Fermion Mass Derivations: I. Neutrino Masses via the Linear Superposition of the 2T, 2O, and 2I Discrete Symmetry Binary Subgroups of SU(2)

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We derive neutrino masses from discrete symmetry binary subgroups of SU(2), 2T for the electron family, 2O for the muon family, and 2I for the tau family, acting collectively to generate the PMNS mixing angles. Using the modulus τ near $\omega = \exp(2\pi i/3)$ in the domain of SU(2) converts the PMNS matrix into the 24th root of unity and produces a factor of 3^{11} to predict neutrino masses: $m_1 = 0.3$ meV, $m_2 = 8.9$ meV, $m_3 = 50.7$ meV.

1 Introduction

One of the most challenging fundamental problems in particle physics is to calculate the mass values of the leptons and quarks. We tackle this problem within the framework of the Standard Model by considering the three specific discrete symmetry binary subgroups of SU(2) that we have established previously [1,2]. The three lepton families represent the binary tetrahedral group 2T for the electron family, the binary octahedral group 2O for the muon family, and the binary icosahedral group 2I for the tau family. The mass values for the quark families will be derived via an identical approach in a separate article.

After a brief review of some of the limitations of the Standard Model, we explain some of the consequences of the discrete symmetry binary subgroups of SU(2), including how we utilized their generators to derive the correct mixing angles for the lepton PMNS[‡] mixing matrix. These subgroups have a domain in the upper half of the complex plane and we use their modulus τ for fractional linear transformations near its symmetry point $\tau_0 = \omega = \exp(2\pi i/3)$ in our procedure to predict the lepton mass values. Note that the modular subgroups of SL(2,Z) used to calculate lepton masses via many parameters [3, 4] are isomorphic to our subgroups of SU(2).

We find that by treating the three lepton families equivalently leads to the circulant matrix method used to derive [5,6] the 1982 Koide formula [7] that accurately predicted the mass value of the tau lepton. We move the value of τ slightly away from ω, thereby introducing CP symmetry breaking, to convert our PMNS mixing matrix into the 24th root of unity, from which we calculate neutrino mass values by using the factor of 3^{11} difference from the charged-lepton mass values.

Finally, we examine how the unique invariant N for each binary subgroup can be used to derive the lepton mass values from geometry. According to F. Klein [8] in 1884, each of the three binary subgroups has an invariant N inversely related to $j(\tau)$ of elliptic modular functions, the N being: 1 for 2T, 108 for 2O, and 1728 for 2I, integer values that have a similar hierarchy to the 0.511 MeV, 105.66 MeV, and 1776.82 MeV

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mass values for the charged leptons!

2 SM limitations

The Standard Model (SM) of leptons and quarks has been an extremely successful effective field theory [9–12] for combining the unified electroweak interaction with the nuclear color interaction since its formulation in the 1970s. Its fundamental particles represent quantum fields, with the SM probably being an approximation to an underlying theory.

The physical world is artificially partitioned into a (3+1)-D spacetime and an internal symmetry space at each point in spacetime. The known fundamental particle quantum states are defined in the internal symmetry space, but the number of dimensions of the internal symmetry space has yet to be established.

The two particle quantum states for each lepton family and for each quark family represent the continuous symmetry group SU(2), i.e. the $\pm 1/2$ weak isospin states which are also called the up and down flavor states. Of the 3 known lepton families, the electron family (ν_e, e^-), the muon family (ν_μ, μ^-), and the tau family (ν_τ, τ^-), the more massive muon and tau charged leptons are known to not be excited higher mass states of the electron. Likewise, the two known quark families beyond the first quark family are not simply higher mass states of the first quark family.

