How to Couple the Space-Time Curvature With the Yang-Mills Theory

Patrick Marquet
Calais, France. E-mail: patrick.marquet6@wanadoo.fr

We suggest here a new approach to couple space-time curvature with the three fundamental forces (interactions) of the standard model described by the Yang-Mills Theory. This is achieved through the extension of the Einstein tensor in the framework of the Weyl formalism (Weyl-Einstein tensor) which is known to exhibit a particular 4-vector referred to as the Weyl-Einstein vector. The Weyl-Einstein manifold so defined admits a tangent Minkowski space at a given point, where this particular vector asymptotically identifies with the Yang-Mills gauge field vectors. As a result, the Weyl-Einstein tensor implicitly interacts with the particles' masses and fields provided by the Yang-Mills equations. Assuming that the principle of equivalence always holds, a very simple grand unification with gravity could be achieved in this way.

Notations

Space-time Greek indices α, β run from 0, 1, 2, 3 for local coordinates.
Latin indices a, b are the group indices.
Space-time signature is -2.
We assume here that $c = 1$.

Introduction

Fields Φ are used to describe the fundamental particles known in modern physics. In Quantum Electrodynamics such fields associated with these particles must be chosen consistent with the symmetries in nature which include for example the space-time symmetries of Special Relativity. The fields Ψ are either scalars (neutral or charged) with spin-zero/spin-1 particles, or fermions with spin-$\frac{1}{2}$ particles. Initially, it was thought that these symmetries should be global symmetries, not depending on the position in space and time. However, it is well known that the laws of electromagnetism possess another *local symmetry*, in which charge is locally conserved, meaning that charged fields have a phase (in the exponent) that varies freely from point to point. This feat led Yang and Mills to suggest that local symmetries be extended from this U(1) group to non *Abelian symmetries* based on *local* gauge invariance which open the way to unify the electromagnetism, weak and strong interactions: U(1) × SU(2) × SU(3) is today known as the *standard model* elaborated by Glashow, Weinberg, Salam and Ward (1979 Nobel Prize). As we know, this theory implies the existence of *gauge fields* $A_\mu(x)$, which are necessarily part of a new covariant derivative $\mathrm{D}_\mu = \partial_\mu - i\mathrm{e}Am(x)$, where e is a coupling constant (see §2.1). In a curved space-time, the classical theory makes use of the Riemann derivative ∇_μ, and D_μ is thus generalized to $\nabla_\mu - i\mathrm{e}A_\mu(x)$ (see, for example, [1, p. 68]). However, the gauge fields $A_\mu(x)$, do not account for the space-time curvature except in the case of the electromagnetic field alone through the Einstein field equations.

Herein, we tackle this problem in a different way:

a) We start by defining a *Weyl connection* that exhibits a particular 4-vector (Weyl-Einstein vector) which induces extended curvature tensors;

b) From these curvatures is inferred the *Weyl-Einstein tensor* which is conceptually conserved like its standard counterpart which it generalizes;

c) A simple relation is established whereby the Weyl-Einstein 4-vector is asymptotically related to the Yang-Mills field vectors.

All three contributions (electromagnetic, weak and strong interactions) are then permitted to interact with the Weyl-Einstein 4-tensor through their respective gauge field vectors alone. A simple grand unification could be achieved through this particular coupling.

1 The Weyl-Einstein tensor

1.1 The curvatures

1.1.1 General issues

Following Lichnerowicz [2], we start by defining the *symmetric Weyl-Einstein connection* on a semi-metric 4-manifold denoted by $\mathfrak{M}$, i.e.

$$W^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \frac{1}{2}\, g^{\alpha\beta}\left(g_{\mu\beta}J_\nu + g_{\nu\beta}J_\mu - g_{\mu\nu}J_\beta\right) \tag{1.1}$$

or, in another form,

$$W^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \frac{1}{2}\left(\delta^\alpha_\mu J_\nu + \delta^\alpha_\nu J_\mu - g_{\mu\nu}J^\alpha\right). \tag{1.1bis}$$

From the point m in the neighbourhood of the Lorentz manifold denoted (M, g), where $\exists$ is a congruence of differentiable lines such that $\forall m' \in (\mathrm{M}, \mathrm{g})$, we may have the conformal metric

$$ds^2_{\mathrm{W}} = e^J ds^2, \tag{1.1ter}$$

where $J = \int_m^{m'} J_\mu dx^\mu$.

