon could be visible. One can notice that the factor i in front of $\boldsymbol{n}_G$ is consistent, as it implies a quadrivector of Minkowskian norm one.

4 Transition from black hole to white hole

The appearance of a state in a zone forbidden by the singularity poses a more delicate problem. Indeed, the Christoffel coefficients involve the parameter

$$r = r_s\left(\mathcal{W}_0\left(\frac{1}{e}\left(X^2 - T^2\right)\right) + 1\right).$$

This critical zone is defined by $X^2 - T^2 < -1$ which is outside the domain of $\mathcal{W}_0$.

The solution proposed here is to use the other part of this function on the real line, namely

$$r = r_s\left(\mathcal{W}_{-1}\left(\frac{1}{e}\left(X^2 - T^2\right)^{-1}\right) + 1\right)$$

by reversing the term $X^2 - T^2$ which enters the domain of W_{-1}. The trajectory is then continued by changing the signs of T and X, which moves the mobile from the black hole to the white hole. This idea is supported by the hyperbolic character of the Kruskal map.

5 Cost of quantification

The evolution of the trajectory during the time quantum is no longer geodesic, and therefore requires some work. The force that appears during this displacement is given by

$$f^\alpha = x''^\alpha + \Gamma^\alpha_{\mu\nu} x'^\mu x'^\nu$$

and its work on the affine segment $\delta\lambda$ is given by

$$\delta W = \int_{\lambda_0}^{\lambda_1} g_{\mu\nu} f^\mu x'^\nu d\lambda .$$

A quick calculation shows that

$$\begin{aligned}\delta W &= \frac{1}{2}\delta\lambda\, x'^\mu x'^\nu x'^\rho \int_0^1 \frac{\partial g_{\mu\nu}}{\partial x^\rho}\left(x + x'\xi\delta\lambda\right) d\xi \\ &= \delta\lambda\, x'^\mu x'^\nu x'^\rho \int_0^1 \Gamma_{\rho\mu\nu}\left(x + x'\xi\delta\lambda\right) d\xi .\end{aligned}$$

This expression makes it possible to estimate the energy needed to quantify the movement.

6 Refutability of the model

Given a time quantum, one can ask which mass M_0 corresponds to a quantum of action equal to Planck's constant. Thus, $M_0 c^2 \delta\tau = \hbar$.

Clearly, the finer the digitisation, the closer the trajectories to the unquantized geodesics, thus deferring the quantum effects mentioned above.

The smaller the quantum, the later the effect, the longer the calculation time. The calculations carried out here allowed us to aim for a time quantum of approximately 10^{-13} s which corresponds to a mass of 10^{-2} eV/c^2. For example, reaching the mass of the neutrino, which is currently estimated at 1.1 eV/c^2, would require a temporal resolution two orders of magnitude lower, resulting in calculation times that are approximately 100 times longer. As the calculations performed here require several days, it is not impossible to think that an optimisation could be achieved up to the level of actually observable particles.

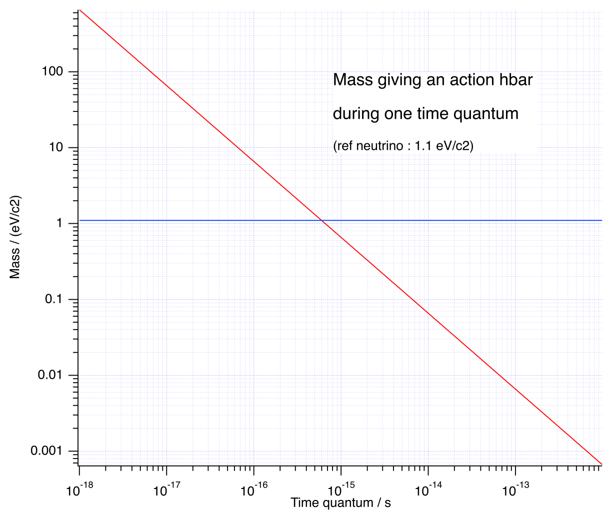

Fig. 4: Mass–time quantum relationship.

7 Two typical trajectories

In general, the trajectories end either with the limiting velocity 1 or at the singularity. Tachyons are short-lived, and return to standard space-time with a final velocity of 1. Two examples are given in Fig. 5 and Fig. 6.

8 Calculation tools

The digital tracking of trajectories requires over several million steps. The standard precision of the current computers (double precision) is 53 bits, which is totally insufficient. The MPFR library [4] implements the calculation with an arbitrary precision, which is only limited by the machine's memory. An interface written by P. Holoborodko [3] then allows the use of the Eigen vector calculation library [6]. The very complete study of F. Johansson [5] on the Lambert function finally makes it possible to carry out the calculation of trajectories, which becomes stable with a precision of 4096 bits (approximately 1200 decimal places).