The SM as presently understood cannot predict the number of lepton families nor the number of quark families. However, the weak interaction Z^0 boson decays suggest that there are exactly the 3 lepton families [13] if there are only neutrinos with mass values below about 90 GeV, which appears to be the case. In addition, there is a cosmological limit of 15 total fundamental leptons plus quarks. There being 12 known fundamental leptons plus quarks, at least one more family of two particles is possible. [14, 15]

Lepton mixing occurs [16–18] when one neutrino type or charged-lepton can change into another on the journey from source to detector. This behavior is in direct conflict with the SM expectation for massless neutrinos. However, most conserved quantities still hold true, such as electric charge conservation with the electromagnetic interaction being equiva-

Table 1: Lepton Family Group Assignments.

Family	Group	Order	3-D	Mass (MeV)
ν_e				< 0.001
e^-	2T	24		0.511
ν_μ				< 0.001
μ^-	2O	48		105.7
ν_τ				< 0.001
τ^-	2I	120		1776.8

Table 2: Quaternion Generators. $\phi = (1+\sqrt{5})/2$

Fam.	Grp.	U3 Generator	Factor	Ang./2
	SU(2)	k		
ν_e, e^-	2T	$-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k$	-0.2642	52.66^o
ν_μ, μ^-	2O	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}j + \frac{1}{2}k$	+0.8012	18.38^o
ν_τ, τ^-	2I	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{1}{2\phi}k$	-0.5370	61.24^o

Table 3: Comparison to NuFit 5.2 values for neutrino observables.

	bfp $\pm 1\sigma$	3σ range	predicted
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \rightarrow 0.341$	0.3172
$\theta_{12}/^o$	$33.41^{+0.75}_{-0.72}$	$31.31 \rightarrow 35.74$	34.29^o
$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \rightarrow 0.603$	0.4627
$\theta_{23}/^o$	$42.2^{+1.1}_{-0.9}$	$39.7 \rightarrow 51.0$	42.85^o
$\sin^2\theta_{13}$	$0.0222^{+0.0006}_{-0.0006}$	$0.0205 \rightarrow 0.0240$	0.0223
$\theta_{13}/^o$	$8.58^{+0.12}_{-0.12}$	$8.23 \rightarrow 8.91$	8.56^o

lent for all electrically charged particles as well as the weak interaction being identical for each of the lepton and quark family particles, a property called weak universality. Further tests challenging this weak interaction lepton flavor universality (LFU) continue to be carried out at many different experiments worldwide.

3 Lepton mixing

In order to better understand the physical behavior of the SM particle states, in the 1990s we introduced [1] specific different discrete symmetry binary subgroups of SU(2) in R^3 for each family of leptons and in R^4 for each family of quarks. This approach has gained in importance in the past decade as other approaches have become less likely or eliminated. The discrete symmetry binary subgroups for the lepton families are the assignments listed in Table 1 along with their 3-D representations as the Platonic solids at the Planck scale. The justification for these specific binary subgroup assignments includes the correct mixing angles for the lepton PMNS matrix that relates the wave functions for the SM flavor states to their mass states.

One major consequence of having the fundamental particles represent specific discrete symmetry binary subgroups is that the lepton and quark SM weak isospin states are not the same as the mass states, in agreement with experimental results. Otherwise, in the traditional SM view with the lepton and quark families representing the continuous symmetry group SU(2), there is no known reason for the mass states to be different from the SM weak isospin states and this difference is simply attributed to a mismatch between the weak isospin states and the mass states!

We proposed [2] that the reason for the difference between the SM weak isospin flavor states and the mass states depends upon the continuous symmetry requirement of quantum field theory (QFT) because the fields are required to be continuous. Having our specific discrete symmetry binary subgroups define their weak isospin states within the framework of the SM violates this QFT continuous symmetry requirement. Therefore, to eliminate this violation, we determined that a linear superposition of the binary subgroup generators was needed so that acting collectively the three discrete symmetry binary subgroups could mimic the continuous symmetry group SU(2).

This linear superposition is achieved separately for the lepton families and for the quark families. The quaternion generators for each of the three lepton binary subgroups are the same for the first two generators, i.e. the quaternions U1 = i and U2 = j of SU(2), but the third generators, the U3's, which should each be k, are different for each subgroup and are listed in Table 2. The normalized contributing factors to the linear superposition for each lepton family binary subgroup are listed in column four of Table 2 as well as their half-angle contributions whose differences determine the PMNS mixing angles.

