Conditions for the Riemannian Description of Maxwell's Source-Free Field Equations

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In this paper, away from the intricate mathematics and philosophy presented in our earlier work [1], we demonstrate that well within Riemann geometry, Maxwell's electrodynamic source-free field equations [2] are indeed susceptible to a geometric description by the metric tensor, provided: (1) the non-linear term of the Riemann curvature tensor is assumed to vanish identically, and (2) the electromagnetic four-vector field A_μ obeys the gauge condition $A^\alpha \partial_\alpha A_\mu$. We strongly believe that this demonstration is important for physics because if the electromagnetic force can be given a geometric description, this most certainly will lead to the opening of new pathways for incorporating the gravitational force into such a scheme.

Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things.

Sir Isaac Newton (1642-1727)

1 Introduction

As far as prevailing wisdom is concerned, there is only one *Force of Nature* that is described geometrically and this is the force of gravity and its geometric description was handed down to us by Albert Einstein [3] in his intellectual masterpiece – the *General Theory of Relativity* (GTR). By geometric description, we here mean the ability of the force in question to submit to a metric description in a manner redolent or akin to the force of gravity in Einstein [3]'s GTR, where the gravitational force is described by the metric tensor $g_{\mu\nu}$. In turn, the metric tensor $g_{\mu\nu}$ evolves and is governed by the laws governing Riemann Geometry (RG).

Given our opening statement, the question naturally suggests itself: *Can Maxwell's electromagnetic force be given a geometric description?* Our answer to this question is that with the *proviso* that the:

1. Riemann curvature tensor is linearized, i.e. $\Gamma^\lambda_{\delta\sigma}\Gamma^\delta_{\mu\nu} - \Gamma^\lambda_{\delta\nu}\Gamma^\delta_{\mu\sigma} \equiv 0$, and the metric tensor $g_{\mu\nu}$ is decomposed into a product of the components of a four-vector, i.e. $g_{\mu\nu} = A_\mu A_\nu$,
2. Electromagnetic four-vector field A_μ obeys the gauge condition $A^\alpha \partial_\alpha A_\mu$,

then, one can successfully give a geometric description of Maxwell's [2] source-free field equations.

Herein, we have for clarity's sake removed most of the intricate mathematics and philosophy (found in [1]) so that our reader(s) will have a much greater appreciation of our ongoing work. We here only deal with the Riemann tensor and its identities and from that only, we demonstrate that a decomposed metric ($g_{\mu\nu} = A_\mu A_\nu$) can successfully lead one to the source-free Maxwell's equation [2]. This we believe is something that will provoke our reader(s) into thinking further (than meet the eye) by asking about the possibility of doing the same for the source-coupled field equations. Not only will this provoke the reader(s) into thinking about the possibility of a geometrically derived source-coupled Maxwell's equation [2], but of the possibility of a unity between gravitation, electricity and possibly the other two forces of Nature – the *weak* and *strong* nuclear forces.

Lastly, this article is organised as follows: in §2, we present the Riemann tensor and in addition to this, we introduce a gauge condition that linearises this tensor. In §3, we present the metric tensor in its decomposed form and some of the necessary gauge conditions. In §4, we write down the affine connection in terms of the decomposed metric tensor and from this exercise, we show that the Maxwellian electrodynamic tensor can be harnessed. In §5, we delve onto the main task of the day whereby we derive the Maxwellian source-free field equations purely from the Riemann tensor and lastly, in §6, we present a general discussion.