In general, the form $dJ = J_\mu dx^\mu$ is non-integrable. The 4-vector J_μ is referred to as the *Weyl-Einstein vector*.

1.1.2 The Weyl-Einstein 4th rank curvature tensor

With the Weyl connection $W^\alpha_{\mu\nu}$ we construct the *Weyl-Einstein curvature tensor* which is assumed to have the standard form of the Riemann-Christoffel tensor

$$(R^{\alpha}_{\ \beta\mu\nu})_{\mathrm{W}} = \partial_\nu W^{\ \alpha}_{\beta\mu} - \partial_\mu W^{\ \alpha}_{\beta\nu} + W^{\ \lambda}_{\beta\mu} W^{\ \alpha}_{\lambda\nu} - W^{\ \lambda}_{\beta\nu} W^{\ \alpha}_{\lambda\mu} . \quad (1.2)$$

Inspection shows that the following identity takes place

$$(R^{\rho}_{\ \alpha\beta\mu})_{\mathrm{W}} + (R^{\rho}_{\ \mu\alpha\beta})_{\mathrm{W}} + (R^{\rho}_{\ \beta\mu\alpha})_{\mathrm{W}} = 0 . \quad (1.3)$$

Using the Riemann covariant derivative denoted using a semi-colon, the Bianchi identity also reads

$$(R^{\rho}_{\ \alpha\beta\mu})_{\mathrm{W}\,;\delta} + (R^{\rho}_{\ \alpha\delta\beta})_{\mathrm{W}\,;\mu} + (R^{\rho}_{\ \alpha\mu\delta})_{\mathrm{W}\,;\beta} = 0 . \quad (1.3\text{bis})$$

Let us now express $(R_{\mu\nu\alpha\beta})_{\mathrm{W}}$ with the metric connection ∇_β. Setting $(\Gamma^\rho_{\nu\alpha})_{\mathrm{J}} = \frac{1}{2}\,(\delta^\rho_\nu J_\alpha + \delta^\rho_\alpha J_\nu - g_{\nu\alpha} J^\rho)$, we obtain

$$\begin{aligned}(R_{\mu\nu\alpha\beta})_{\mathrm{W}} = R_{\mu\nu\alpha\beta} &+ g_{\mu\rho}\nabla_\beta\,(\Gamma^\rho_{\nu\alpha})_{\mathrm{J}} - \\ &- \frac{1}{2}\, g_{\mu\rho}\left[\nabla_\alpha\,(\Gamma^\rho_{\nu\beta})_{\mathrm{J}} + \nabla_\nu\,(\Gamma^\rho_{\alpha\beta})_{\mathrm{J}}\right] + \\ &+ g_{\mu\rho}\left[(\Gamma^\rho_{\lambda\beta})_{\mathrm{J}}(\Gamma^\lambda_{\nu\alpha})_{\mathrm{J}} - (\Gamma^\rho_{\lambda\alpha})_{\mathrm{J}}(\Gamma^\lambda_{\nu\beta})_{\mathrm{J}}\right] + \\ &+ g_{\mu\nu}\left[\partial_\alpha(\Gamma^\rho_{\beta\rho})_{\mathrm{J}} - \partial_\beta(\Gamma^\rho_{\alpha\rho})_{\mathrm{J}}\right].\end{aligned} \quad (1.4)$$

1.1.3 The Weyl-Einstein 2nd rank tensor

Relation (1.4) eventually leads to the contracted tensor

$$\begin{aligned}(R^{\delta}_{\ \alpha\beta\delta})_{\mathrm{W}} = (R_{\alpha\beta})_{\mathrm{W}} = R_{\alpha\beta} &+ \nabla_\nu\,(\Gamma^\nu_{\alpha\beta})_{\mathrm{J}} - \nabla_\beta\,(\Gamma^\nu_{\alpha\nu})_{\mathrm{J}} + \\ &+ (\Gamma^\lambda_{\alpha\beta})_{\mathrm{J}}(\Gamma^\nu_{\lambda\nu})_{\mathrm{J}} - (\Gamma^\lambda_{\alpha\rho})_{\mathrm{J}}(\Gamma^\rho_{\lambda\beta})_{\mathrm{J}}\end{aligned}$$

