Twin Universes Confirmed by General Relativity

Patrick Marquet
Calais, France. E-mail: patrick.marquet6@wanadoo.fr

The twin universes hypothesis was initially proposed by A. D. Sakharov followed by several astrophysicists in order to explain some unsolved questions mainly the current dark matter issue. However, no one could provide a physical justification as to the origin and existence of the second universe. We show here that general relativity formally yields two coupled field equations exhibiting an opposite sign, which lends support to Sakharov's conjecture. To this end, we use Cartan's calculus, in order to derive the differential form of Einstein's field equations. This procedure readily leads to a particular representation whereby the Einstein's field equation is classically inferred from the "Landau-Lifshitz superpotential". Since this superpotential is a fourth rank tensor (density-like), a second field equation naturally arises from the derivation, a result which has been so far totally obscured and overlooked in all classical treatments.

Notations

Space-time Greek indices α, β run from 0, 1, 2, 3 for local coordinates.
Space-time Latin indices a, b run from 0, 1, 2, 3 for a general basis.
Space-time signature is -2.
Einstein's constant is denoted by $\varkappa$.
We assume here that $c = 1$.

1 Differential calculus

1.1 The field equations in GR (short overview)

In General Relativity, the line element on the 4-dimensional pseudo-Riemannian manifold (M, g) is given by the interval $ds^2 = g_{ab}\, dx^a dx^b$. By varying the action $\mathcal{S} = \mathcal{L}_{\mathrm{E}}\, d^4x$ with respect to g_{ab} where the Lagrangian density is given by

$$\mathcal{L}_{\mathrm{E}} = g^{ab}\sqrt{-g}\left(\left\{{e \atop ab}\right\}\left\{{d \atop de}\right\} - \left\{{d \atop ae}\right\}\left\{{e \atop bd}\right\}\right), \tag{1}$$

one infers the symmetric Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R\,, \tag{2}$$

where, as is well-known,

$$R_{bc} = \partial_a \left\{{a \atop bc}\right\} - \partial_c \left\{{a \atop ba}\right\} + \left\{{d \atop bc}\right\}\left\{{a \atop da}\right\} - \left\{{d \atop ba}\right\}\left\{{a \atop dc}\right\} \tag{3}$$

is the (symmetric) Ricci tensor whose contraction gives the curvature scalar R, and $\left\{{e \atop ab}\right\}$ denote the Christoffel symbols of the second kind.

The source free field equations are

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\, R + \Lambda\, g_{ab} = 0\,, \tag{4}$$

where Λ is usually called the cosmological constant. The second rank tensor G_{ab} is symmetric and is only function of the metric tensor components g_{ab} and their first and second order derivatives. Due to the Bianchi's identities the Einstein tensor is conceptually conserved

$$\nabla_a\, G^a_b = 0\,, \tag{5}$$

where ∇_a is the Riemann covariant derivative.

When a massive source is present, the field equations become

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab}\,(R - 2\Lambda) = \varkappa\, T_{ab}\,. \tag{6}$$

If ρ is the matter density, T_{ab} is here the tensor describing the pressure of a free fluid

$$T_{ab} = \rho\, u_a u_b\,. \tag{7}$$

1.2 The general structures on a manifold

Let us now consider a 4-manifold M referred to a vector basis e_α. A locally defined set of four linearly independent vector fields, determined by the dual basis θ^β of the local coordinates

$$\theta^\beta = \mathrm{a}^\beta_a\, dx^a \tag{8}$$

is called a *tetrad field* or *vierbein* [1].

On this manifold, it is well known that the connection coefficients $\Gamma^\gamma_{\alpha\beta}$ can be decomposed in the most general sense as

$$\Gamma^\gamma_{\alpha\beta} = \left\{{\gamma \atop \alpha\beta}\right\} + \mathrm{K}^\gamma_{\alpha\beta} + (\Gamma^\gamma_{\alpha\beta})_{\mathrm{S}}\,, \tag{9}$$

where $\mathrm{K}^\gamma_{\alpha\beta}$ is the *contorsion tensor* which is built from the *torsion tensor* $T^\gamma_{\alpha\beta} = \frac{1}{2}\,(\Gamma^\gamma_{[\beta\alpha]} - \Gamma^\gamma_{\alpha\beta})$, and

$$(\Gamma^\gamma_{\alpha\beta})_{\mathrm{S}} = \frac{1}{2}\, g^{\gamma\mu}\left(\mathrm{D}_\beta\, g_{\alpha\mu} + \mathrm{D}_\alpha\, g_{\beta\mu} - \mathrm{D}_\mu\, g_{\alpha\beta}\right) \tag{10}$$

is the *segment connection* formed with the general covariant derivatives of the metric tensor (denoted here by D instead of the Riemann symbol ∇)

$$\mathrm{D}_\gamma\, g_{\alpha\beta} = \partial_\gamma\, g_{\alpha\beta} - \Gamma_{\alpha\gamma\beta} - \Gamma_{\beta\gamma\alpha} \neq 0\,. \tag{11}$$

The connection $(\Gamma^{\gamma}_{\alpha\beta})_{\mathrm{S}}$ characterizes a particular property of the manifold related to a second type of structure, called the *segment curvature*. This additional curvature results from the variation of the parallel transported vector around a small closed path.