The exploration of the various trajectories is programmed in C++ and uses a 128-processors machine running in the *Gnu-Linux Ubuntu 20.4* environment.

The trajectories presented here generally require several days of parallel CPU.

9 Analogy with quantum measurement

As we have seen, some of the effects emerging in a time quantum $\eth\tau$ of a relativistic motion are due to the presence of the observer. In summary, the motion naturally follows a geodesic; then during the time of observation, it follows a tangent, and it resumes its natural trajectory, but on another, neighbouring geodesic.

This sequence is similar to the Copenhagen version of quantum measurement, in which two types of evolution coexist in a quantum system. The first, known as unitary (U-type), is governed by the Schrödinger or Dirac equation. The second, which appears when the system is measured, is called wave packet reduction (type R), and consists of projecting the wave function onto an eigenspace associated with the observable to be measured.

Let $\hat{A}$ be the self-adjoint operator translating an observable. To measure A according to Geneva's school [8], the observer asks a series of questions whose answers are *yes* or *no*. A question about A is for example: "Will the value of A appear in a certain interval Δ of the real line?".

Let $\mathrm{Sp}\,\hat{A}$ be the spectrum of the operator $\hat{A}$. This question is represented by the projection operator $J_\Delta = \sum_{a\in\Delta\cap\mathrm{Sp}\hat{A}} J^A_a$ where J^A_a is the projector onto the eigenspace of eigenvalue a. The result of the measurement, i.e. the answer to the question, will be *yes* with probability $p_1 = \langle\psi|J_\Delta|\psi\rangle$ and the system will then be in the state $|1\rangle = J_\Delta\psi \,/\, \|J_\Delta\psi\|$. The answer *no* is treated in the same way, but with the projector $J_{\complement\Delta}$ and gives the final state $|0\rangle$.

One can imagine that the measurement lasts for a time interval $\eth\tau$ and, after the response has been randomly chosen, the wave function evolves "linearly" towards its final state.

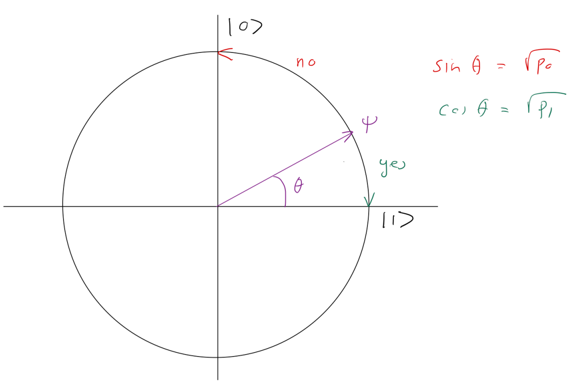

Fig. 7: Evolution of the quantum probability amplitude in R mode.

For example, the path in Fig. 7

$$t \mapsto \psi_t = \cos\left(\theta\left(1-\frac{t}{\eth\tau}\right)\right)|1\rangle + \sin\left(\theta\left(1-\frac{t}{\eth\tau}\right)\right)|0\rangle$$

where $\cos\theta = \sqrt{p_1}$, moves in a uniform and unitary manner from ψ to $|1\rangle$ in case of a *yes* answer.

For the Schrödinger equation, this evolution is governed in the $(|1\rangle, |0\rangle)$ basis by the Hamiltonian operator

$$\hat{H}_1 = \frac{\theta\hbar}{\eth\tau}\sigma_2 \quad \theta = \arccos\sqrt{p_1}$$

where σ_2 is the second Pauli matrix. It can be seen that $\langle\hat{H}_1\rangle = 0$, and that we have

$$\psi_t = \exp\frac{\theta t}{i\eth\tau}\sigma_2\psi = \exp\frac{\theta t}{\eth\tau}(|1\rangle\langle 0| - |0\rangle\langle 1|)\psi\,.$$

Initially, the wave function follows a trajectory U given by a Hamiltonian $\hat{H}$. During the measurement, the reduction R is replaced by a trajectory U with a Hamiltonian proportional to σ_2. It then resumes the trajectory U given by $\hat{H}$.

10 From the quantum to the infinitesimal

The infinitesimals of Leibnitz and Newton were only recently given a consistent axiomatic basis. They have been used systematically by mathematicians such as Euler, Lagrange or Wallis with success and without rigorous justification. Physicists use these devices without further ado on a daily basis. The axiomatization of continuity by d'Alembert, Cauchy and Weierstrass almost sounded the death knell of these quantities, as small as one likes, but nonzero.

