Lamb Shift in Discrete Time

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Lamb shift is the energy difference between the two energy levels of $2S_{1/2}$ and $2P_{1/2}$ of a hydrogen atom. This cannot be explained by the existing relativistic quantum mechanics, but was explained by the interaction of electrons and vacuum in quantum field theory. However, in this paper, I tried to explain the Lamb shift as a result of the previous paper [1] that causal delay in a discrete time perspective causes the charge change. As a result, the charge change caused an additional energy change in the existing fine structure of hydrogen, and the value was approximated.

1 Introduction

In my previous paper [1], I showed that the concept of causal delay in discrete time provides a correction for minimal coupling in electromagnetic interactions, and that this correction causes energy-scale-dependent changes in the charge and mass of elementary particles. An application example of such a result was attempted to explain the anomalous magnetic moment. In this paper, I will try to explain Lamb shift as another application example.

Like the anomalous magnetic moment, the Lamb shift is not explained by the existing relativistic quantum mechanics, but by the quantum field theory, a completely different paradigm. However, the changes in charge and mass due to the concept of causal delay open the possibility that these can be explained within the scope of modified relativistic quantum mechanics.

2 Nonrelativistic approximation of the modified Dirac equation

In the previous paper [1], it was shown that the Hamiltonian of electromagnetic interacting particles in terms of causal delay and the newly defined charge and mass dependent on energy scales are as follows

$$\mathrm{H} - q'\phi = \vec{\alpha}\cdot\left(\vec{p} - q'\vec{A}\right) + \beta m' , \tag{1}$$

where

$$\begin{aligned} m' &= f_{1r} m \\ q' &= (1 - f_{2r})\, q \end{aligned} \tag{2}$$

$$\begin{aligned} f_{1r} &= \mathrm{Re}\,(f_1) = \frac{1}{3}\,\mathrm{Re}\left(\frac{\mathrm{e}^{-i\Delta x\cdot p}}{\mathrm{e}^{-i\Delta x\cdot p} + 2\left(\mathrm{e}^{-i\Delta x\cdot \Delta p} - 1\right)}\right) \\ f_{2r} &= \mathrm{Re}\,(f_2) = \frac{1}{3}\,\mathrm{Re}\left(\frac{2\mathrm{e}^{-i\Delta x\cdot \Delta p}}{\mathrm{e}^{-i\Delta x\cdot p} + 2\left(\mathrm{e}^{-i\Delta x\cdot \Delta p} - 1\right)}\right) . \end{aligned} \tag{3}$$

The Dirac equation satisfied by the electromagnetic interacting particle with mass m' and charge q' is as follows when expressed with two-component spinors ψ_A and ψ_B

$$(H - q'\phi)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \vec{\alpha}\cdot\left(\vec{p} - q'\vec{A}\right)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} + \beta m'\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} . \tag{4}$$

Eq. (4) becomes the following system of equations

$$\begin{aligned} (H - q' - m')\,\psi_A &= \vec{\sigma}\cdot\left(\vec{p} - q'\vec{A}\right)\psi_B \\ (H - q' + m')\,\psi_B &= \vec{\sigma}\cdot\left(\vec{p} - q'\vec{A}\right)\psi_A . \end{aligned} \tag{5}$$

Since $\vec{A} = 0$ and ϕ is static in a hydrogen atom,

$$\psi\,(\vec{r}, t) = \mathrm{e}^{-iEt}\psi\,(\vec{r}), \quad E = m' + \varepsilon . \tag{6}$$

Then, in the second of (5), the following expression is obtained

$$\begin{aligned} (E - q'\phi + m')\,\psi_B\,(\vec{r}) &= \vec{\sigma}\cdot\vec{p}\,\psi_A\,(\vec{r}) \\ \psi_B\,(\vec{r}) &= (2m' + \varepsilon - q'\phi)^{-1}\,\vec{\sigma}\cdot\vec{p}\,\psi_A\,(\vec{r}) \\ &\cong \frac{1}{2m'}\left(1 - \frac{\varepsilon - q'\phi}{2m'}\right)\vec{\sigma}\cdot\vec{p}\,\psi_A\,(\vec{r}) . \end{aligned} \tag{7}$$