The absolute values of our predicted mixing angles for the lepton PMNS mixing matrix are listed in Table 3, showing that they agree with the empirically determined ranges of values. Note that we predict the θ_{23} angle of 42.85^o to be in the first quadrant, in agreement with some of the empirical values but in contrast to other results that suggest the second quadrant.

The PMNS matrix for the lepton families is the product

of the charged-lepton and the neutrino matrices

$$U_{PMNS} = U_e^\dagger \, U_\nu \,. \qquad (1)$$

If the charged-lepton states do not mix, or their mixing is minimal, U_e is diagonal, an assumption that is discussed in a later section, then the PMNS mixing matrix represents neutrino mixing only. Therefore, the PMNS matrix relates the neutrino mass states (ν_1, ν_2, ν_3) to the SM weak isospin states $(\nu_e, \nu_\mu, \nu_\tau)$ as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \qquad (2)$$

Keeping a phase factor δ for CP violation consideration, our PMNS matrix in the standard popular 3x3 formulation is

$$\begin{pmatrix} 0.817 & 0.557 & -0.1491 e^{-i\delta} \\ -0.413 - 0.084 e^{i\delta} & 0.606 - 0.057 e^{i\delta} & -0.669 \\ -0.383 + 0.051 e^{i\delta} & 0.558 + 0.062 e^{i\delta} & 0.725 \end{pmatrix}. \qquad (3)$$

Therefore, we have established the very important result that each lepton family represents its own specific discrete symmetry binary subgroup of SU(2) because our assigned groups lead directly to correct predictions of the mixing angles for the PMNS mixing matrix. And we know that the origin of this mixing is the QFT requirement for continuous symmetry behavior. So the discrete symmetries of the lepton families mix collectively via a linear superposition to mimic the continuous symmetry group SU(2).

Our binary subgroups of SU(2) have their fundamental domain $\mathcal{D}$ in the upper half of the complex plane between $-1/2$ and $+1/2$ as shown in Fig. 1 with three symmetric points $\tau_{sym} = i\infty$, i, and $\omega = \exp(2\pi i/3)$. Although no value of the modulus τ preserves the full symmetry of SU(2) (or its isomorphic modular group SL(2,Z)), at the three τ_{sym} values, specific $\mathbb{Z}_N$ symmetries are preserved, with $N = 2, 3$, or 4. When τ lies on the border of $\mathcal{D}$, CP symmetry is preserved [3,4], but small deviations expressed by $|\tau - \tau_{sym}|$ lead to CP symmetry being broken and hierarchial mass patterns emerging according to the sequence $(1, \epsilon, \epsilon^2)$, where $\epsilon \ll 1$. See Appendix A for the details which were introduced in a modular group analysis.

4 Circulant matrix approach

We know from the collective action dictated by the continuous symmetry constraint of QFT that perhaps the three lepton families should be treated as equals, a symmetry that suggests they obey the group U(3). If we assume U(3) symmetry for this equal treatment, we can utilize its expression as a 3x3 circulant matrix [5,6], from which the famous Koide formula [7] has been derived.
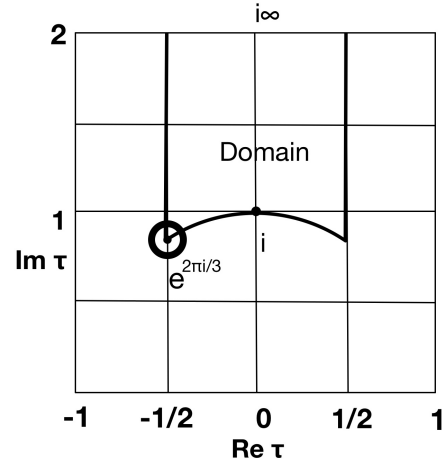

Fig. 1: The fundamental domain of our three SU(2) subgroups 2T, 2O, and 2I (and of the modular group Γ = SL(2,Z)) with its three symmetric points $\tau_{sym} = i\infty, i, \omega$, where $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$, with a small ring of acceptable values around ω.