In a dual basis θ^{α}, the following 2-forms can be associated with any parallel transported vector along the closed path:

— a rotation curvature form

$$\Omega^{\alpha}_{\beta} = \frac{1}{2}\, R^{\alpha}{}_{\beta\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta}, \tag{12}$$

— a torsion form

$$\Omega^{\alpha} = \frac{1}{2}\, T^{\alpha}_{\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta}, \tag{13}$$

— a segment curvature form

$$\Omega = -\frac{1}{2}\, R^{\alpha}{}_{\alpha\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta}. \tag{14}$$

These are the maximum admissible mathematical structures defining a general manifold.

1.3 The Cartan structure equations

We now introduce the *Cartan procedure*. This is a powerful coordinate calculus extensively used in the foregoing.

Let us first define the connection forms

$$\Gamma^{\alpha}_{\beta} = \left\{ {\alpha \atop \gamma\beta} \right\} \theta^{\gamma}. \tag{15}$$

The *first Cartan structure equation* is related to the torsion by [2, p. 40]

$$\Omega^{\alpha} = \frac{1}{2}\, T^{\alpha}_{\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta} = d\theta^{\alpha} + \Gamma^{\alpha}_{\gamma} \wedge \theta^{\gamma}\,. \tag{16}$$

and the *second Cartan structure equation* is [2, p. 42]

$$\Omega^{\alpha}_{\beta} = \frac{1}{2}\, R^{\alpha}{}_{\beta\gamma\delta}\, \theta^{\gamma} \wedge \theta^{\delta} = d\Gamma^{\alpha}_{\beta} + \Gamma^{\alpha}_{\gamma} \wedge \Gamma^{\beta}_{\gamma}\,. \tag{17}$$

and $R^{\alpha}{}_{\beta\gamma\delta}$ are here the components of the curvature tensor in the most general sense.

Within the Riemannian framework alone (torsion free), $R^{\alpha}{}_{\beta\gamma\delta}$ reduces to the Riemann curvature tensor components and the first structure equation (16) becomes

$$d\theta^{\alpha} = -\Gamma^{\alpha}_{\gamma} \wedge \theta^{\gamma}. \tag{18}$$

We shall now define the absolute exterior differential D of a tensor valued p-form of type (r, s)

$$(\mathrm{D}\phi)^{i_1\ldots i_r}_{j_1\ldots j_s} = d\phi^{i_1\ldots i_r}_{j_1\ldots j_s} + \Gamma^{i_1}_{k} \wedge \phi^{k\, i_1\ldots i_r}_{\;\; j_1\ldots j_s} + \ldots - \Gamma^{k}_{j_1} \wedge \phi_{k\;\; j_2\ldots j_s}^{\; i_1\ldots i_r} - \ldots$$

As a simple example, the Bianchi identities can be simply written with the exterior differential as

$$\mathrm{D}\Omega^{\alpha} = \Omega^{\alpha}_{\beta} \wedge \theta^{\beta} \quad \text{(1st Bianchi identity)},$$

$$\mathrm{D}\Omega^{\alpha}_{\beta} = 0 \qquad \text{(2nd Bianchi identity)}.$$

2 The differential Einstein equations

2.1 The Einstein action

We first recall the definition of the Hodge star operator for an oriented n-dimensional pseudo-Riemannian manifold (M, g), wherein the volume element is determined by g

$$\eta = \sqrt{-g}\, \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Let $\Lambda_k(\mathrm{E})$ be the subspace of completely antisymmetric multilinear forms on the real vector space E. The *Hodge star operator* * is a linear isomorphism $\Lambda_k(\mathrm{M}) \to \Lambda_{n-k}(\mathrm{M})$, where $k \leqslant n$. If $\{\theta^0, \theta^1, \theta^2, \theta^3\}$ is an oriented basis of 1-forms, this operator is defined by

$$^*(\theta^{i_1} \wedge \theta^{i_2} \wedge \ldots \theta^{i_k}) =$$

$$= \frac{\sqrt{-g}}{(n-k)!}\, \varepsilon_{j_1\ldots j_n}\, g^{j_1 i_1} \ldots g^{j_k i_k}\, \theta^{j_{k+1}} \wedge \ldots \wedge \theta^{j_n}.$$

With this preparation, the Einstein action simply reads

$$^*R = R\,\eta\,. \tag{19}$$

To show this, we express this action in terms of tetrads. With $\eta^{\mu\nu} = {}^*(\theta^{\mu} \wedge \theta^{\nu})$ and taking into account (17) we have

$$\eta_{\beta\gamma} \wedge \Omega^{\beta\gamma} = \frac{1}{2}\, \eta_{\beta\gamma}\, R^{\beta\gamma}_{\;\;\mu\nu}\, \theta^{\mu} \wedge \theta^{\nu},$$

$$^*(\theta^{\mu} \wedge \theta^{\nu}) = \frac{1}{2}\, \eta_{\beta\gamma\sigma\rho}\, g^{\beta\gamma} \theta^{\sigma} \wedge \theta^{\rho},$$

i.e., we have

$$\eta_{\beta\gamma} = \frac{1}{2}\, \eta_{\beta\gamma\sigma\rho}\, \theta^{\sigma} \wedge \theta^{\rho}. \tag{20}$$