Since m' and q' are only parameters, the first of (5) is as follows

$$\begin{aligned} (\varepsilon - q'\phi)\,\psi_A &= \vec{\sigma}\cdot\vec{p}\,\frac{1}{2m'}\left(1 - \frac{\varepsilon - q'\phi}{2m'}\right)\vec{\sigma}\cdot\vec{p}\,\psi_A \\ &= \frac{q'}{4m'^2}\,(\vec{\sigma}\cdot\vec{p}\phi)\,(\vec{\sigma}\cdot\vec{p}\psi_A) + \\ &\quad + \frac{1}{2m'}\left(1 - \frac{\varepsilon - q'\phi}{2m'}\right)(\vec{\sigma}\cdot\vec{p})^2\,\psi_A . \end{aligned} \tag{8}$$

Therefore, the results of the relativistic correction of the modified Dirac equation can be obtained as follows

$$\begin{aligned} \varepsilon\psi_A &= \left\{\frac{\vec{p}^2}{2m'} + q'\phi - \frac{\vec{p}^4}{8m'^3} - \frac{q'}{4m'^2}\nabla\phi\cdot\nabla + \right. \\ &\quad \left. + \frac{q'}{4m'^2}\,\vec{\sigma}\cdot(\nabla\phi\times\vec{p})\right\}\psi_A \\ &= \left\{H'_0 + H'_{rel} + H'_D + H'_{SO}\right\}\psi_A . \end{aligned} \tag{9}$$

Eq. (9) is the same as just replacing m and q with m' and q' in the existing equation. In Section 3, I briefly review the fine structure of hydrogen, and in Section 4, how m' and q' of each term in (9) change the fine structure will be discussed. This discussion will be limited to only $2S_{1/2}$ and $2P_{1/2}$.

3 Fine structure of hydrogen

The Hamiltonian representing the fine structure of hydrogen is as follows

$$\begin{aligned} H &= \frac{\vec{p}^2}{2m} - \frac{\alpha}{r} - \frac{\vec{p}^4}{8m^3} + \frac{1}{8m^2}\nabla^2 V_C + \frac{\alpha}{2m^2}\frac{\vec{S}\cdot\vec{L}}{r^3} \\ &= H_0 + H_{rel} + H_D + H_{SO}\,. \end{aligned} \tag{10}$$

The changes in the energy of $2S_{1/2}$ and $2P_{1/2}$ by the last three terms of (10) are known as follows. Eq. (11) is the expectation value of each Hamiltonian, and the subscripts S and P denote $2S_{1/2}$ and $2P_{1/2}$

$$\begin{aligned} \Delta_{rel} &= \langle H_{rel}\rangle_S - \langle H_{rel}\rangle_P = -\frac{1}{12}\,m\alpha^4 \\ \Delta_D &= \langle H_D\rangle_S - \langle H_D\rangle_P = \langle H_D\rangle_S = \frac{1}{16}\,m\alpha^4 \\ \Delta_{SO} &= \langle H_{SO}\rangle_S - \langle H_{SO}\rangle_P = -\langle H_{SO}\rangle_P = \frac{1}{48}\,m\alpha^4\,. \end{aligned} \tag{11}$$

According to (11), the relativistic correction term lowers the energy of both $2S_{1/2}$ and $2P_{1/2}$, but the energy value of $2S_{1/2}$ has a lower energy value than $2P_{1/2}$ by $m\alpha^4/12$, and the Darwin term increases only the energy of $2S_{1/2}$ by $m\alpha^4/16$, and spin-orbit term lowers the energy of only $2P_{1/2}$ by $m\alpha^4/48$. The sum of all three effects is 0, so the energies of $2S_{1/2}$ and $2P_{1/2}$ are the same as a result. In other words, the Lamb shift cannot be explained by the existing relativistic quantum mechanics.

However, as we will see in the next chapter, the change in charge due to the causal delay effect causes a slight change in the expectation value of each Hamiltonian, which may explain the Lamb shift.