we then have the splitting

$$(R_{\alpha\beta})_{\mathrm{W}} = (R_{(\alpha\beta)})_{\mathrm{W}} + (R_{[\alpha\beta]})_{\mathrm{W}} , \quad (1.5)$$

where

$$\begin{aligned}(R_{(\alpha\beta)})_{\mathrm{W}} = R_{\alpha\beta} &+ \nabla_\nu\,(\Gamma^\nu_{\alpha\beta})_{\mathrm{J}} - \frac{1}{2}\left[\nabla_\beta\,(\Gamma^\nu_{\alpha\nu})_{\mathrm{J}} + \nabla_\alpha\,(\Gamma^\nu_{\beta\nu})_{\mathrm{J}}\right] + \\ &+ (\Gamma^\lambda_{\alpha\beta})_{\mathrm{J}}(\Gamma^\nu_{\lambda\nu})_{\mathrm{J}} - (\Gamma^\lambda_{\alpha\rho})_{\mathrm{J}}(\Gamma^\rho_{\lambda\beta})_{\mathrm{J}} ,\end{aligned} \quad (1.6)$$

$$(R_{[\alpha\beta]})_{\mathrm{W}} = \partial_\alpha\,(\Gamma^\nu_{\beta\nu})_{\mathrm{J}} - \partial_\beta\,(\Gamma^\nu_{\alpha\nu})_{\mathrm{J}} . \quad (1.6\text{bis})$$

So forth, we check that $(\Gamma^\rho_{\nu\rho})_{\mathrm{J}} = \frac{1}{2}\,(\delta^\rho_\nu J_\rho + \delta^\rho_\rho J_\nu - g_{\nu\rho} J^\rho) = \frac{1}{2}\,(J_\nu + 4J_\nu - J_\nu) = 2J_\nu$. Thus we get

$$(R_{(\alpha\beta)})_{\mathrm{W}} = R_{\alpha\beta} - \frac{1}{2}\,(g_{\alpha\beta}\nabla_\nu J^\nu + J_\alpha J_\beta) , \quad (1.7)$$

$$(R_{[\alpha\beta]})_{\mathrm{W}} = 2\,(\partial_\alpha J_\beta - \partial_\beta J_\alpha) = 2J_{\alpha\beta} . \quad (1.8)$$

1.1.4 The Weyl-Einstein curvature scalar

Applying the contraction $R_{\mathrm{W}} = g^{\nu\alpha}(R_{\nu\alpha})_{\mathrm{W}}$, one obtains

$$\begin{aligned}R_{\mathrm{W}} = R &- \nabla_\rho\left[g^{\nu\alpha}(\Gamma^\rho_{\nu\alpha})_{\mathrm{J}}\right] - \nabla_\rho\left[g^{\nu\rho}(\Gamma^\rho_{\nu\rho})_{\mathrm{J}}\right] - \\ &- g^{\nu\alpha}\left[(\Gamma^\rho_{\nu\alpha})_{\mathrm{J}}(\Gamma^\nu_{\nu\rho})_{\mathrm{J}} - (\Gamma^\lambda_{\nu\rho})_{\mathrm{J}}(\Gamma^\rho_{\lambda\alpha})_{\mathrm{J}}\right],\end{aligned} \quad (1.9)$$

i.e.,

$$R_{\mathrm{W}} = R - \left(\nabla_\rho J^\rho + \frac{1}{2}\, J^2\right). \quad (1.10)$$

1.2 The Weyl-Einstein tensor

Here we omit the subscript $_{\mathrm{W}}$ for clarity. Unlike the Riemann-Christoffel curvature tensor, the Weyl curvature tensor is no longer antisymmetric on the pair of indices $\mu\nu$

$$R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta} = g_{\mu\nu} J_{\alpha\beta} , \quad (1.11)$$

or, in another form,

$$R^{\mu\nu}_{\ \ \alpha\beta} + R^{\nu\mu}_{\ \ \alpha\beta} = g^{\mu\nu} J_{\alpha\beta} . \quad (1.11\text{bis})$$