Thus, we have

$$\eta_{\beta\gamma} \wedge \theta^{\mu} \wedge \theta^{\nu} = \frac{1}{2}\, \eta_{\beta\gamma\sigma\rho}\, \theta^{\sigma} \wedge \theta^{\rho} \wedge \theta^{\mu} \wedge \theta^{\nu} = (\delta^{\mu}_{\beta}\, \delta^{\nu}_{\gamma} - \delta^{\mu}_{\gamma}\, \delta^{\nu}_{\beta})\, \eta\,,$$

$$\eta_{\beta\gamma} \wedge \Omega^{\beta\gamma} = \frac{1}{2}\, (\delta^{\mu}_{\beta}\, \delta^{\nu}_{\gamma} - \delta^{\mu}_{\gamma}\, \delta^{\nu}_{\beta})\, R^{\beta\gamma}_{\;\;\mu\nu}\, \eta = R\,\eta = {}^*R\,.$$

Taking also into account (20), we compute the absolute exterior differential $\mathrm{D}\,\eta_{\beta\gamma} = \frac{1}{2}\, \mathrm{D}\,(\eta_{\beta\gamma\sigma\rho}\, \theta^{\sigma} \wedge \theta^{\rho})$. In an orthonormal frame $\eta_{\beta\gamma\sigma\rho}$ is constant and $\mathrm{D}\,\eta_{\beta\gamma\sigma\rho} = 0$. This manifests the fact that in the *Riemannian framework* (metric connection), orthonormality is preserved under parallel transfer. Therefore, $\mathrm{D}\,\eta_{\beta\gamma} = \eta_{\beta\gamma\sigma\rho}\, \mathrm{D}\,\theta^{\sigma} \wedge \theta^{\rho}$.

Now, keeping in mind that the basis θ^{σ} is a tensor 1-form of the type (1,0), the first structure equation reads

$$\mathrm{D}\,\theta^{\sigma} = \Omega^{\sigma},$$

$$\mathrm{D}\,\eta_{\beta\gamma} = \eta_{\beta\gamma\sigma\rho}\, \Omega^{\sigma} \wedge \theta^{\rho} = \Omega^{\sigma} \wedge \eta_{\beta\gamma\sigma}\,.$$

The latter equation is zero for the Riemannian connection $\mathrm{D}\,\eta_{\beta\gamma} = 0$. In the same way, we can show that

$$\mathrm{D}\,\eta^{\beta\gamma}_{\;\;\alpha} = d\eta^{\beta\gamma}_{\;\;\alpha} + \Gamma^{\beta}_{\delta} \wedge \eta^{\delta\gamma}_{\;\;\alpha} + \Gamma^{\alpha}_{\delta} \wedge \eta^{\beta\delta}_{\;\;\alpha} - \Gamma^{\delta}_{\alpha} \wedge \eta^{\beta\gamma}_{\;\;\delta} = 0 \tag{21}$$

with $\eta^{\beta\gamma}_{\;\;\alpha} = {}^*(\theta^{\beta} \wedge \theta^{\gamma} \wedge \theta_{\alpha})$ (all indices are raised or lowered with $g_{\alpha\beta}$ from $g = g_{\alpha\beta}\, \theta^{\alpha} \otimes \theta^{\beta}$).

2.2 The Einstein field equations

From (20), we infer that

$$\eta_{\beta\gamma\delta} = \eta_{\beta\gamma\delta\lambda}\,\theta^{\lambda}. \tag{22}$$

Under the variation $\delta\theta^{\beta}$ of the orthonormal tetrad fields θ^{β}, we have

$$\delta\,(\eta_{\beta\gamma}\wedge\Omega^{\beta\gamma}) = \delta\eta_{\beta\gamma}\wedge\Omega^{\beta\gamma} + \eta_{\beta\gamma\delta}\wedge\delta\Omega^{\beta\gamma\delta}.$$

Now, using (20) and (22) yields

$$\delta\eta_{\beta\gamma} = \frac{1}{2}\,\delta\,(\eta_{\beta\gamma\delta\lambda}\,\theta^{\delta}\wedge\theta^{\lambda}) = \delta\theta^{\delta}\wedge\eta_{\beta\gamma\delta}\,.$$