4 Corrections of fine structure

4.1 Modified Coulomb potential energy

First, let's try to find the charges q'_e and q'_p of the electron and the proton interacting in a hydrogen atom from (2)

$$\begin{aligned} q'_p &= \left(1 - f^p_{2r}\right) e \\ q'_e &= -\left(1 - f^e_{2r}\right) e\,. \end{aligned} \tag{12}$$

In a reference frame where the proton is at rest, first about the proton*,

$$\begin{aligned} &p_\mu = \left(E_p = m_p, 0, 0, 0\right) \quad , \quad \Delta p_\mu = 0 \\ &\Delta x \cdot p = E_p\,\Delta t_p = 1. \end{aligned} \tag{13}$$

$$\begin{aligned} q'_p &= \left(1 - \frac{1}{3}\,\mathrm{Re}\left(\frac{2\mathrm{e}^{-i\Delta x\cdot\Delta p}}{\mathrm{e}^{-i\Delta x\cdot p} + 2\left(\mathrm{e}^{-i\Delta x\cdot\Delta p} - 1\right)}\right)\right) e \\ &= \left(1 - \frac{2}{3}\,\cos 1\right) e \equiv de\,. \end{aligned} \tag{14}$$

*See the definition of causal delay time $\Delta t \equiv 1/m$ in the previous paper [1].

About the electron,

$$\begin{aligned} \Delta x \cdot p &= \Delta t_e \left(E - \frac{\vec{p}^2}{\gamma m_e}\right) \cong \Delta t_e\, m_e = 1 \\ \Delta x \cdot \Delta p &= \Delta t_e \left(\Delta E - \frac{\vec{p}\cdot\Delta\vec{p}}{\gamma m_e}\right) = \Delta t_e\,\Delta V\,. \end{aligned} \tag{15}$$

Using (15) and $\Delta p \ll p$, we get

$$\begin{aligned} q'_e &= -\left(1 - \frac{1}{3}\,\mathrm{Re}\left(\frac{2\mathrm{e}^{-i\Delta x\cdot\Delta p}}{\mathrm{e}^{-i\Delta x\cdot p} + 2\left(\mathrm{e}^{-i\Delta x\cdot\Delta p} - 1\right)}\right)\right) e \\ &= -\frac{2 - \frac{2}{3}\,\cos(\Delta x\cdot p)}{9 - 8\,\cos(\Delta x\cdot\Delta p)}\, e = \frac{-(d+1)\, e}{9 - 8\,\cos(\Delta t_e\,\Delta V)}\,. \end{aligned} \tag{16}$$

If the potential due to the proton is defined as (17), the $q'\phi$ related to the potential energy of the electron in (9) becomes (18)

$$\phi \equiv \frac{q'_p}{r}\,. \tag{17}$$

$$\begin{aligned} q'_e\phi &= -\frac{k\alpha}{r\,(9 - 8\,\cos(\Delta t_e\,\Delta V))}\,, \\ k &\equiv d\,(d+1) = 1.049\,. \end{aligned} \tag{18}$$

What we now need to do is to find the explicit expression for ΔV in (18). In (3), Δp represents the change in the momentum of an electron due to the interaction, which means that when the momentum of a free electron is p, the electromagnetic field is "turned on" and the momentum after the interaction is $p + \Delta p$. Therefore, in (18), ΔV means the value obtained by subtracting the potential energy of the free electron from the potential energy of the electron in a hydrogen atom, that is, Coulomb potential energy $-\alpha/r$. And, since $\Delta t_e = 1/m$,

$$q'_e\phi = -\frac{k\alpha}{r\,(9 - 8\,\cos(\alpha/mr))}\,. \tag{19}$$

At (19), $q'_e\phi$ is not exactly equal to the potential energy of the electron. Eq. (19) becomes $-k\alpha/r$ for large r, so it is somewhat different from the Coulomb potential energy $-\alpha/r$. So, to be equal to the Coulomb potential energy at a large r, the modified Coulomb potential energy must be defined as follows

$$V_m \equiv \frac{q'_e\phi}{k} = -\frac{\alpha}{r\,(9 - 8\,\cos(\alpha/mr))}\,. \tag{20}$$

Eq. (20) approximates the Coulomb potential energy well at large r. For example, at Bohr radius $a_0 = 1/m\alpha$, the ratio of modified Coulomb potential energy to Coulomb potential energy is as follows

$$\frac{V_m}{V_C} = \frac{1}{9 - 8\,\cos\alpha^2} = 0.99999999\,. \tag{21}$$

However, for small r, especially around $r = b \equiv \alpha/m = a_0\,\alpha^2 = 2.82\times10^{-15}$ m, that is, the closer to the proton (proton radius $r_p = 0.84 \times 10^{-15}$ m), the more it deviates from the Coulomb potential energy.