We now paraphrase a 2006 article by C. Brannen [5], which shows how to use this type of equality to derive the Koide formula for the charged-lepton mass values from a circulant matrix and then proceeds to derive the mathematical relations that lead to the prediction of reasonable neutrino mass values in the meV energy range.

The 3x3 1-circulant matrix

$$G(A, B, C) = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}, \qquad (4)$$

where A, B and C are complex constants, has eigenvectors of the form

$$|n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \exp(+2ni\pi/3) \\ \exp(-2ni\pi/3) \end{pmatrix}, \qquad (5)$$

with $n = 1, 2, 3$. By requiring the eigenvalues λ_n to be real, the circulant matrix can be rewritten in the form

$$G(\mu, \eta, \beta) = \mu \begin{pmatrix} 1 & \eta \exp(+i\beta) & \eta \exp(-i\beta) \\ \eta \exp(-i\beta) & 1 & \eta \exp(+i\beta) \\ \eta \exp(+i\beta) & \eta \exp(-i\beta) & 1 \end{pmatrix}, \qquad (6)$$

with η assumed to be non-negative. The η and β are pure numbers, whereas μ will scale with the eigenvalues given by

$$G(\mu, \eta, \beta)|n\rangle = \lambda_n|n\rangle = \mu \, (1 + 2\eta \cos(\beta + 2n\pi/3)) \, |n\rangle. \qquad (7)$$

From the traces of G and G^2 one derives the eigenvalue relationships

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\mu \qquad (8)$$

and

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3\mu^2 (1 + 2\eta^2). \tag{9}$$

From here one obtains the Koide formula by setting $\eta^2 = 0.5$:

$$\frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \frac{3}{2}. \tag{10}$$

By setting the eigenvalues $\lambda_i = \sqrt{m_i}$, the 1982 formula proposed by Koide for the masses of the charged leptons is:

$$\frac{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}{m_e + m_\mu + m_\tau} = \frac{3}{2}. \tag{11}$$

Using the known mass values of the electron and the muon, the mass value of the tau was predicted [7] to be in agreement with future experimental results to better than two decimal places!

Consequently, from knowing the masses of the charged leptons, one determines [5] that

$$\mu_1 = 17716.13(109) \text{ eV}^{0.5}$$
$$\eta_1^2 = 0.500003(23) \tag{12}$$
$$\beta_1 = 0.2222220(19)$$

where the subscript 1 has been added to distinguish these parameters from the future neutrino parameters. Notice that β_1 is essentially 2/9 and perhaps could be related to the phase $\phi = -2\pi/9$ of the scalar potential in the modular group approach introduced in the Appendix.

5 Lepton mass hierarchy

Before there was any evidence of tau neutrino mixing with the electron neutrino, the tribimaximal matrix with its zero value in the (1,3) position was thought by researchers to be the PMNS matrix that best represented the neutrino data:

$$\begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{13}$$

Of course, we will substitute our PMNS matrix for this approximate matrix, but first we shall continue to follow the original article [5] in order to reveal its amazing result.

Left-multiplying this tribimaximal matrix by a matrix of the circulant eigenvectors achieves a simple product with the value of $\tau = \omega$, i.e. the lower left corner at $\tau_0 = \exp(2\pi i/3)$ in the domain region:

$$\alpha = \omega = e^{2\pi i/3} = -0.5 + 0.866i : \tag{14}$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \alpha & \alpha^* \\ 1 & 1 & 1 \\ 1 & \alpha^* & \alpha \end{pmatrix} \begin{pmatrix} 0.8165 & 0.5773 & 0 \\ -0.4082 & 0.5773 & -0.7071 \\ -0.4082 & 0.5773 & 0.7071 \end{pmatrix} = \tag{15}$$

$$= \begin{pmatrix} 0.7071 & 0 & -0.7071i \\ 0 & 1 & 0 \\ 0.7071 & 0 & 0.7071i \end{pmatrix}.$$

This resulting matrix is the 24th root of unity! That is, its 24th power is the unit matrix.