Hence, applying the varied second structure equation

$$\delta\Omega^{\beta\gamma} = d\,\delta\Gamma^{\beta\gamma} + \delta\Gamma^{\beta}_{\eta}\wedge\Gamma^{\eta\gamma} + \Gamma^{\beta}_{\eta}\wedge\delta\Gamma^{\eta\gamma},$$

we obtain

$$\begin{aligned}\delta\,(\eta_{\beta\gamma}\wedge\Omega^{\beta\gamma}) &= \delta\theta^{\delta}\wedge(\eta_{\beta\gamma\delta}\wedge\Omega^{\beta\gamma}) + d\,(\eta_{\beta\gamma}\wedge\delta\Gamma^{\beta\gamma}) - \\ &- d\eta_{\beta\gamma}\wedge\delta\Gamma^{\beta\gamma} + \eta_{\beta\gamma}\wedge(\delta\Gamma^{\beta}_{\eta}\wedge\Gamma^{\eta\gamma} + \Gamma^{\beta}_{\eta}\wedge\delta\Gamma^{\eta\gamma})\,,\end{aligned} \tag{23}$$

and from the second line we extract $d\eta_{\beta\gamma} + \eta_{\beta\gamma}\wedge(\Gamma^{\eta}_{\gamma} + \Gamma_{\beta\eta})$, which is just $\mathrm{D}\eta_{\beta\gamma}$.

However, we know that $\mathrm{D}\,\eta_{\beta\gamma} = 0$, and finally, the varied Einstein action is

$$\begin{aligned}\delta\,(\eta_{\beta\gamma}\wedge\Omega^{\beta\gamma}) &= \delta\theta^{\beta}\wedge(\eta_{\beta\gamma\delta}\wedge\Omega^{\gamma\delta}) + d\,(\eta_{\beta\gamma}\wedge\delta\Gamma^{\beta\gamma}) + \\ &+ (\text{exact differential}).\end{aligned} \tag{24}$$

The global Lagrangian density $\mathcal{L}$ in the presence of matter is written as

$$\mathcal{L} = -\,\frac{1}{2\varkappa}\,{}^{*}R + \mathcal{L}_{\text{matter}}\,.$$

Setting up ${}^{*}T_{\beta}$ as the energy-momentum 3-form for *bare matter* we have the Lagrangian density for the varied matter

$$\delta\mathcal{L}_{\text{matter}} = -\,\delta\theta^{\beta}\wedge{}^{*}T_{\beta}$$

and taking into account (24), the global variation is

$$\delta\mathcal{L} = -\,\delta\theta^{\beta}\wedge\left(\frac{1}{2\varkappa}\,\eta_{\beta\gamma\delta}\wedge\Omega^{\gamma\delta} + {}^{*}T_{\beta}\right) + (\text{exact differential}).$$

We eventually arrive at the field equations in the differential form

$$-\,\frac{1}{2}\,\eta_{\beta\gamma\delta}\wedge\Omega^{\gamma\delta} = \varkappa\,{}^{*}T_{\beta}\,, \tag{25}$$

where T_{α} is related to the energy-momentum tensor $T_{\alpha\beta}$ by $T_{\alpha} = T_{\alpha\beta}\,\theta^{\beta}$.

In the same manner, we can obtain $G_{\alpha} = G_{\alpha\beta}\,\theta^{\beta}$ for the Einstein tensor $G_{\alpha\beta}$ (see Appendix A).

2.3 The energy-momentum tensor

It is well known however, that $G_{\alpha\beta}$ is intrinsically conserved while the massive tensor $T_{\alpha\beta}$ is not. This is because the gravitational field is not included in $T_{\alpha\beta}$. To restore conservation for the energy-momentum tensor, we start by reformulating (25) in the form

$$-\,\frac{1}{2}\,\Omega_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha} = \varkappa\,{}^{*}T_{\alpha}\,. \tag{25bis}$$

Then, we use the second structure equation under the following form

$$\Omega_{\beta\gamma} = d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta}\wedge\Gamma^{\mu}_{\gamma} \tag{26}$$

so that we obtain

$$d\Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha} = d(\Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha}) + \Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha}\,. \tag{27}$$

Then, using (21) in (26), we infer

$$\begin{aligned}d\Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha} &= d\,(\Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha}) + \\ &+ \Gamma_{\beta\gamma}\wedge(-\Gamma^{\beta}_{\delta}\wedge\eta^{\delta\gamma}{}_{\alpha} - \Gamma^{\gamma}_{\delta}\wedge\eta^{\beta\delta}{}_{\alpha} + \Gamma^{\delta}_{\alpha}\wedge\eta^{\beta\gamma}{}_{\delta})\,.\end{aligned} \tag{28}$$

Adding the second contribution of (26) to (28), we obtain the Einstein field equations in a new form, which is

$$-\,\frac{1}{2}\,d(\Gamma_{\beta\gamma}\wedge\eta^{\beta\gamma}{}_{\alpha}) = \varkappa\,({}^{*}T_{\alpha} + {}^{*}t_{\alpha})\,, \tag{29}$$

where we denote

$${}^{*}t_{\alpha} = -\,\frac{1}{2\varkappa}\,\Gamma_{\beta\gamma}\wedge(\Gamma_{\delta\alpha}\wedge\eta^{\beta\gamma\delta} - \Gamma^{\gamma}_{\delta}\wedge\eta^{\beta\delta}{}_{\alpha})\,, \tag{30}$$

and the quantity ${}^{*}t_{\alpha}$ is interpreted as the energy-momentum (pseudo-tensor) of the gravitational field generated by this distributed matter.