Considering the potential energy in the proton, assuming that the charges are uniformly distributed, the potential energy is a linear function with respect to r^2, so the overall potential energy function is as follows

$$
\begin{aligned}
r < r_p : V_{in} &= \frac{\alpha}{2r_p}\left[\left(\frac{r}{r_p}\right)^2 - 1.12\right] \\
r \geq r_p : V_m &= -\frac{\alpha}{r(9 - 8\cos(b/r))}\,.
\end{aligned}
\tag{22}
$$

At $r < r_p$, the effect of fine structure by (10) is negligible, and the same is true for (9).

4.2 Mass change effect

As can be seen from (2), the mass also changes according to the energy scale. We discuss how the change in mass affects the energy of the electron in hydrogen. The energy of the electron is

$$
E = \sqrt{m^2 + \vec{p}^2} + V \cong m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} + V\,. \tag{23}
$$

In (23), $\vec{p}^2/2m$ and V are in order of $m\alpha^2$, and $-\vec{p}^4/8m^3$ is in order of $m\alpha^4$. Meanwhile, $\vec{p}^2/2m + V$ is invariant with respect to mass change. The reason is that, when the charge is constant, $mv^2/r = -e|\vec{E}|$, the change in mass cancels out the change in velocity. Thus, the energy change due to mass change appears in the term $-\vec{p}^4/8m^3$, which is α^2 times smaller than $\vec{p}^2/2m$ or V. That is, the energy change given by the mass change is α^2 times the energy change due to the charge change, so it can be ignored. Therefore, mass will be treated as a constant from now on.

Now, let's examine how each of the terms in (9) changes the fine structure.

4.3 Nonrelativistic term

$$
H_0' = \frac{\vec{p}^2}{2m} + kV_m \Rightarrow \frac{\vec{p}^2}{2m} + V_m\,. \tag{24}
$$

$P^2/2m + V_m$ in (24) is used to converge to the nonrelativistic Hamiltonian H_0 at large r. This is possible because the physics is invariant to the gauge transformation of electromagnetic potential energy. Also, convergence to the nonrelativistic Hamiltonian H_0 at large r means that each term of H' can be considered as a perturbation to H_0.

Now we need to find the expectation value $\langle H_0' \rangle_{S,P}$. In (10), the expectation value of H_0 is $\langle H_0 \rangle = \langle V_C \rangle/2$ by the virial theorem, which does not strictly apply to H_0'. However, since the expectation value of H_0' mostly contributes to the large r part, and $V_m \cong V_C$ in the large r, the virial theorem can be approximately applied to the expectation value of H_0'. Thus

$$
\left\langle H_0' \right\rangle_{S,P} \cong \frac{\langle V_m \rangle_{S,P}}{2}\,. \tag{25}
$$

What we want to calculate is

$$
\Delta_0' = \left\langle H_0' \right\rangle_S - \left\langle H_0' \right\rangle_P = \frac{1}{2}\left\{\langle V_m \rangle_S - \langle V_m \rangle_P\right\}. \tag{26}
$$

And the function of the eigenstates $2S_{1/2}$ and $2P_{1/2}$ to be used in the calculation, that is, the solution of the Schrödinger equation is as follows.

$$
\begin{aligned}
\psi_{n=2,l=0,m=0} &= \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \\
\psi_{n=2,l=1,m=0} &= \frac{1}{4\sqrt{2\pi a_0^3}}\,\frac{r}{a_0}\, e^{-r/2a_0}\,\cos\theta\,.
\end{aligned}
\tag{27}
$$