Note there exists many mathematical relationships from here which we could list, such as relationships to the expansions of the j-invariant $j(\tau)$, the eta function, etc., which involve 24th powers or 24th roots, but we do not need these mathematical functions to derive the neutrino mass values. However, these functions would be needed for expressing the wave functions of the particles.

Continuing onward, we know that the true PMNS mixing matrix is not the tribimaximal matrix but our PMNS matrix determined by our binary subgroups. We can achieve the same result, i.e. the 24th root of unity matrix, by using a value of τ slightly different from ω. After trying several different values, using this value of τ:

$$\alpha = \tau = -0.496 + 0.877i, \tag{16}$$

to multiply the values in our PMNS matrix leads to

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \alpha & \alpha^* \\ 1 & 1 & 1 \\ 1 & \alpha^* & \alpha \end{pmatrix} \begin{pmatrix} 0.817 & 0.557 & -0.149 \\ -0.4213 & 0.6084 & -0.669 \\ -0.3936 & -0.5654 & 0.7248 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.7014 - 0.0731i & 0.021 & 0.021 - 0.7059i \\ 0.008 & 0.927 & 0.116 \\ 0.7014 - 0.0731i & 0.021 & 0.0707 + 0.7075i \end{pmatrix}. \tag{17}$$

The result is within 1% of the 24th root of unity when using our PMNS mixing matrix and this value of α. A slight adjustment in the α value could make the fit closer.

As shown in the Appendix, the modular subgroup approach agrees that this value of α is a universal fit for the SU(2) subgroups or their equivalent modular subgroups.

Therefore, we will consider our α to be close enough and continue with this approach in order to establish the relationship between the charged-lepton states and the neutrino states as well as to determine the neutrino mass values.

Following the procedure, we define the mass operator M associated with the eigenvalue $\lambda_i = \sqrt{m_i}$ to take left-handed states to right-handed states and vice-versa:

$$M |R\rangle = |L\rangle$$
$$M |L\rangle = |R\rangle. \tag{18}$$

In general, M^2 picks up a Berry-Panchartnam or topological phase to become complex upon returning to the original state, so we can express

$$M^2 |L\rangle = p^2 \exp(2i\kappa) |L\rangle. \tag{19}$$

Note that if $\kappa = 2\pi/24 = \pi/12$, then the state $|L\rangle$ is brought back to a multiple of $|L\rangle$ by

$$M^{24}|L\rangle = p^{24}|L\rangle. \qquad (20)$$

Therefore, if M^2 operates on the left-handed electron as

$$M^2|e_L\rangle = p^2|e_L\rangle, \qquad (21)$$

then we would have

$$M^{24}|\nu_L\rangle = p^{24}|\nu_L\rangle, \qquad (22)$$

meaning that the masses of the two particle states in the lepton family differ by a factor of p^{22}.

The mass scale factors μ_1 for charged leptons and μ_0 for neutrinos are therefore related by

$$\mu_0^2 = \mu_1^2/3^{22} = 0.100^2, \qquad (23)$$

where the factor of three comes from the square of the normalization factor $1/\sqrt{3}$ for the three eigenvectors in the matrix multiplying the PMNS matrix above. Likewise, there is a phase difference

$$\beta_1 - \beta_0 = -\frac{\pi}{12}. \qquad (24)$$

Neutrino mass predictions using $\mu_1/\mu_0 = 3^{11}$ and the phase difference $\beta_1 - \beta_0 = -\pi/12$ in the eigenvalue $G(\mu, \eta, \beta)$ results in these reasonable predicted neutrino mass values:

$$
\begin{aligned}
m_1 &= 0.3\,\text{meV} \\
m_2 &= 8.9\,\text{meV} \qquad (25) \\
m_3 &= 50.7\,\text{meV},
\end{aligned}
$$

assuming still that $\eta^2 = 0.5$. Although these predicted mass values fit the neutrino values estimated from experimental results, we will need to wait for confirmation from ongoing and future experiments.