Raising the index α in the equation (1.3bis) and contracting on α and μ as well as on μ and δ, we obtain

$$R^{\mu\delta}_{\ \ \beta\mu\,;\delta} + R^{\mu\delta}_{\ \ \mu\delta\,;\beta} = 0 . \quad (1.12)$$

We next replace $R^{\mu\delta}_{\ \ \delta\beta}$ by its value taken from (1.11bis), and we eventually find

$$R^{\mu\delta}_{\ \ \mu\delta\,;\beta} + 2R^{\mu\delta}_{\ \ \beta\mu\,;\delta} + 2g^{\mu\delta} J_{\delta\beta\,;\mu} = 0 , \quad (1.13)$$

$$\left(R^{(\delta)}_{\ (\beta)} - \frac{1}{2}\,\delta^\delta_\beta R\right)_{;\delta} = -J^{\ \delta}_{\beta\,;\delta} , \quad (1.14)$$

which is just the conservation law for the tensor (re-instating the subscript $_{\mathrm{W}}$ and changing the indices)

$$(G_{\alpha\beta})_{\mathrm{W}} = (R_{(\alpha\beta)})_{\mathrm{W}} - \frac{1}{2}\left(g_{\alpha\beta} R_{\mathrm{W}} - 2J_{\alpha\beta}\right). \quad (1.15)$$

We call $(G_{\alpha\beta})_{\mathrm{W}}$ the *Weyl-Einstein tensor* expressed with the Riemannian derivatives. Lets us note that $(G_{\alpha\beta})_{\mathrm{W}}$ is no longer symmetric. In the pure Riemannian regime, this tensor obviously reduces to the usual Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R . \quad (1.16)$$

2 The unification

2.1 A short overview of the Yang-Mills theory

2.1.1 The principle of gauge invariance

Let us recall that a general Lie group G is defined by the representation of a group element denoted U in terms of its generators T^a

$$\mathrm{U} = \exp\left(-i\mathrm{e}\sum_{a=1}^{n} \mathrm{T}^a \mathrm{k}_a\right), \quad (2.1)$$

where e is a coupling constant generalizing the fundamental electronic charge *e* in the electromagnetic case. The group element U is defined by the values of the N constants k_a, and T^a are hermitian generators satisfying the associated Lie algebra

$$[\mathrm{T}^a, \mathrm{T}^b] = i\,\mathrm{C}^{abc}\,\mathrm{T}_c\,, \tag{2.2}$$

where C^{abc} are the real antisymmetric structure constants defining the algebra.

The SU(2) group is defined in terms of the set of all *unitary unimodular matrices* with (2×2) complex elements. The related constraints are known to be

$$\det \|\mathrm{U}\| = 1\,, \tag{2.3}$$

$$\mathrm{U}^{+}\mathrm{U} = \mathrm{U}\mathrm{U}^{+} = \mathrm{I}\,, \tag{2.4}$$

where I is the unit matrix, and U^{+} is the Hermitian conjugate of the matrix U.

2.1.2 Electromagnetism and local gauge invariance U(1)

Consider non-hermitian complex charged scalar fields written in terms of the real fields $\Phi_1(x)$ and $\Phi_2(x)$

$$\Phi(x) = \frac{1}{\sqrt{2}}\,[\Phi_1(x) + i\Phi_2(x)]\,, \tag{2.5}$$

$$\Phi^{+}(x) = \frac{1}{\sqrt{2}}\,[\Phi_1(x) - i\Phi_2(x)]\,.$$

The classical Lagrangian for this charged scalar field is

$$\mathcal{L} = \partial^{\mu}\Phi^{+}\,\partial_{\mu}\Phi - m^2\Phi^{+}\Phi\,, \tag{2.6}$$

where the first term corresponds to the *kinetic energy* of the scalar field, and the second the *potential energy* of the massive field (mass of the charged particle).

Noether's theorem states that the symmetry of charge conservation is equivalent to the invariance of $\mathcal{L}$ under the group U(1) of continuous phase rotations, specified by a single parameter k.

We then check that this Lagrangian is invariant under the continuous group of phase rotations of Φ called the *global Abelian gauge group* U(1)

$$\Phi(x) \to \Phi(x)\exp i\,\mathrm{k}\,, \tag{2.7}$$

$$\Phi^{+}(x) \to \Phi(x)\exp(-i\,\mathrm{k})\,, \tag{2.7bis}$$

with the real parameter k.