Eq. (26) is calculated as follows

$$
\begin{aligned}
\langle V_m \rangle_S &= \int_{r_p}^{\infty} 4\pi r^2\, \frac{-\alpha}{r\,(9 - 8\cos(b/r))}\,\frac{1}{8\pi a_0^3} \times \\
&\quad \times \left(1 - \frac{r}{a_0} + \frac{r^2}{4a_0^2}\right) e^{-r/a_0}\, dr \\
&= -\frac{m\alpha^6}{2}\int_{0.3}^{\infty} \frac{1}{9 - 8\cos(1/r')} \times \\
&\quad \times \left(r' - \alpha^2 r'^2 + \frac{\alpha^4}{4} r'^3\right) e^{-\alpha^2 r'} dr' \;\; (r = br') \\
\langle V_m \rangle_P &= \int_{r_p}^{\infty} 4\pi r^2\, \frac{2}{3}\,\frac{-\alpha}{r(9 - 8\cos(b/r))} \times \\
&\quad \times \frac{1}{32\pi a_0^3}\,\frac{r^2}{a_0^2}\, e^{-r/a_0}\, dr \\
&= -\frac{m\alpha^{10}}{24}\int_{0.3}^{\infty} \frac{1}{9 - 8\cos((1/r')} r'^3 e^{-\alpha^2 r'}\, dr' \\
\Delta_0' &= -\frac{m\alpha^6}{4}\int_{0.3}^{\infty} \frac{1}{9 - 8\cos(1/r')} \times \\
&\quad \times \left(r' - \alpha^2 r'^2 + \frac{\alpha^4}{6} r'^3\right) e^{-\alpha^2 r'}\, dr'\,.
\end{aligned}
\tag{28}
$$

Unfortunately, the integral of (28) cannot be calculated analytically, but can be approximated. In the above integral, the factor $1/(9 - 8\cos(b/r))$ converges to 1 at large r. Its shape resembles a step function. This means that the integral is dominant at large r, so it can be calculated with the factor $1/(9 - 8\cos(b/r)) \cong 1$. So

$$
\begin{aligned}
&\langle V_m \rangle_S \cong \langle V_C \rangle_S\,,\; \langle V_m \rangle_P \cong \langle V_C \rangle_P \\
&\therefore \Delta_0' \cong \frac{1}{2}\left\{\langle V_C \rangle_S - \langle V_C \rangle_P\right\} = 0\,.
\end{aligned}
\tag{29}
$$

Consequently, it can be said that the energy difference between $2S_{1/2}$ and $2P_{1/2}$ by V_m is very small.

4.4 Relativistic correction term

$$H'_{rel} = -\frac{\vec{p}^4}{8m^3} = -\frac{1}{2m}(E - V_m)^2 \,. \tag{30}$$

In (30), E is the expectation value of H'_0, so the desired value is

$$\begin{aligned}\Delta'_{rel} &= \left\langle H'_{rel}\right\rangle_S - \left\langle H'_{rel}\right\rangle_P \\ &= -\frac{1}{2m}\Big\{\left(E_S^2 - E_P^2\right) - 2\left(E_S \left\langle V_m\right\rangle_S - E_P \left\langle V_m\right\rangle_P\right) + \\ &\quad + \left(\left\langle V_m^2\right\rangle_S - \left\langle V_m^2\right\rangle_P\right)\Big\} \\ &\cong -\frac{1}{2m}\left\{\left\langle V_m^2\right\rangle_S - \left\langle V_m^2\right\rangle_P\right\}.\end{aligned} \tag{31}$$

The first and second terms in the second line of (31) can be ignored by the results in the previous chapter $\langle V_m\rangle_S \cong \langle V_m\rangle_P, E_S \cong E_P$

$$\begin{aligned}\left\langle V_m^2\right\rangle_S &= \int_{r_p}^{\infty} 4\pi r^2 \frac{\alpha^2}{r^2\,(9 - 8\cos(b/r))^2}\,|\psi_{200}|^2\,dr \\ \left\langle V_m^2\right\rangle_P &= \int_{r_p}^{\infty} 2\pi r^2\,\frac{2}{3}\,\frac{\alpha^2}{r^2\,(9 - 8\cos(b/r))^2}\,|\psi_{210}|^2\,dr\,.\end{aligned} \tag{32}$$