However, do these values produce the 3/2 value in the Koide formula? Not with the original version, but they do agree if we utilize the valid alternative version in which the square root of the lowest mass neutrino m_1 is preceded by a negitive sign [5,6].

Our results for the leptons being 3-D objects with the discrete symmetries of the binary subgroups 2T, 2O, 2I of SU(2) not only predict reasonable neutrino mass values but also predict the normal mass hierarchy NH of the neutrino mass states as $m_1 < m_2 < m_3$. However, the present experimental data also allows for an inverse hierarchy IH as $m_3 < m_1 < m_2$.

In addition, we have exactly 3 lepton families, in agreement with the Z^0 decay results, but there continues to be speculation about an additional lepton, such as a sterile neutrino. And, our approach treats the 3 lepton families as symmetrical contributors as eigenvectors of a 1-circulant matrix, whereas all other analyses place the charged leptons and the neutrinos into irreducible representations of a subgroup, usually in a 3 or 3′ irreducible representation. Which method Nature has chosen will be determined by experiments in the near future.

6 Review of the derivation

Our sequence of steps to neutrino mass predictions were:

1. We first established that the three discrete symmetry 3-D binary subgroups 2T, 2O, 2I of SU(2) are represented by the three lepton families and are 3-D objects instead of point particles at the Planck scale. These are the correct groups because when they collectively mimic the continuous group SU(2) to satisfy the requirement of QFT they produce the correct mixing angles for the PMNS mixing matrix.

2. By treating the lepton families as equals in symmetry group U(3), the famous Koide formula is derived via a 3x3 1-circulant matrix, revealing that the important mass quantity is the square root of the mass values $\lambda = \sqrt{m}$ instead of the mass value itself. Three parameters μ_1, η_1, and β_1 for calculating the mass eigenvalues of the charged leptons were determined.

3. With the value $\alpha = \tau = -0.496 + 0.877i$ in the domain of SU(2), we derived the 24th root of the unity matrix by multiplying our PMNS matrix by the appropriate 1-circulant eigenvector matrix. This specific value of α agreed with the findings of the modular group approach that uses subgroups of SL(2,Z), i.e. that the value applies equally to our three subgroups of SU(2).

4. By inserting the Berry-Panchartnam phase factor when returning a left-handed lepton state back to its original state for the mass-squared operator M^2, there resulted a factor of p^{22} difference between the charged-lepton states and the neutrino states as well as a phase difference of $\pi/12$.

5. Finally, using the factor of 3^{11} that connected the neutrino mass values to the charged-lepton mass values for the parameter ratio μ_1/μ_0, with the eigenvalue expression $G(\mu, \eta, \beta)$ we predicted reasonable neutrino mass values in the meV range in NH: $m_1 = 0.3\,\text{meV}, m_2 = 8.9\,\text{meV}, m_3 = 50.7\,\text{meV}$.

In the next section, our goal is to relate the above results to the invariants $N = 1, 108, 1728$ of the three lepton family binary subgroups 2T, 2O, 2I respectively. Therefore, we should be able to understand how the lepton family mass values originate from their 3-D geometric properties.

7 Invariant theory connection

Invariant theory connects the elliptic modular function $j(\tau)$ to invariants of our specific discrete symmetry binary subgroups. Each invariant N is related by

$$j(\tau) = \frac{W_1}{N W_2}, \qquad (26)$$

where W_1 is expressed in two complex variables for the vertices and W_2 for the face centers of the polyhedrons [8] for

the binary groups 2T, 2O, and 2I, with $N = 1, 108$, and 1728, respectively.