2 Riemann curvature tensor

From the view point of tensors, the Riemann curvature tensor $R^\lambda_{\mu\sigma\nu}$ has two components to it – i.e. the linear and non-linear parts which are themselves tensors. That is to say:

$$R^\lambda_{\mu\sigma\nu} = \underbrace{\overbrace{\Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu}}^{\hat{R}^\lambda_{\mu\sigma\nu}}}_{\textbf{linear terms}} + \underbrace{\overbrace{\Gamma^\lambda_{\delta\sigma}\Gamma^\delta_{\mu\nu} - \Gamma^\lambda_{\delta\nu}\Gamma^\delta_{\mu\sigma}}^{\check{R}^\lambda_{\mu\sigma\nu}}}_{\textbf{non-linear terms}} = \hat{R}^\lambda_{\mu\sigma\nu} + \check{R}^\lambda_{\mu\sigma\nu}\,, \quad (1)$$

where $\hat{R}^\lambda_{\mu\sigma\nu}$ and $\check{R}^\lambda_{\mu\sigma\nu}$ are the linear and non-linear components of the Riemann curvature tensor and these are defined as follows:

$$\hat{R}^\lambda_{\mu\sigma\nu} = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu}\,, \quad (2a)$$

$$\check{R}^\lambda_{\mu\sigma\nu} = \Gamma^\lambda_{\delta\sigma}\Gamma^\delta_{\mu\nu} - \Gamma^\lambda_{\delta\nu}\Gamma^\delta_{\mu\sigma}\,. \quad (2b)$$

Because $R^{\lambda}_{\mu\sigma\nu}$ and $\hat{R}^{\lambda}_{\mu\sigma\nu}$ are tensors, it directly follows that $\breve{R}^{\lambda}_{\mu\sigma\nu}$ is a tensor too. If, as proposed in [1], we are to choose as a natural gauge condition on our desired spacetime the condition $\breve{R}^{\lambda}_{\mu\sigma\nu} = 0$, then, in any subsequent system of coordinates and/or reference frame, this condition will hold because $\breve{R}^{\lambda}_{\mu\sigma\nu}$ is a tensor. What we now have is a linear Riemann world. Insofar as computations are concerned, such a world is certainly much easier to deal with. Besides this, one is able to obtain exact solutions from the resultant field equations. Whether or not this is the world that we live in, we can only compare our final results with what obtains in Nature.

3 Decomposition of the metric tensor

As is well known, the metric tensor $g_{\mu\nu}$ of RG has a total of sixteen components and as a result of the symmetry in its $\mu\nu$-indices, i.e. $g_{\mu\nu} = g_{\nu\mu}$, it has ten independent components. Starting in [4], we realised that the number of independent terms can be reduced from ten to four by way of casting this metric as a product of a four-vector A_μ, i.e.

$$g_{\mu\nu} = A_\mu A_\nu \,. \tag{3}$$

With the metric now written in this manner, we where able to write down a curved spacetime Dirac equation [4] using the same approach used by Dirac to arrive at the Dirac equation.

This four-vector A_μ is assumed to have unit magnitude throughout all of spacetime, i.e.

$$A^{\alpha} A_{\alpha} = 1 \,. \tag{4}$$

In [1], we have called this condition (4), the *Normalization Gauge Condition* (NGC). Differentiating this NGC with respect to: x^μ, we obtain the following corollary condition:

$$A^{\alpha} \partial_\mu A_{\alpha} = 0. \tag{5}$$

As will be seen in §5, this corollary condition (5) and the NGC, are necessary for the derivation that we shall carry out.

Apart from the NGC (4) and its corollary (5), we will also need the following condition for our derivation, i.e.

$$A^{\alpha} \partial_\alpha A_{\mu} = 0 \,. \tag{6}$$

At present, we have no ready natural justification for this condition, i.e. where it originates from, except that it is a necessary condition for our derivation.

4 Recomposition of the affine connection

The Christoffel three-symbol [5] (affine connection) is given by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(\partial_\nu g^{\lambda}{}_{\mu} + \partial_\mu g^{\lambda}{}_{\nu} - \partial^{\lambda} g_{\mu\nu}\right) . \tag{7}$$

Under the new decomposition of the metric given in (3), this affine connection can be recomposed or redefined by substituting the decomposed metric tensor. So doing, we obtain:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left[\partial_\nu\left(A^{\lambda}A_\mu\right) + \partial_\mu\left(A^{\lambda}A_\nu\right) - \partial^{\lambda}\left(A_\mu A_\nu\right)\right] . \tag{8}$$