Equation (29) readily implies the conservation law

$$d\,({}^{*}T_{\alpha} + {}^{*}t_{\alpha}) = 0\,. \tag{31}$$

Writing

$$t_{\alpha} = t_{\alpha\beta}\,\theta^{\beta}\,, \tag{32}$$

we see that $t_{\alpha\beta}$ describes the gravitational field, which can be expressed, for example, by the Einstein-Dirac pseudo-tensor [3, p. 61].

From (30) we verify that $t_{\alpha\beta}$ is not symmetric. To correct this problem, we shall not apply the Belinfante symmetrization procedure [4]. Instead, we will modify the field differential equations. We first revert to the field equations (25) in which we insert $\eta^{\alpha\beta\gamma} = \eta^{\alpha\beta\gamma\delta}\,\theta_{\delta}$. With (26) this yields

$$-\,\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\theta_{\delta}\wedge(d\Gamma_{\beta\gamma} - \Gamma_{\mu\beta}\wedge\Gamma^{\mu}_{\gamma}) = \varkappa\,{}^{*}T^{\alpha} \tag{33}$$

leading to

$$-\,\frac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,d(\Gamma_{\beta\gamma}\wedge\theta_{\delta}) = \varkappa\,({}^{*}T^{\alpha} + {}^{*}t^{\alpha})\,, \tag{34}$$

where

$$ {}^{*}t^{\alpha} = -\frac{1}{2}\,\varkappa\,\eta^{\alpha\beta\gamma\delta}(\Gamma_{\mu\beta}\wedge\Gamma^{\mu}_{\gamma}\wedge\theta_{\delta} - \Gamma_{\beta\gamma}\wedge\Gamma_{\mu\delta}\wedge\theta^{\mu})\,. \qquad (35)$$

We see that ${}^{*}t^{\alpha}$ is unaffected by the exterior product terms in the bracket, therefore $t_{\alpha\beta}$ is now symmetric. In that case, we idendify ${}^{*}t^{\alpha}$ with the Landau-Lifshitz 3-form ${}^{*}t^{\alpha}_{\text{L-L}}$, which yields the corresponding pseudo-tensor $t^{\alpha\beta}_{\text{L-L}}$.

3 The 4th rank tensor equation

3.1 The first set of Einstein's field equations

Multiply (34) by $\sqrt{-g}$. Then, taking $\eta_{\alpha\beta\gamma\delta} = -\frac{1}{2}\sqrt{-g}\,\varepsilon_{\alpha\beta\gamma\delta}$ into account, we find a new form for the field equations

$$ -d\,(\sqrt{-g}\,\eta^{\alpha\beta\gamma\delta}\,\Gamma_{\beta\gamma}\wedge\theta_{\delta}) = 2\varkappa\sqrt{-g}\,({}^{*}T^{\alpha} + {}^{*}t^{\alpha}_{\text{L-L}}) \qquad (36)$$

or

$$ d\,(\sqrt{-g}\,\Gamma^{\beta\gamma}\wedge\eta^{\alpha}_{\ \beta\gamma}) = 2\sqrt{-g}\,({}^{*}T^{\alpha} + {}^{*}t^{\alpha}_{\text{L-L}})\,. \qquad (37)$$

From these equations follows immediately the differential conservation law

$$ d\left[\sqrt{-g}\,({}^{*}T^{\alpha} + {}^{*}t^{\alpha}_{\text{L-L}})\right] = 0\,. \qquad (38)$$

A tedious calculation eventually shows that

$$ d\,(\sqrt{-g}\,\Gamma^{\beta\gamma}\wedge\eta^{\alpha}_{\ \ \beta\gamma}) = \frac{1}{\sqrt{-g}}\,H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\beta\gamma}\,\eta_{\nu}\,, \qquad (39)$$

where

$$ H^{\alpha\beta\nu\gamma} = -g\,(g^{\alpha\nu}g^{\beta\gamma} - g^{\beta\nu}g^{\gamma\alpha}) \qquad (40)$$

is the "Landau-Lifshitz superpotential" [5, eq. 101.2]. Therefore the field equations read here

$$ H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\beta\gamma} = 2\varkappa\left[-g\,(T^{\alpha\nu} + t^{\alpha\nu}_{\text{L-L}})\right]. \qquad (41)$$

Explicitly, we have

$$ H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\beta\gamma} = \partial_{\beta}\left\{\partial_{\gamma}\left[-g\,(g^{\alpha\nu}g^{\beta\gamma} - g^{\beta\nu}g^{\gamma\alpha})\right]\right\}. \qquad (42)$$

Remark: It is essential to note that the quantities $t^{\alpha\nu}_{\text{L-L}}$ do not represent a true tensor. Indeed, the gravitational field can be transformed away at any point and its energy is not localizable. This is why the left hand side of (41) and (42) exhibits ordinary derivatives instead of covariant ones.