Nevertheless, they have made a surprising reappearance through topos, equipped with their not necessarily Boolean logic. For smooth infinitesimal analysis, for example, they are defined by the subset of the line $\Delta = \{\varepsilon\,|\,\varepsilon^2 = 0\}$ which is no longer reduced to $\{0\}$. One then speaks of nilpotent real numbers. This has the effect of eliminating all powers greater than or equal to 2 in the Taylor developments on this set. In other words, any function becomes linear on Δ or: Δ is a representation of the tangent space in zero which is included in the real line.

The above analysis performs this integration with the idea that the time quantum could ideally be understood from the nilpotents.

The quantum effects in the vicinity of singularities are reminiscent of John Lane Bell's formula:

Vale *ict*, ave *iε* !* [7]

as an extension of the discussion of the introduction of imaginary time by Minkowski [1, Box 2.1 p. 51].

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*Goodbye *ict*, welcome *iε* !

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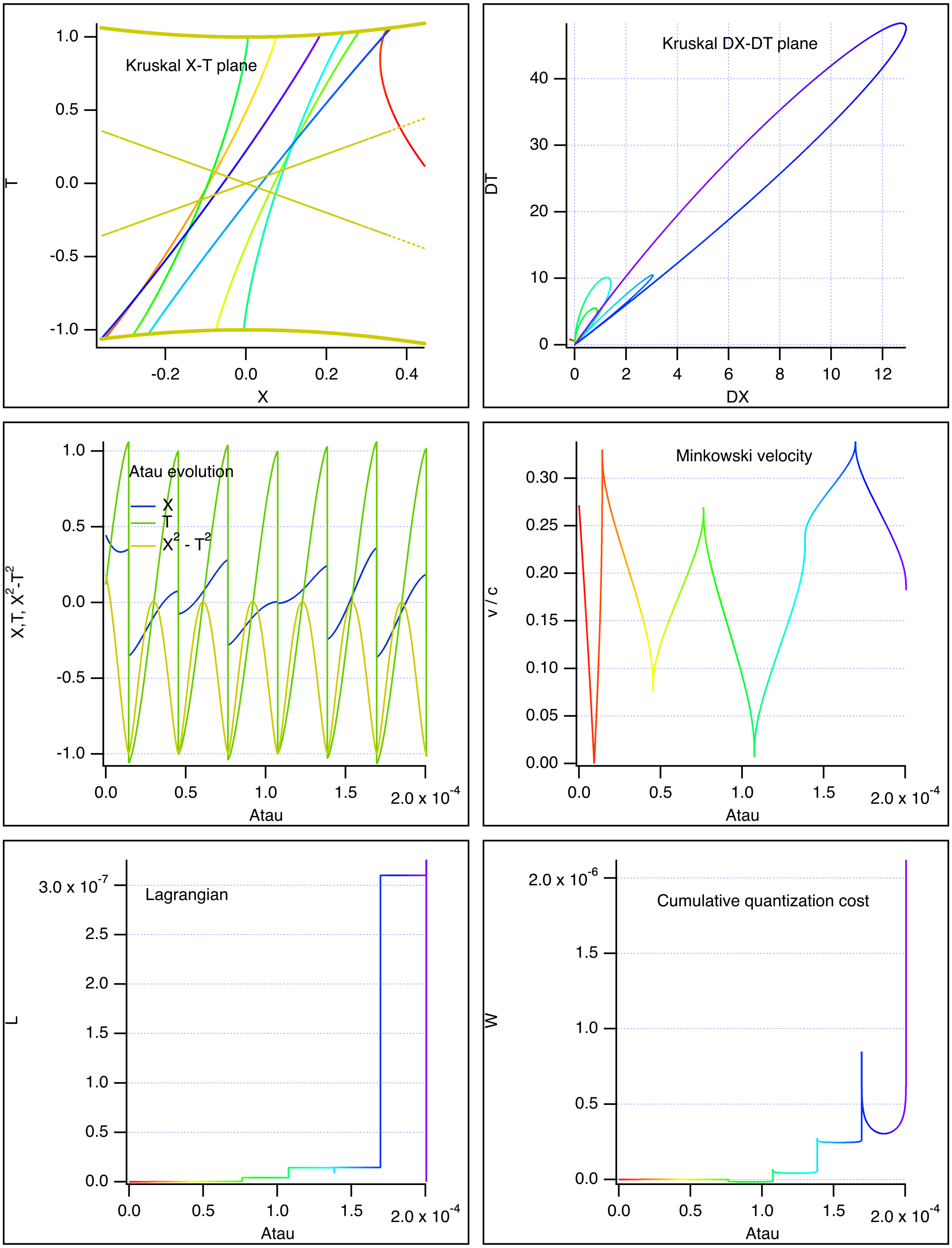

Fig. 5: Trajectory evolving towards the singularity. The variable Atau is simply the addition of the two real and imaginary clocks. The imaginary time is identified by a negative Lagrangian. The start of the trajectory is in red and its end in dark blue. The passages through the singularity are located at the points where $X^2 - T^2 < -1$.