These invariants are similar to the charged-lepton mass values in MeV, i.e. 0.511, 105.66, and 1776.82, but they have no energy units, so we would naturally consider their ratios instead. However, the question remains, why is there a change from the original geometrical values N that are invariant under all fractional linear transformations to the experimentally determined universal values for the charged lepton masses?

One possible answer could be related to the change of the value of τ from $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$ to the nearby value, $\alpha = \tau_0 = -0.496 + 0.877i$ in the domain. However, we realize that we have simply changed the question without providing the reason for the change.

However, recall that the lepton PMNS mixing matrix

$$U_{PMNS} = U_e^\dagger \, U_\nu \,, \qquad (27)$$

relates the wave functions, so we can speculate that there could be a slight mixing among the charged-lepton states, particularly among the electron and the muon states. Experiments are being planned specifically to check for this mixing possibility.

If we want the tentative geometrical state mass values suggested by the SM binary group N values to become the measured mass state values, one would have a mass matrix very close to being the unitary matrix but containing some small off-diagonal terms. Such a mass matrix might look like

$$\begin{pmatrix} 1 & -0.0274 & 0 \\ 0.0274 & 1 & -0.0033 \\ 0 & 0.0558 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1} \\ \sqrt{108} \\ \sqrt{1728} \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{0.511} \\ \sqrt{105.66} \\ \sqrt{1776.82} \end{pmatrix}, \qquad (28)$$

in which we have used the square root of the mass values as determined by the Koide relationship. That is, the slight mixing among the charged-lepton wave functions could be carried over to a mass matrix relating our N values to the measured mass values. Of course, mass ratios would be preferred. But we are still left with determining an energy scale for these mass values.

8 Conclusions

We have been able to calculate the mass values of the neutrinos by following a series of steps beginning with the correct identification of the discrete symmetry binary subgroups of SU(2), which are equivalent to subgroups of the modular group SL(2,Z). The three lepton families represent 2T, 2O, and 2I, and we derived their PMNS mixing matrix for

their wave functions from their quaternion generators in order to agree with a continuous symmetry constraint dictated by quantum field theory (QFT).

Assuming that these binary subgroups together act as a U(3) symmetry, the famous Koide formula follows directly via a 1-circulant matrix approach that also relates the PMNS matrix to the 24th root of unity matrix by using a modulus τ value slightly different from the symmetry point value $\omega = \exp(2\pi i/3) = -0.5 + 0.866i$ in the fundamental domain of SU(2) and its isomorphic modular group SL(2,Z). That is, we set $\tau = -0.496 + 0.877i$. This method then produced a factor of 3^{11} difference in the mass values of the charged leptons and the neutrinos, which led directly to the predicted neutrino mass values being $m_1 = 0.3\,\mathrm{meV}, m_2 = 8.9\,\mathrm{meV}, m_3 = 50.7\,\mathrm{meV}$.

Although we assumed that the charged-lepton mixing matrix was diagonal, the invariants $N = 1, 108$, and 1728 from geometry and invariant theory for the electron family, muon family, and tau family binary subgroups, respectively, indicated that there is a slight mixing of the charged leptons also. We suggested a matrix that has unit values on the diagonal but also has a few very small off-diagonal terms to relate the N values to the actual charged lepton universal mass values 0.511 MeV, 105.66 MeV, and 1776.82 MeV. Of course, the mass scale would still remain to be determined.

In a future article, i.e. part II, we determine the origin of the quark mass values. We will establish that a similar approach succeeds for modulus τ values near to the other symmetric point $\tau = i$ within the fundamental domain. In the quark case, we predict 4 quark families, (u,d), (c,s), (t,b), and (t',b'), which represent [1, 2] the discrete symmetry binary subgroups [333], [433], [343], and [533], respectively, in R^4. QFT dictates a continuous symmetry group behavior, so the linear superposition of their generators to mimic SU(2) produces the CKM4 mixing matrix with CKM[‡] submatrix values.