Eqs. (2.7) and (2.7bis) should be true even when the parameter k depends on x^{μ}, thus the phase difference between distinct space-time points is *unobservable*: it is called the *local gauge invariance principle*. However inspection shows that the kinetic energy Lagrangian $\partial^{\mu}\Phi^{+}\partial_{\mu}\Phi$ is not invariant under the local gauge transformation

$$\Phi(x) \to \Phi(x)\exp(-i\,\mathrm{k})\,\mathrm{Q}(x)\,. \tag{2.8}$$

This is because the derivative may now operate on the variable parameter $\mathrm{k}(x)$. To remedy this problem one is forced to introduce a new covariant derivative

$$\mathrm{D}_{\mu} = \partial_{\mu} - i\,\mathrm{e}\,A_{\mu}(x)\,, \tag{2.9}$$

where Q is the quantity of the charges of the fields Φ which is proportional to the fundamental electronic unit e.

Here, the vector field $A_{\mu}(x)$ transforms as

$$A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\,\mathrm{k}(x)\,. \tag{2.10}$$

Hence, it is also necessary to include a kinetic energy term in $\mathcal{L}$ which takes into account the introduction of the new gauge field $A_{\mu}(x)$. This is achieved by adding the term

$$(\mathcal{L})^{\mathrm{kin}}_{A} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}\,, \tag{2.11}$$

where we retrieve the electromagnetic field strength tensor

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \tag{2.12}$$

The new Lagrangian is now

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \mathcal{L}'\,[\Phi,\ \Phi^{+}\,\mathrm{D}_{\mu}\Phi\,\mathrm{D}_{\mu}\Phi^{+}]\,. \tag{2.13}$$

The tensor $F_{\mu\nu}$ is obviously invariant under the gauge transformation of (2.8), so $(\mathcal{L})^{\mathrm{kin}}_{A}$ is also gauge invariant. This symmetry group is the Abelian group U(1) with a single commuting generator $\mathrm{T}^1 = \mathrm{Q}$ satisfying

$$[\mathrm{T}^1, \mathrm{T}^1] = 0\,. \tag{2.14}$$

Unlike the classical theory, the equations of motion are obtained by varying the action $\mathcal{L}$ with respect to A_{μ} for the fixed Φ, i.e.,

$$\partial_{\nu}\left[\frac{\mathcal{L}}{\partial\left(\partial_{\nu}A_{\mu}(x)\right)}\right] - \frac{\partial\mathcal{L}}{\partial A_{\mu}(x)} = 0\,, \tag{2.15}$$

or, in another form,

$$\partial_{\nu}F^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial A_{\mu}(x)}\,. \tag{2.16}$$

From this equation, the current density is easily inferred

$$\mathrm{I}^{\mu}(x) = -\frac{1}{\mathrm{e}}\,\frac{\partial\mathcal{L}}{\partial A_{\mu}(x)}\,, \tag{2.17}$$

$$\mathrm{I}^{\mu}(x) = i\left[\Phi^{+}(x)\frac{\partial\mathcal{L}}{\partial\left(\mathrm{D}_{\mu}\Phi^{+}\right)} - \Phi^{\sigma+}(x)\frac{\partial\mathcal{L}}{\mathrm{D}_{\mu}\Phi^{+}}\right], \tag{2.18}$$

which is conserved

$$\partial_{\mu}\mathrm{I}^{\mu} = 0\,. \tag{2.19}$$

The associated charge is given by

$$\mathrm{Q} = \int I^{0}(x)\,d^3x = \int i\left\{\Phi^{+}\,\mathrm{D}_{\mu}\Phi - \mathrm{D}^{\mu}\,\Phi^{+}\Phi\right\}d^3x\,, \tag{2.20}$$

which also remains unchanged with time

$$\frac{d\mathrm{Q}}{dt} = 0\,, \tag{2.21}$$

$$\int \partial x_0\, \mathrm{I}^0(x)\, d^3x = 0\,, \tag{2.22}$$

or, equivalently, $\int \partial_\mu \mathrm{I}^\mu(x)\, d^3x = 0$.

This result is formally equivalent to the classical theory, but it also shows that this new approach remains a particular case of a higher symmetry principle which rules modern physics.