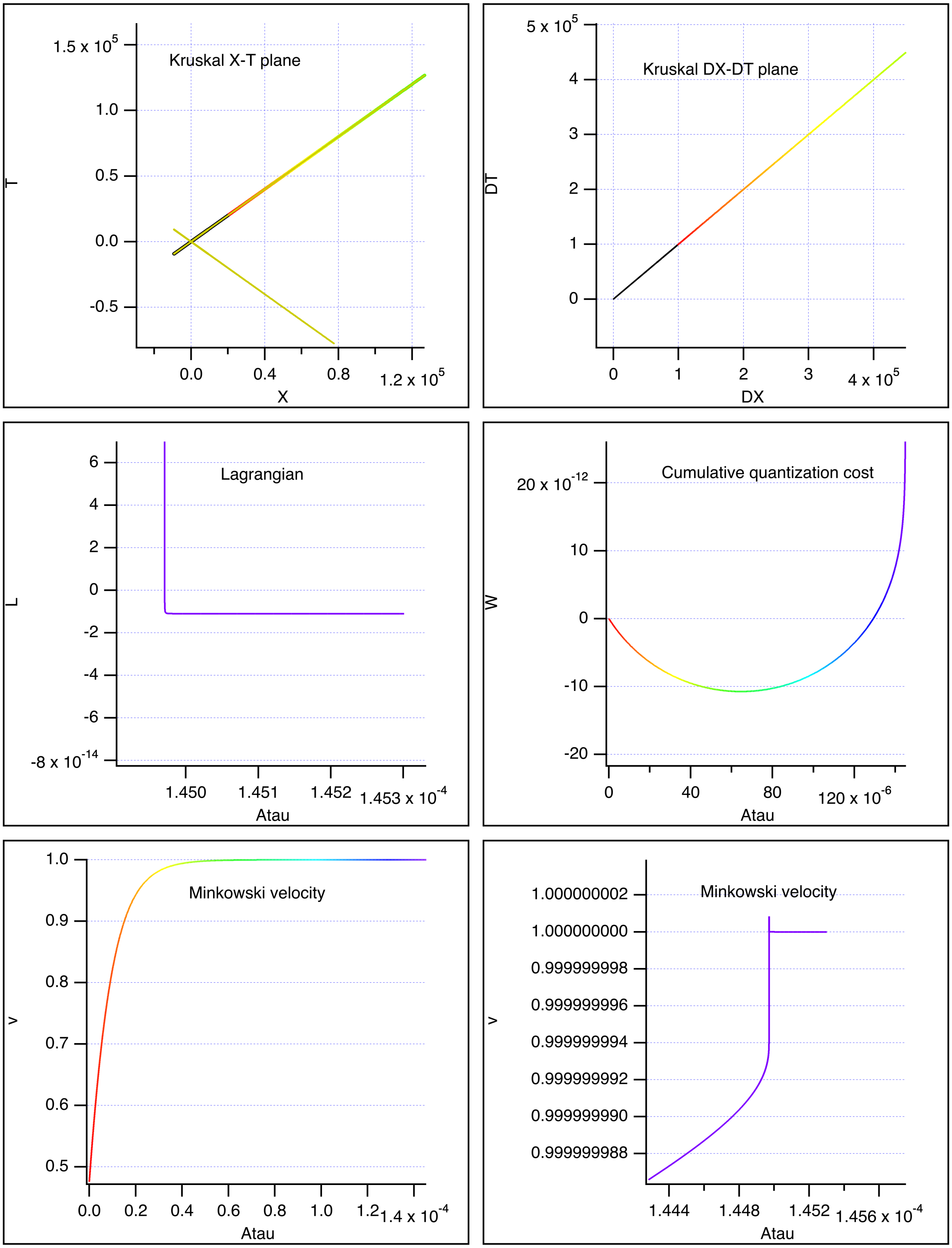

Fig. 6: Trajectory leading to a tachyon, before ending on the singularity. The colouring of the top two graphs is given by the imaginary clock from the black part. The calculation was redone by increasing the precision from 4096 to 8192 bits, with no significant difference.