Differentiating the terms of the metric in (8), we will have:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\Big(&\underbrace{A^{\lambda}\partial_\nu A_\mu}_{\textbf{Term I}} + \underbrace{A_\mu\partial_\nu A^{\lambda}}_{\textbf{Term II}} + \underbrace{A^{\lambda}\partial_\mu A_\nu}_{\textbf{Term III}} + \underbrace{A_\nu\partial_\mu A^{\lambda}}_{\textbf{Term IV}} - \\ &- \underbrace{A_\nu\partial^{\lambda}A_\mu}_{\textbf{Term V}} - \underbrace{A_\mu\partial^{\lambda}A_\nu}_{\textbf{Term VII}}\Big) .\end{aligned} \tag{9}$$

Rearranging the differentiated terms of the metric tensor labelled in (9) above as: Term I, II, III, *etc*, we will have:

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\Big[&\Big(\underbrace{A_\mu\partial_\nu A^{\lambda}}_{\textbf{Term II}} - \underbrace{A_\mu\partial^{\lambda}A_\nu}_{\textbf{Term VII}}\Big) + \\ &+ \Big(\underbrace{A_\nu\partial_\mu A^{\lambda}}_{\textbf{Term IV}} - \underbrace{A_\nu\partial^{\lambda}A_\mu}_{\textbf{Term V}}\Big) + \Big(\underbrace{A^{\lambda}\partial_\nu A_\mu}_{\textbf{Term I}} + \underbrace{A^{\lambda}\partial_\mu A_\nu}_{\textbf{Term III}}\Big)\Big] .\end{aligned} \tag{10}$$

From (10), we can now write the Christoffel as follows:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(A_\mu F^{\lambda}{}_{\nu} + A_\nu F^{\lambda}{}_{\mu} + A^{\lambda}H_{\mu\nu}\right) , \tag{11}$$

where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{12a}$$

$$H_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu \,. \tag{12b}$$

The object $F_{\mu\nu}$ in (12a) can easily be identified with Maxwell's electromagnetic field tensor [2] while the object $H_{\mu\nu}$ is a new object which may appear to be unrelated to Maxwell's electromagnetic field tensor [2]. As will be seen in the next section where we are going to derive the electrodynamic source-free field equations, this seemingly unrelated object $H_{\mu\nu}$ is what shall lead us to our desideratum.

Now, on the corollary to the NGC, i.e. (5) and (6), an application of these to (12), leads to the following corollary gauge conditions:

$$A^{\alpha}F_{\alpha\nu} = A^{\alpha}F_{\nu\alpha} = 0 \,, \tag{13a}$$

$$A^{\alpha}H_{\alpha\nu} = A^{\alpha}H_{\nu\alpha} = 0 \,. \tag{13b}$$

The above completes the necessary package of conditions needed to derive Maxwell's source-free field equations.

We shall now make a further reduction in the symbols by writing the affine connection as follows:

$$\Gamma^{\lambda}_{\mu\nu} = F^{\lambda}_{\mu\nu} + H^{\lambda}_{\mu\nu} \,, \tag{14}$$

where:

$$F^{\lambda}_{\mu\nu} = \frac{1}{2}\left(A_\mu F^{\lambda}{}_{\nu} + A_\nu F^{\lambda}{}_{\mu}\right) , \tag{15a}$$

$$H^{\lambda}_{\mu\nu} = \frac{1}{2}A^{\lambda}H_{\mu\nu} \,. \tag{15b}$$

With the affine connection written as we have written it in (14), we can now write the linear Riemann tensor as follows:

$$R^{\lambda}_{\mu\sigma\nu} = F^{\lambda}_{\mu\sigma\nu} + H^{\lambda}_{\mu\sigma\nu} \,, \tag{16}$$

where the new curvature tensors $F^{\lambda}_{\mu\sigma\nu}$ and $H^{\lambda}_{\mu\sigma\nu}$ are such that:

$$F^{\lambda}_{\mu\sigma\nu} = \partial_\sigma F^{\lambda}_{\mu\nu} - \partial_\nu F^{\lambda}_{\mu\sigma}\,, \tag{17a}$$

$$H^{\lambda}_{\mu\sigma\nu} = \partial_\sigma H^{\lambda}_{\mu\nu} - \partial_\nu H^{\lambda}_{\mu\sigma}\,. \tag{17b}$$

We are now ready to demonstrate that deeply embedded in the Riemann metric under the present metric decomposition (3), are the Maxwell source-free field equations [2].