The 4th rank tensor $H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\beta\gamma}$ can be regarded as a special choice of $R^{\alpha\nu}$ — the Ricci tensor, where all first derivatives of the metric tensor cancel out at this given point.

The Landau-Lifshitz pseudo-tensor has the form

$$ \begin{aligned} (-g)\,t^{\alpha\nu}_{\text{L-L}} = \frac{1}{2\varkappa}\Big\{ & {}^{\#}g^{\alpha\nu}_{\ \ ,\lambda}\,{}^{\#}g^{\lambda\mu}_{\ \ ,\mu} - {}^{\#}g^{\alpha\lambda}_{\ \ ,\lambda}\,{}^{\#}g^{\nu\mu}_{\ \ ,\mu} + \\ & + \frac{1}{2}\,g^{\alpha\nu}g_{\lambda\mu}\,{}^{\#}g^{\lambda\theta}_{\ \ ,\rho}\,{}^{\#}g^{\rho\mu}_{\ \ ,\theta} + g_{\mu\lambda}\,g^{\theta\rho}\,{}^{\#}g^{\alpha\lambda}_{\ \ ,\theta}\,{}^{\#}g^{\nu\mu}_{\ \ ,\rho} - \\ & - \left(g^{\alpha\lambda}g_{\mu\theta}\,{}^{\#}g^{\nu\theta}_{\ \ ,\rho}\,{}^{\#}g^{\mu\rho}_{\ \ ,\lambda} + g^{\nu\lambda}g_{\mu\theta}\,{}^{\#}g^{\alpha\theta}_{\ \ ,\rho}\,{}^{\#}g^{\mu\rho}_{\ \ ,\lambda}\right) + \\ & + \frac{1}{8}\,(2g^{\alpha\lambda}g^{\nu\mu} - g^{\alpha\nu}g^{\lambda\mu})(2g_{\theta\rho}\,g_{\delta\tau} - g_{\rho\delta}\,g_{\theta\tau})\,{}^{\#}g^{\theta\tau}_{\ \ ,\lambda}\,{}^{\#}g^{\rho\delta}_{\ \ ,\mu}\Big\}, \end{aligned} \qquad (43)$$

where ${}^{\#}g^{\alpha\nu} = \sqrt{-g}\,g^{\alpha\nu}$.

When velocities are low and the gravitational field is weak (42) reduces to

$$ H^{0i0j}_{\ \ \ \ \ \ ,ij} = \partial_{i}\left\{\partial_{j}\left[-g\,(g^{00}g^{ij} - g^{i0}g^{j0})\right]\right\}, \qquad (44)$$

where $i, j, \ldots = 1, 2, 3$ are the spatial indices. We can write this equation in mixed indices by lowering one of the space indices

$$ H^{0\ 0j}_{\ i\ \ ,ij} = \partial_{i}\,\partial_{j}\,(-g\,g^{00}\delta^{j}_{i})\,. \qquad (45)$$

When $i = j$, the Newton law is retrieved through the weak potential $g^{00} = 1 + 2\psi$ as (45) reduces to the Laplacian

$$ \partial_{i}\,\partial_{i}\,g^{00} = \Delta\psi\,, \qquad (46)$$

so that we obtain the well-known Poisson equation

$$ \Delta\psi = G\rho\,,$$

where G is Newton's constant.

Therefore, at the Newtonian approximation, we can write the *generalized Poisson equation*, which has the form

$$ H^{0i0j}_{\ \ \ \ \ \ ,ij} = 2\varkappa\sqrt{-g}\,(T^{00} + t^{00}_{\text{L-L}})\,, \qquad (47)$$

where the Newtonian pseudo-tensor $t^{00}_{\text{L-L}}$ reads

$$ \begin{aligned} (-g)\,t^{00}_{\text{L-L}} = \frac{1}{2\varkappa}\Big\{ & {}^{\#}g^{00}_{\ \ ,k}\,{}^{\#}g^{kn}_{\ \ ,n} - {}^{\#}g^{0k}_{\ \ ,k}\,{}^{\#}g^{0n}_{\ \ ,n} + \\ & + \frac{1}{2}\,g^{00}g_{kn}\,{}^{\#}g^{kr}_{\ \ ,l}\,{}^{\#}g^{ln}_{\ \ ,r} + g_{nk}\,g^{rl}\,{}^{\#}g^{0k}_{\ \ ,r}\,{}^{\#}g^{0k}_{\ \ ,l} - \\ & - \left(g^{0k}g_{nr}\,{}^{\#}g^{0r}_{\ \ ,l}\,{}^{\#}g^{nl}_{\ \ ,k} + g^{0k}g_{nr}\,{}^{\#}g^{0r}_{\ \ ,l}\,{}^{\#}g^{nl}_{\ \ ,k}\right) + \\ & + \frac{1}{8}\,(2g^{0k}g^{0n} - g^{00}g^{kn})(2g_{rl}\,g_{sm} - g_{ls}\,g_{rm})\,{}^{\#}g^{rm}_{\ \ ,k}\,{}^{\#}g^{ls}_{\ \ ,n}\Big\}. \end{aligned} \qquad (48)$$

Unlike the classical Newtonian theory, the static bare mass density generally produces a gravitational field, which is described by $t^{00}_{\text{L-L}}$ at the considered point.