2.2 The unification

2.2.1 The gauge invariance of the Weyl-Einstein connection

If we were to define a Weyl-Einstein covariant derivative just as in (2.9), the connection coefficients $\mathrm{W}^\tau_{\ \mu\sigma}$ should be invariant under the conformal relation

$$g_{\alpha\beta} \to \mathrm{U}\, g_{\alpha\beta}\,, \tag{2.23}$$

where $U(x) > 0$ is a real scalar. Conformal invariance is here simply achieved by implementing the additional gauge condition

$$J_\mu \to J_\mu - \partial_\mu \mathrm{U} \tag{2.24}$$

as oneself can be easily convinced.

2.2.2 The Weyl-Einstein-Yang-Mills relation

Let us consider the time-like geodesic ds_W spanned by the connexion coefficients $\mathrm{W}^\tau_{\ \mu\sigma}$ (1.1ter). To this geodesic is associated the 1-form $dJ = J_\mu dx^\mu$. Likewise, we write the Minkowskian line element as ds to which we associate the Yang-Mills 1-form $dA = A_\mu dx^\mu$ where A_μ is the generic term that stands for every gauge field of any of the first three Yang-Mills interactions. A specific unification between the Yang-Mills theory and space-time curvature can be thus achieved through the interaction between the Yang-Mills gauge field and vectors and the Weyl-Einstein vector J_μ. Such a relation can be set so as to maintain the euclidean character of the Yang-Mills theory within the Weyl-Einstein formalism. To this end, we write

$$\frac{dJ}{dA} = 1 + \ln\left(\frac{ds_\mathrm{W}}{ds}\right), \tag{2.25}$$

$$dJ = dA\left[1 + \ln\left(\frac{ds_\mathrm{W}}{ds}\right)\right]. \tag{2.26}$$

When $ds_\mathrm{W} \to ds$, the 4-vector J_μ identifies with the Yang-Mills gauge field vector.

The Yang-Mills physics always takes place in the Minkowski space that is asymptotic to the genuine Weyl-Einstein manifold $\mathfrak{M}$. In this way, the vector J_μ inherent to space-time curvature is regarded as "embedding" all the Yang-Mills gauge fields thereby providing a specific unification as described below.

2.3 Application to the Yang-Mills interactions

2.3.1 The weak interaction (SU(2) symmetry)

Writing classically the group element as

$$\mathrm{U} = \exp\left[\,-\,i\,\mathrm{h}\,\mathrm{T}^a \mathrm{k}_a\right], \qquad a = 1, 2, 3\,, \tag{2.27}$$

with the generators

$$\mathrm{T}^a = \frac{\sigma^a}{2}\,, \tag{2.28}$$

where σ^a are the three 2×2 Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.29}$$

which satisfy [4, p. 2]

$$\mathrm{Tr}\left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right) = \frac{1}{2}\,\delta^{ab}, \tag{2.30}$$

$$\mathrm{Tr}\,\frac{\sigma^a}{2} = 0\,. \tag{2.31}$$

Here we must introduce three vector gauge fields B^a_μ, which are conveniently represented by the vector field

$$B_\mu(x) = \mathrm{T}^a B_{a\mu}(x)\,. \tag{2.32}$$

The transformation properties of B_μ are obtained from :

$$B_\mu(x) \to B_\mu(x) - \mathrm{T}^a \partial_\mu \mathrm{k}^a(x) + i\,\mathrm{h}\,\mathrm{k}^a(x)\left[\mathrm{T}^a, B_\mu(x)\right], \tag{2.33}$$

where h is the relevant coupling constant.

Here T^a satisfy the commutation relations with different structure constants

$$[\mathrm{T}^a, \mathrm{T}^b] = i\,\mathrm{f}^{abc}\,\mathrm{T}_c\,. \tag{2.34}$$

Using (2.30) in (2.33), then multiplying by T^b and taking the trace, we have the transformations laws of the individual gauge field $B^a_\mu(x)$

$$B^a_\mu(x) \to B^a_\mu(x) - \partial_\mu \mathrm{k}^a(x) + \mathrm{h}\,\mathrm{f}^a_{\ bc}\,\mathrm{k}^b(x)\, B^c_\mu(x)\,, \tag{2.35}$$

and the general form of the covariant derivative is

$$D_\mu = \partial_\mu - i\,\mathrm{h}\, B_\mu\,. \tag{2.36}$$

The SU(2) group relevant for matter representation is determined by the generators T^a, so that (2.36) is expressed by

$$\mathrm{D}_\mu = \partial_\mu - i\,\mathrm{h}\, B_{a\mu}\mathrm{T}^a, \tag{2.37}$$

where B_μ is here related to J_μ through equation (2.26).