5 Derivation

We know that the Riemann tensor satisfies the following first Bianchi identity:

$$R^{\lambda}_{\mu\sigma\nu} + R^{\lambda}_{\nu\mu\sigma} + R^{\lambda}_{\sigma\nu\mu} \equiv 0\,. \tag{18}$$

Multiplying this identity (18) throughout by A_γ, and then contracting the $\gamma\lambda$-indices of the resulting tensor, i.e. $\gamma = \lambda = \alpha$, (19) will reduce to:

$$A_\alpha R^{\alpha}_{\mu\sigma\nu} + A_\alpha R^{\alpha}_{\nu\mu\sigma} + A_\alpha R^{\alpha}_{\sigma\nu\mu} \equiv 0\,. \tag{19}$$

From the decomposition of $R^{\lambda}_{\mu\sigma\nu}$ into the curvature tensors $F^{\lambda}_{\mu\sigma\nu}$ and $H^{\lambda}_{\mu\sigma\nu}$ given in (16), it follows that we can decompose (19) into two corresponding parts as follows:

$$\begin{aligned} &\left(A_\alpha F^{\alpha}_{\mu\sigma\nu} + A_\alpha F^{\alpha}_{\nu\mu\sigma} + A_\alpha F^{\alpha}_{\sigma\nu\mu}\right) + \\ &+ \left(A_\alpha H^{\alpha}_{\mu\sigma\nu} + A_\alpha H^{\alpha}_{\nu\mu\sigma} + A_\alpha H^{\alpha}_{\sigma\nu\mu}\right) \equiv 0\,. \end{aligned} \tag{20}$$

In our calculation of (19), we shall first compute:

$$A_\alpha F^{\alpha}_{\mu\sigma\nu} + A_\alpha F^{\alpha}_{\nu\mu\sigma} + A_\alpha F^{\alpha}_{\sigma\nu\mu}\,,$$

followed by:

$$A_\alpha H^{\alpha}_{\mu\sigma\nu} + A_\alpha H^{\alpha}_{\nu\mu\sigma} + A_\alpha H^{\alpha}_{\sigma\nu\mu}\,.$$

5.1 Part I

We know that:

$$\begin{aligned} 2F^{\lambda}_{\mu\sigma\nu} = &\left(\partial_\sigma A_\mu F^{\lambda}{}_{\nu} + A_\mu \partial_\sigma F^{\lambda}{}_{\nu} + \partial_\sigma A_\nu F^{\lambda}{}_{\mu} + A_\nu \partial_\sigma F^{\lambda}{}_{\mu}\right) \\ &- \left(\partial_\nu A_\mu F^{\lambda}{}_{\sigma} + A_\mu \partial_\nu F^{\lambda}{}_{\sigma} + \partial_\nu A_\sigma F^{\lambda}{}_{\mu} + A_\sigma \partial_\nu F^{\lambda}{}_{\mu}\right). \end{aligned} \tag{21}$$

Multiplying $F^{\lambda}_{\mu\sigma\nu}$ by A_γ, and then contracting the $\gamma\lambda$-indices of the resulting tensor, i.e. $\gamma = \lambda = \alpha$, and taking into account the gauge condition $A_\alpha F^{\alpha}{}_{\mu} = 0$, (21) will reduce to:

$$\begin{aligned} 2A_\alpha F^{\alpha}_{\mu\sigma\nu} = &\left(A_\alpha A_\mu \partial_\sigma F^{\alpha}{}_{\nu} - A_\alpha A_\mu \partial_\nu F^{\alpha}{}_{\sigma}\right) + \\ &+ \left(A_\alpha A_\nu \partial_\sigma F^{\alpha}{}_{\mu} - A_\alpha A_\sigma \partial_\nu F^{\alpha}{}_{\mu}\right). \end{aligned} \tag{22}$$