Remark: A slightly variable cosmological term L term induces a stress energy tensor of vacuum which restores a conserved property of the r.h.s. of equation (6) thus avoiding the use of the ill-defined gravitational field pseudo-tensor as shown in [6, 7].

3.2 The second set of Einstein's field equations

The *second rank* tensor field equations have been inferred from a *fourth rank* tensor density like. It is then natural to consider a second set of field equations which is contained in the former.

A close inspection of the "Landau-Lifshitz superpotential" (40) leads to the obvious choice for this second field equation

$$ d\,(\sqrt{-g}\,\Gamma^{\gamma\alpha}\wedge\eta^{\beta}_{\ \gamma\alpha}) = \frac{1}{\sqrt{-g}}\,H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\gamma\alpha}\,\eta_{\nu}\,, \qquad (49)$$

$$ H^{\alpha\beta\nu\gamma}_{\ \ \ \ \ \ ,\gamma\alpha} = 2\varkappa\sqrt{-g}\,(T^{\beta\nu} + t^{\beta\nu}_{\text{L-L}}). \qquad (50)$$

Note that (41) and (50) are linked using a common index. Furthermore, each set of field equations differ from a sign.

Proof: Let us label the "negative" equation as

$$^{(-)}H^{\alpha\beta\nu\gamma}{}_{,\gamma\alpha} = \partial_\alpha \left\{ \partial_\gamma \left[-g \left(g^{\alpha\nu} g^{\beta\gamma} - g^{\beta\nu} g^{\gamma\alpha} \right) \right] \right\}. \qquad (51)$$

Now in the same manner as for (44), equation (51) reduces to

$$^{(-)}H^{i\,00}{}_{,ij} = \partial_i \left\{ \partial_j \left[-g \left(g^{i0} g^{0j} - g^{00} g^{ij} \right) \right] \right\}. \qquad (52)$$

Lowering one of the space indices we obtain

$$^{(-)}H^{i\,00}{}_{j\,,ij} = \partial_i \left(\partial_j \, g g^{00} \delta^j_i \right), \qquad (53)$$

which is just the opposite to $H^{0\,0j}_{\;i\;,ij} = \partial_i \partial_j \left(-g g^{00} \delta^j_i \right)$ (45).

Had we set $i = j$, we would have found

$$^{(-)}\Delta\psi = -\Delta\psi\,. \qquad (54)$$

As a consequence, the right member of the Poisson equation (in our orthonormal frame) should also reverse sign

$$^{(-)}(G\rho) = -\,G\rho\,. \qquad (55)$$

Since the Einstein constant is here a common factor, we infer that mass densities of each field equations differ from a sign as well as the gravitation potential ψ.

Therefore, in the framework of the Newtonian approximation we find two opposite field tensors which induce two opposite energy density tensors which we label as

$$^{(+)}\left(T^{00} + t^{00}_{\text{L-L}} \right) \quad \text{and} \quad ^{(-)}\left(T^{00} + t^{00}_{\text{L-L}} \right). \qquad (56)$$

3.3 Two antagonist manifolds

Conservation properties lead to the following evident corresponding equivalences

$$H^{\alpha\beta\nu\gamma}{}_{,\beta\gamma} \rightarrow {}^{(+)}G^{\alpha\nu} = \varkappa \left[^{(+)}(T^{\alpha\nu} + t^{\alpha\nu}_{\text{L-L}}) \right]. \qquad (57)$$

$$H^{\alpha\beta\nu\gamma}{}_{,\gamma\alpha} \rightarrow {}^{(-)}G^{\beta\nu} = \varkappa \left[^{(-)}(T^{\beta\nu} + t^{\beta\nu}_{\text{L-L}}) \right]. \qquad (58)$$

Hence, the field equation (57) can be regarded as being defined on a "positive" manifold with respect to the "negative" manifold on which is defined the field equation (58).

Remark: One should always bear in mind that both $^{(+)}G^{\alpha\nu}$ and $^{(-)}G^{\beta\nu}$ are coupled through the 4th rank tensor $H_{\beta\alpha\gamma\mu}$, which necessarily imposes that indices must keep their respective label. The "intertwined" metrics are then

$$^{(+)}ds^2 = {}^{(+)}g_{\alpha\nu}\, dx^\alpha dx^\nu, \qquad ^{(-)}ds^2 = {}^{(-)}g_{\beta\nu}\, dx^\beta dx^\nu, \qquad (59)$$

and, in the "vierbein" (tetrad) formalism, we have

$$^{(+)}g_{\alpha\nu} = \mathrm{e}^a_\alpha\, \mathrm{e}^b_\nu\, \eta_{ab}\,, \qquad ^{(-)}g_{\beta\nu} = \mathrm{e}^a_\beta\, \mathrm{e}^b_\nu\, \eta_{ab}\,, \qquad (60)$$

where η_{ab} is the Minkowski tensor.