2.3.2 The SU(3) symmetry

We finally illustrate the strong interaction (gluons) by defining the non-Abelian symmetry SU(3) whose elementary group element with 8 real parameters reads

$$\mathrm{U} = \exp\left[-i\,\mathrm{g}\,\frac{\lambda_a}{2}\,\mathrm{k}^a\right], \qquad a = 1, \ldots, 8\,. \tag{2.38}$$

The λ^a are the eight Gell-Mann 3×3 Hermitian traceless matrices [5]

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and the representation of SU(3) acting on the matter field triplet

$$\psi(x) = \begin{matrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{matrix} \tag{2.39}$$

is just the group element U. Accordingly, the Lagrangian for the SU(3) gauge bosons interacting with the above fermion triplet can be computed to give

$$\mathcal{L} = -\frac{1}{4}\, F^{\mu\nu}_{\ \ k} F_{\mu\nu}^{\ \ k} + i\,{}^\circ\psi_a \gamma^\mu_k \left[\partial_\mu - i\,\mathrm{g}\,S^k_\mu \left(\frac{\lambda}{2}\right)^a_{a'}\right] \psi^{a'}, \tag{2.40}$$

where ${}^\circ\psi_a$ is the complex conjugate spinor and where the field strength tensor is

$$F^{\mu\nu}_{\ \ k} = \partial^\mu S^\nu_k(x) + \mathrm{g}\,\mathrm{e}^{ln}_k\,\mathrm{k}_n(x)\, F^\mu_l\, F^\nu_n\,, \quad k, l, n = 1, \ldots, 8\,. \tag{2.41}$$

Here, we have the correspondence $S^\mu \to J^\mu$.

2.3.3 Example of the gauge group U(1) × SU(2)

Using (2.29), we can construct explicit examples of the generators T^a needed to describe the transformation of matter multiplet under SU(2) which we will couple with the electromagnetic boson under U(1). We first introduce three vector gauge fields B^a_μ which may be written in the form [6, p. 53, eq. 2.91]

$$B_\mu = \left(\frac{\sigma_a}{2}\right) B^a_\mu = \frac{1}{2} \begin{vmatrix} B^3_\mu & B^1_\mu - i\,B^2_\mu \\ B^1_\mu + i\,B^2_\mu & -B^3_\mu \end{vmatrix}. \tag{2.42}$$

These are the gauge bosons transforming as the adjoint of SU(2) we couple with the gauge boson transforming as U(1).

The kinetic term of the resulting Lagrangian is given by

$$(\mathcal{L})^{\mathrm{kin}} = -\frac{1}{4}\,(B^{\ a}_{\mu\nu} B^{\ \mu\nu}_a + F^{\mu\nu} F_{\mu\nu})\,. \tag{2.43}$$

Here, the combination $C_\mu = B_\mu + A_\mu$ which takes place in the Euclidean tangent space is identified to the Weyl-Einstein 4-vector J_μ at this point.

All these examples illustrate how the Yang-Mills gauge field vectors actually interact with the Weyl-Einstein 4-vector through equation (2.26).

Conclusion

In this short paper, we have only sketched a possible representation of how space-time curvature can couple with the Yang-Mills Theory in a non-trivial way.

For each type of interaction, we show that the Yang-Mills gauge fields are asymptotically connected to the space-time curvature through the Weyl-Einstein 4-vector. This amounts to state that the first three interactions are defined in the euclidean space-time which is tangent to the Weyl-Einstein manifold at the point where this 4-gauge vector is chosen.