Writing $A_\alpha A_\mu = \mathrm{g}_{\alpha\mu}$, $A_\alpha A_\nu = \mathrm{g}_{\alpha\nu}$ and $A_\alpha A_\sigma = \mathrm{g}_{\alpha\sigma}$, we will have:

$$\begin{aligned} 2A_\alpha F^{\alpha}_{\mu\sigma\nu} = &\left(\mathrm{g}_{\alpha\mu} \partial_\sigma F^{\alpha}{}_{\nu} - \mathrm{g}_{\alpha\mu} \partial_\nu F^{\alpha}{}_{\sigma}\right) + \\ &+ \left(\mathrm{g}_{\alpha\nu} \partial_\sigma F^{\alpha}{}_{\mu} - \mathrm{g}_{\alpha\sigma} \partial_\nu F^{\alpha}{}_{\mu}\right), \end{aligned} \tag{23}$$

hence, lowering the indices in (23) where applicable, we will have:

$$2A_\alpha F^{\alpha}_{\mu\sigma\nu} = \left(\partial_\sigma F_{\mu\nu} - \partial_\nu F_{\mu\sigma}\right) + \left(\partial_\sigma F_{\nu\mu} - \partial_\nu F_{\sigma\mu}\right). \tag{24}$$

Using in (24) the antisymmetry property of the electromagnetic field tensor, namely $F_{\nu\mu} = -F_{\mu\nu}$ and $F_{\sigma\mu} = -F_{\mu\sigma}$, we will have:

$$2A_\alpha F^{\alpha}_{\mu\sigma\nu} = 0 \;\;\Rightarrow\;\; A_\alpha F^{\alpha}_{\mu\sigma\nu} = 0\,, \tag{25}$$

hence:

$$A_\alpha F^{\alpha}_{\mu\sigma\nu} + A_\alpha F^{\alpha}_{\nu\mu\sigma} + A_\alpha F^{\alpha}_{\sigma\nu\mu} = 0\,. \tag{26}$$

Next, we need to calculate $A_\alpha H^{\alpha}_{\mu\sigma\nu} + A_\alpha H^{\alpha}_{\nu\mu\sigma} + A_\alpha H^{\alpha}_{\sigma\nu\mu}$.

5.2 Part II

We know that:

$$\begin{aligned} 2H^{\lambda}_{\mu\sigma\nu} = &\left(A^\lambda \partial_\sigma H_{\mu\nu} - A^\lambda \partial_\nu H_{\mu\sigma}\right) + \\ &+ \left(H_{\mu\nu} \partial_\sigma A^\lambda - H_{\mu\sigma} \partial_\nu A^\lambda\right). \end{aligned} \tag{27}$$

Multiplying $H^{\lambda}_{\mu\sigma\nu}$ by A_γ, and then contracting the $\gamma\lambda$-indices of the resulting tensor, i.e. $\gamma = \lambda = \alpha$, (27) will reduce to:

$$\begin{aligned} 2A_\alpha H^{\alpha}_{\mu\sigma\nu} = &\left(A_\alpha A^\alpha \partial_\sigma H_{\mu\nu} - A_\alpha A^\alpha \partial_\nu H_{\mu\sigma}\right) + \\ &+ \left(H_{\mu\nu} A_\alpha \partial_\sigma A^\alpha - H_{\mu\sigma} A_\alpha \partial_\nu A^\alpha\right). \end{aligned} \tag{28}$$