In (32), it is a rough approximation, but if we put factor $1/(9 - \cos(b/r))^2 \simeq 1$

$$\begin{aligned}&\left\langle V_m^2\right\rangle_S \simeq \left\langle V_C^2\right\rangle_S,\ \left\langle V_m^2\right\rangle_P \simeq \left\langle V_C^2\right\rangle_P \\ &\therefore \Delta'_{rel} \simeq -\frac{1}{2m}\left\{\left\langle V_C^2\right\rangle_S - \left\langle V_C^2\right\rangle_P\right\} = \Delta_{rel}\,.\end{aligned} \tag{33}$$

According to (33), the correction by the relativistic correction term is also expected to be small.

4.5 Spin-orbit term

$$\begin{aligned}H'_{SO} &= \frac{q'_e}{4m^2}\,\vec{\sigma}\cdot(\nabla\phi\times\vec{p}) \\ &= \frac{k\alpha}{2m^2}\,\frac{1}{9 - 8\cos(b/r)}\,\frac{\vec{S}\cdot\vec{L}}{r^3} \\ &= \frac{1}{2m^2}\,\frac{kV_m}{r^2}\,\vec{S}\cdot\vec{L}\,.\end{aligned} \tag{34}$$

In (34), the spin-orbit term H'_{SO} is also expressed as modified Coulomb potential energy. This means that the gauge transformation can be performed so that H'_{SO} also converges to H_{SO} at large r. Thus

$$H'_{SO} = \frac{1}{2m^2}\,\frac{V_m}{r^2}\,\vec{S}\cdot\vec{L}\,. \tag{35}$$

On the other hand, using (36),

$$\begin{aligned}&\left\langle nljm_j\left|\vec{S}\cdot\vec{L}\right|nljm_j\right\rangle = \frac{1}{2}\left\{j\,(j+1) - l\,(l+1) - \frac{3}{4}\right\} \\ &\left\langle\vec{S}\cdot\vec{L}\right\rangle_S = 0\,,\ \left\langle\vec{S}\cdot\vec{L}\right\rangle_P = -1\,.\end{aligned} \tag{36}$$

Expectation values are:

$$\begin{aligned}\langle H'_{SO}\rangle_S &= 0 \\ \langle H'_{SO}\rangle_P &= -\frac{\alpha}{2m^2}\left\langle\frac{1}{9 - 8\cos(b/r)}\,\frac{1}{r^3}\right\rangle_P \\ &\cong -\frac{\alpha}{2m^2}\left\langle\frac{1}{r^3}\right\rangle_P = \langle H_{SO}\rangle_P\,.\end{aligned} \tag{37}$$

The difference between the spin-orbit term before and after charge correction is

$$\Delta'_{SO} - \Delta_{SO} = -\langle H'_{SO}\rangle_P + \langle H_{SO}\rangle_P \cong 0\,. \tag{38}$$

Therefore, the charge change has little contribution to the spin-orbit term.

4.6 Darwin term

$$H'_D = -\frac{q'_e}{4m^2}\,\nabla\phi\cdot\nabla\,. \tag{39}$$

If we get the expectation value of (39), we get

$$\begin{aligned}\langle\psi\,|\,H'_D\,|\,\psi\rangle &= -\frac{1}{4m^2}\int\psi^\dagger\,(q'_e\nabla\phi\cdot\nabla)\,\psi\,d^3\vec{r} \\ &= -\frac{1}{8m^2}\int q'_e\nabla\phi\cdot\nabla\left(\psi^\dagger\psi\right)d^3\vec{r} \\ &= \frac{1}{8m^2}\int\psi^\dagger\psi\nabla\cdot(q'_e\nabla\phi)\,d^3\vec{r}.\end{aligned} \tag{40}$$

Consequently

$$H'_D = \frac{1}{8m^2}\,\nabla\cdot(q'_e\nabla\phi)\,. \tag{41}$$

where

$$q'_e = -\frac{(d+1)\,e}{9 - 8\cos(b/r)}\,,\quad \phi = \frac{de}{r}\,. \tag{42}$$