This particular interaction appears as a new coupling between the Weyl-Einstein space-time geometry and the various particles/fields satisfying the Yang-Mills theory. In a sense, such a coupling could be regarded as the realization of a new representation of Einstein's field equations with a source. In the classical General Relativity, the Riemannian field equations disregard the Weyl-Einstein vector and they just display an energy-momentum tensor on the right hand side as a source. The insertion of such a tensor was never entirely satisfactory to Einstein's opinion who always claimed that the right hand side of his equations was somewhat "clumsy". Einstein's argument should not be hastily dismissed: indeed, while his tensor exhibits a conceptually conserved property, the energy-momentum tensor as a source does not, which leaves the theory with a major inconsistency [7]. For a massive tensor, the problem has been cured by introducing the so-called pseudo-tensor that conveniently describes the gravitational field of the mass so that the 4-momentum vector of both matter and its gravity is conserved (for example, the Einstein-Dirac pseudo-density) [8, 9]. Unfortunately by essence, this pseudo-tensor can be transformed away at any point by a change of coordinates that naturally shows the non-localizability of the gravitational energy [10]. At any rate, a pseudo-tensor is not suitable to be represented on the right hand side of the field equations. This is of course a stumbling-block which has plagued General Relativity for more than a century. Moreover, unlike the Einstein tensor, the energy-momentum tensors are mainly antisymmetric and symmetrization is thus always required "afterwards" through the Belinfante procedure. To evade the initial problem one is led to introduce a vacuum energy-momentum field energy that is "excited" in the vicinity of a mass to produce the gravitational field [11, 12]. Far from the mass, this (real) vacuum energy tensor never vanishes and guarantees the conservation of the source tensor on the right hand side of the field equations. However, several constraints are needed to be implemented which might be viewed as a loss of generality of the theory [13].

Let us note in passing that the most important Einstein solutions are derived from source-free equations as for example the famous Schwarzschild metric [14]. In the frame of our theory, the field equations in the post-Newtonian approximation should certainly deserve further scrutiny which is beyond the scope of this paper. In conclusion, we suggest here to correlate gauge fields so that unification of the three fundamental interactions with Einstein's General Theory of Relativity can be achieved in a very simple way. The principle of equivalence implies that gravity is thus indirectly related to each type of particles described in the Yang-Mills Theory.

Many topics such as the fermion and scalar quantum numbers in the electroweak model, or the spontaneous symmetry breakdown and the Higgs mechanism have not been discussed here.

We are however convinced that the introduction of the Weyl-Einstein formalism in the theory does not conflict with these results, and that it constitutes one of the permissible unifying theory between gauge theories.

Submitted on June 2, 2022

References

1. Hawking S. W., Ellis G. F. R. The Large Scale Structure of Space-Time. Cambridge University Press, 1987.
2. Lichnerowicz A. Les espaces variationnels généralisés. *Annales scientifiques de l'École Normale Supérieure*, série 3, t .62, 1945, 339–384.
3. Weyl H. On generalized Riemann matrices. *Annals of Mathematics*, 1934, v. 35, no. 4, 714–729.
4. Greiner W., Müller B. Gauge Theory of Weak Interaction. Springer-Verlag, 2009.
5. Gell-Mann M. The eightfold way: a theory of strong interaction symmetry. *Cal. Inst. of Technology*, TID 12608 CSTL-20, 1961.
6. Ross G. R. Grand Unified Theories. Benjamin/Cummings Publ. Co., Menlo Park (California), 1985.
7. Landau L., Lifshitz E. The Classical Theory of Fields. Addison-Wesley, Reading, 1962.
8. Tonnelat M. A. Les Théories unitaires de l'Électromagnétisme et de la Gravitation. Gauthier-Villars, Paris, 1965.
9. Dirac P.A.M. General Theory of Relativity. Princeton University Press, 2nd edition, 1975.
10. Straumann N. General Relativity and Relativistic Astrophysics. Springer-Verlag, 1984.
11. Marquet P. The gravitational field: a new approach. *Progress in Physics*, 2013, v. 3, 62–66.
12. Marquet P. Vacuum background field in General Relativity. *Progress in Physics*, 2016, v. 12, 314–316.
13. Marquet. P. Some insights on the nature of the vacuum background field in General Relativity. *Progress in Physics*, 2016, v. 12, 366–367.
14. Kramer D., Stephani H., Hertl E., MacCallum M. Exact Solutions of Einstein's Field Equations. Cambridge University Press, 1979.