From the normalization gauge ($A_\alpha A^\alpha = 1$), and the corollary of this gauge, namely $A_\alpha \partial_\mu A^\alpha = 0$, (28) reduces to:

$$2A_\alpha H^{\alpha}_{\mu\sigma\nu} = \partial_\sigma H_{\mu\nu} - \partial_\nu H_{\mu\sigma} = \partial_\sigma F_{\nu\mu} + \partial_\nu F_{\mu\sigma}\,, \tag{29}$$

hence:

$$2A_\alpha H^{\alpha}_{\mu\sigma\nu} = \partial_\sigma F_{\nu\mu} + \partial_\nu F_{\mu\sigma} \neq 0\,, \tag{30a}$$

$$2A_\alpha H^{\alpha}_{\nu\mu\sigma} = \partial_\mu F_{\sigma\nu} + \partial_\sigma F_{\nu\mu} \neq 0\,, \tag{30b}$$

$$2A_\alpha H^{\alpha}_{\sigma\nu\mu} = \partial_\nu F_{\mu\sigma} + \partial_\mu F_{\sigma\nu} \neq 0\,. \tag{30c}$$

From (30), it is clear that:

$$\begin{aligned} &A_\alpha H^{\alpha}_{\mu\sigma\nu} + A_\alpha H^{\alpha}_{\nu\mu\sigma} + A_\alpha H^{\alpha}_{\sigma\nu\mu} = \\ &= \partial_\mu F_{\sigma\nu} + \partial_\nu F_{\mu\sigma} + \partial_\sigma F_{\nu\mu}. \end{aligned} \tag{31}$$

Now, we can put everything together.

5.3 Summary

Putting everything together, i.e. (19), (26) and (31), we will have:

$$\begin{aligned} &A_\alpha R^{\alpha}_{\mu\sigma\nu} + A_\alpha R^{\alpha}_{\nu\mu\sigma} + A_\alpha R^{\alpha}_{\sigma\nu\mu} = \\ &= 0 + \left(\partial_\mu F_{\sigma\nu} + \partial_\nu F_{\mu\sigma} + \partial_\sigma F_{\nu\mu}\right) \equiv 0\,, \end{aligned} \tag{32}$$

hence:

$$\partial_\mu F_{\sigma\nu} + \partial_\nu F_{\mu\sigma} + \partial_\sigma F_{\nu\mu} \equiv 0\,. \tag{33}$$

Of course, (33) is indeed Maxwell's source-free field equations [2].

6 Discussion

Given a linearized Riemann curvature tensor, we have herein demonstrated, in clear and no uncertain terms, the conditions under which Maxwell's electrodynamic source-free field equations [2] are readily susceptible to a geometric description by a metric tensor in much the same way that the force of gravity is described by the metric tensor in Einstein's GTR [3]. This description has come at the following cost:

1. A decomposed metric $g_{\mu\nu} = A_\mu A_\nu$. This reduces the number of independent fields from ten to four. In accordance with *Occam's Razor*, this is a welcome development in any theory, especially if the new theory does not destroy the old but enriches and engenders it.
2. A linearised (i.e. $\Gamma^\lambda_{\delta\sigma}\Gamma^\delta_{\mu\nu} - \Gamma^\lambda_{\delta\nu}\Gamma^\delta_{\mu\sigma} \equiv 0$) Riemann curvature tensor. This eliminates the computational complexity that ensues from these non-linear terms.
3. A normalization (i.e. $A^\alpha A_\alpha = 1$) gauge on the four-vector. A corollary to this normalization gauge is that $A^\alpha \partial_\mu A_\alpha = 0$.
4. Introduction of an extra exo-gauge condition to the metric four-vector, i.e. $A^\alpha \partial_\alpha A_\mu = 0$.

Having demonstrated the susceptibility of Maxwell [2]'s source-free field equation to a geometric description, the value of this work is that it indicates that Maxwell's equations [2] may very well be embedded deep inside the labyrinth of Riemann geometry. In addition to this, we strongly believe that this work is important for physics because if the electromagnetic force can be given a geometric description, this most certainly will lead to the opening of new pathways for incorporating the gravitational force into such a scheme.

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