In (41), if $q'_e = -e$ and $q'_p = e$, it becomes H_D. H'_D is

$$\begin{aligned}H'_D &= \frac{1}{8m^2}\left(\nabla q'_e\cdot\nabla\phi + q'_e\nabla^2\phi\right) \\ &= \frac{k\alpha}{8m^2}\left\{\frac{\partial}{\partial r}\,\frac{1}{(9 - 8\cos(b/r))}\,\frac{1}{r^2} + \right. \\ &\quad\left. + \frac{4\pi\delta(\vec{r})}{9 - 8\cos(b/r)}\right\}.\end{aligned} \tag{43}$$

In (43), the Darwin term is expressed as a quantity related to the second order derivative of the modified Coulomb potential energy. This means that there is no gauge degree of freedom in the Darwin term, so the value of k in the equation must be maintained.

Expectation values are:

$$
\begin{aligned}
\langle H'_D \rangle_S &= \frac{k\alpha}{8m^2} \int_{r_p}^{\infty} 4\pi r^2 \, \frac{\partial}{\partial r} \frac{1}{(9 - 8\,\cos(b/r))} \times \\
&\times \frac{1}{r^2} \left|\psi_{200}\left(\vec{r}\right)\right|^2 \, dr \\
&+ \frac{k\alpha}{8m^2} \int_{\vec{r}_p}^{\infty} \frac{4\pi\,\delta\left(\vec{r}\right)}{9 - 8\,\cos(b/r)} \left|\psi_{200}\left(\vec{r}\right)\right|^2 d^3\vec{r} \\
&= \frac{4\pi k\alpha}{8m^2} \left\{ \left[\frac{1}{9 - 8\,\cos(b/r)} \left|\psi_{200}(\vec{r})\right|^2 \right]_{r_p}^{\infty} - \right. \\
&\left. - \int_{r_p}^{\infty} \frac{1}{9 - 8\,\cos(b/r)} \frac{\partial}{\partial r} \left|\psi_{200}(\vec{r})\right|^2 \, dr \right\} \\
&+ \frac{4\pi k\alpha}{8m^2} \frac{\left|\psi_{200}\left(\vec{r}_p\right)\right|^2}{9 - 8\,\cos(b/r_p)} \\
&\cong -\frac{4\pi k\alpha}{8m^2} \int_{r_p}^{\infty} \frac{\partial}{\partial r} \left|\psi_{200}\left(\vec{r}\right)\right|^2 dr \\
&= \frac{4\pi k\alpha}{8m^2} \left|\psi_{200}\left(\vec{r}_p\right)\right|^2 \cong \frac{km\alpha^4}{16}\,. \\
\langle H'_D \rangle_P &\cong \frac{4\pi k\alpha}{8m^2} \left|\psi_{210}\left(\vec{r}_p\right)\right|^2 = O\left(m\alpha^8\right)\,.
\end{aligned}
\tag{44}
$$

As can be seen from (44), the Darwin term by charge correction works mostly in the $2S_{1/2}$ state. Thus

$$\Delta'_D = \langle H'_D \rangle_S - \langle H'_D \rangle_P \cong \langle H'_D \rangle_S \cong \frac{km\alpha^4}{16}\,. \tag{45}$$

Therefore, the difference between the Darwin term before and after charge correction is

$$\Delta'_D - \Delta_D \cong (k-1)\,\frac{m\alpha^4}{16} = 57.67\,m\alpha^6\,. \tag{46}$$

In (11), the existing Darwin term acts only on the $2S_{1/2}$ state to increase its energy by $m\alpha^4/16$, and in (46), the effect of charge correction by causal delay further increases the energy of the $2S_{1/2}$ state by $57.67\,m\alpha^6$.

5 Conclusions

From a discrete time point of view, causal delay gives energy scale-dependent changes to the mass and charge of elementary particles. In this paper, as a result of applying it to the Lamb shift, it was obtained that the change in the charge value increases the energy of $2S_{1/2}$ by about $57.67\,m\alpha^6$ =1076 MHz mostly by the Darwin term. This is slightly different from the experimental value of 1057.86 MHz, but it is a good result as an approximation. If numerical integration can be done accurately, I think it will be close to the actual value.

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