The quark mass values fit a four term Koide formula separately for the up and the down states, and a 4x4 circulant matrix defines eigenvectors. The predicted t' quark should have a mass value of about 3 TeV, a mass value large enough to gain a factor of about 10^{13} multiplying the present Jarlskog constant, thereby providing a value large enough to help explain the baryon asymmetry of the Universe [BAU] in terms of CP violation [19].

Appendix: Modular group

A brief look into what researchers in the past decade have achieved using subgroups of the modular group SL(2,Z) in order to calculate neutrino mass values will demonstrate some agreement with our results. We therefore provide a summary of their research by paraphrasing a recent article [3, 4] to illustrate how our bottoms-up approach from the binary sub-

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[‡]Cabibbo-Kobayashi-Maskawa

groups of SU(2) can relate to the top-down calculations using modular groups related to superstring theory. The modulus τ of SL(2,Z) is the single field quantity associated with the fermion particle states.

Our three discrete symmetry binary subgroups 2T, 2O, and 2I of SU(2) for the lepton families are isomorphic to these modular double cover subgroups:

$$2\text{T} = \Gamma'_3, \ \ 2\text{O} = \Gamma'_4, \ \ 2\text{I} = \Gamma'_5 \,. \tag{29}$$

Therefore, their modular mathematical properties apply to our discrete symmetry binary subgroups of SU(2) as well.

Lepton flavor models based upon the modular symmetry group $\Gamma = $ SL(2,$\mathbb{Z}$) utilize its subgroups $\Gamma'_N = $ SL(2,$\mathbb{Z}_N$), such as the double covers $\Gamma'_2 = S'_3$, $\Gamma'_3 = A'_4$, $\Gamma'_4 = S'_4$, $\Gamma'_5 = A'_5$ of the permutation groups S_3, A_4, S_4, A_5. With significant fine-tuning and a number of coupling constants, the mass hierarchies of the leptons can be reproduced in terms of a small parameter when the three lepton families are assigned to an irreducible representation of a modular subgroup, such as $\Gamma'_3 = A'_4$.

The modular group's fundamental domain $\mathcal{D}$ shown in Fig. 1 has the three symmetric points $\tau_{sym} = i\infty, i$, and $\omega = \exp(2\pi i/3)$ with its three τ_{sym} values preserving specific $\mathbb{Z}_N$ symmetries, i.e. those with $N = 2, 3$, or 4. When τ lies on the border, CP symmetry is preserved, but small deviations lead to CP symmetry being broken and hierarchial mass patterns emerging according to the sequence $(1, \epsilon, \epsilon^2)$.

This recent research has revealed that the lepton data suggests a value of τ near the cusp $\tau_0 = \omega = -0.5 + 0.866i$, with the best fit being

$$\tau = -0.496 + 0.877i \tag{30}$$

with a viable region being a small ring of values around the cusp ω, as shown in Fig. 1. The result is universal, meaning that its value is independent of which modular subgroup is being considered.

The research defined a scalar potential V_m near $\tau_0 = \omega$ that depends upon an integer parameter m and a phase angle ϕ, with a minimum in the scalar potential at

$$\frac{0.0145}{m + 0.0025} \,. \tag{31}$$

If the phase angle is included, the minimum occurs at

$$\phi_{min} = \frac{-2\pi}{9} \tag{32}$$

independent of m, producing for $m = 2$ the result

$$\frac{0.0145}{2 + 0.0025} \exp\left(\frac{-2\pi i}{9}\right) \leftrightarrow \tau_{min} = -0.492 + 0.875i \,. \tag{33}$$

The scalar potential V_m has a deep trench from ω upward from ω in the first quadrant direction that depends upon the quantity

$$[j(\tau) - 1728]^{m/2} \tag{34}$$

where $j(\tau)$ is the j-invariant of elliptic modular functions.

Therefore the modular group approach has revealed some very important results, particularly telling us that there seems to be no dependence upon which modular subgroup Γ'_N is being used as the modular subgroup for lepton flavor symmetry! Whence, the above results apply to all the modular subgroups equally or, equivalently, to our specific binary subgroups of SU(2) for the lepton families.

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