One thus writes the *Pfaffian metrics* as

$$^{(+)}ds^2 = \eta_{ab}\, {}^{(+)}\eta^a \eta^b, \qquad ^{(-)}ds^2 = \eta_{ab}\, {}^{(-)}\theta^a \theta^b, \qquad (61)$$

$$^{(+)}\theta^\alpha = \mathrm{e}^a_\alpha\, dx^\alpha, \qquad ^{(-)}\theta^a = \mathrm{e}^a_\beta\, dx^\beta, \qquad (62)$$

The common basis 1-form $\theta^b = \mathrm{e}^b_\nu\, dx^\nu$ outlines the coupling between the metrics.

Obviously, in a flat space-time, $^{(+)}g_{\alpha\nu}$ and $^{(-)}g_{\beta\nu}$ coincide with η_{ab} meaning that the twin universes emerge from curvature.

Conclusions and outlook

The twin universe hypothesis recently saw a revived interest.

Several astrophysicists conjectured that it could provide an appropriate explanation to the puzzle of the dark energy and dark matter issues and other unsolved observational data questions [8–15]. However, all these theories do not justify the origin of the double universe which remains a pure arbitrary statement, not relying on any sound physical grounds. In here we showed that General Relativity formally confirms the existence of two coupled Einstein's field equations characterizing two co-existing antagonist manifolds.

General Relativity further shows that there exists at most two such field equations [16].

We hope that this formal demonstration will help to substantiate the current research in astrophysics.

Appendix. Classical Einstein tensor retrieved from the differential equations

Using (12), the field equations

$$-\frac{1}{2}\, \eta_{\beta\gamma\delta} \wedge \Omega^{\gamma\delta} = \varkappa\, {}^*T_\beta \qquad (A1)$$

can be written in the form

$$-\frac{1}{4}\, \eta_\alpha{}^\mu{}_\nu\, \theta^\sigma \wedge \theta^\delta R^\nu{}_{\mu\sigma\delta} = \varkappa\, T_{\alpha\beta}\, \eta^\beta. \qquad (A2)$$

We first use the following relations

$$\eta^\alpha \wedge {}^*\theta^\alpha, \qquad (A3)$$

$$\eta_\alpha = \frac{1}{3!} \left(\eta_{\alpha\beta\gamma\delta}\, \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta \right) = \frac{1}{3!}\, \theta^\beta \wedge \eta_{\alpha\beta}\,. \qquad (A4)$$

Then, applying the following Riemannian identities

$$\theta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha\, \eta\,,$$

$$\theta^\gamma \wedge \eta_{\alpha\beta} = \delta^\gamma_\beta\, \eta_\alpha - \delta^\gamma_\alpha\, \eta_\beta\,,$$

$$\theta^\delta \wedge \eta_{\alpha\beta\gamma} = \delta^\delta_\gamma\, \eta_{\alpha\beta} + \delta^\delta_\beta\, \eta_{\gamma\alpha} + \delta^\delta_\alpha\, \eta_{\beta\gamma}\,,$$

$$\theta^\varepsilon \wedge \eta_{\alpha\beta\gamma\delta} = \delta^\varepsilon_\delta\, \eta_{\alpha\beta\gamma} - \delta^\varepsilon_\gamma\, \eta_{\delta\alpha\beta} + \delta^\varepsilon_\beta\, \eta_{\gamma\delta\alpha} - \delta^\varepsilon_\alpha\, \eta_{\beta\gamma\delta}\,,$$

we obtain

$$- \frac{1}{4} R^{\mu\nu}{}_{\sigma\tau} \Big[\delta^{\tau}_{\nu} (\delta^{\sigma}_{\mu} \eta_{\alpha} - \delta^{\sigma}_{\nu} \eta_{\mu}) + \\ + \delta^{\tau}_{\mu} (\delta^{\sigma}_{\alpha} \eta_{\nu} - \delta^{\sigma}_{\nu} \eta_{\alpha}) + \delta^{\tau}_{\alpha} (\delta^{\sigma}_{\nu} \eta_{\mu} - \delta^{\sigma}_{\mu} \eta_{\nu}) \Big] = \\ = - \frac{1}{2} R^{\mu\nu}{}_{\mu\nu} \eta_{\alpha} + R^{\mu\nu}{}_{\alpha\nu} \eta_{\mu} = \\ = - \frac{1}{2} R^{\beta\nu}{}_{\alpha\nu} \eta_{\beta} - \frac{1}{2} \eta^{\beta}_{\alpha} R^{\mu\nu}{}_{\mu\nu} \eta_{\beta} = \\ = \left(R^{\beta}{}_{\alpha} - \frac{1}{2} \delta^{\beta}_{\alpha} R \right) \eta_{\beta} \,. \qquad (A5)$$

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