# Gamow Theory for Diproton Decays of Proton-Rich Heavy Nuclei $^{45}$Fe and $^{67}$Kr

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A diproton is an unusual particle, made of only two protons, which are believed to be unbound. In the core of a main sequence star such as the Sun, protons first combine to form diprotons in order for the proton-proton chain nuclear fusion reactions to occur. Exploring properties and activities of diprotons plays an important role in understanding the physics of stellar energy generation by nuclear fusion. In laboratories, it has been shown experimentally that some proton-rich radioactive heavy nuclei such as $^{45}$Fe and $^{67}$Kr can decay with emissions of diprotons and longer lifetimes in comparison with lighter nuclei. In this study, we investigate diproton decays of proton-rich or neutron-rare radioactive heavy nuclei. We first quantum-mechanically analyze to formulate the expressions for the transmission probability and lifetime of the diproton decays. Then, we numerically calculate the transmission probabilities and lifetimes of the diproton decays. The numerical results obtained for the diproton decays of the two typical proton-rich radioactive heavy nuclei $^{45}$Fe and $^{67}$Kr are plotted as functions of the energy of the emitted diproton and further compared with the measurements. It is shown that the transmission probabilities rapidly increase with the energy of the emitted diprotons, while the lifetimes for the diproton decays decrease with the energy of the emitted diproton and can be consistent with the laboratory measurements.

## 1 Introduction

A diproton is the nucleus of a rare isotope of helium, $^2$He, and consists of only two protons. It is extremely unstable and believed to be in an unbound state with a negative binding energy due to the spins of the two protons to be anti-aligned according to the Pauli exclusive principle [1, 2]. A diproton does not stably exist in nature but can be formed temporarily in two ways (see Fig. 1): (1) combination of two separate protons and (2) decay of proton-rich radioactive heavy nuclei. Two separate protons, when they collide with enough energy to tunnel through the Coulomb barrier between them, form a diproton, $^1\mathrm{H} + {}^1\mathrm{H} + \text{Energy} \longrightarrow {}^2\mathrm{He}$. This frequently occurs in the core of the Sun or any star in the main sequence. Approximately, there are about $10^{63}$ diprotons formed every second in the core of the Sun. Most of them quickly separate back to protons, ${}^2\mathrm{He} \longrightarrow {}^1\mathrm{H} + {}^1\mathrm{H}$, and only a very small part rarely get fused into deuterons via positron decays with emissions of neutrinos, ${}^2\mathrm{He} \longrightarrow {}^2\mathrm{H} + e^+ + \nu_e$. The fusion rate of the Sun should be about $10^{39}$ protons per second according to its luminosity, which is about $10^{24}$ times lower than the rate of diproton formation. In addition to the rareness of positron decays and difficultness of Coulomb barrier penetrations or quantum tunneling, plasma waves or oscillations may also play a significant role in the reduction of the rate of fusion in the core of the Sun [3, 4].

In laboratories, on the other hand, scientists have experimentally discovered that some proton-rich (or neutron-rare) heavy nuclei can emit diprotons [5, 6]. For instance, the proton-rich radioactive nuclei $^{15}$Ne, $^{45}$Fe and $^{67}$Kr can decay into $^{13}$O, $^{43}$Cr, and $^{65}$Se, respectively, after emitting a diproton [7–9]. The emission of a diproton from a proton-rich heavy nucleus is usually called diproton decay. The diproton decay is a rare decay mode found in a few nuclei beyond the proton drip line [10]. It is found, on the basis of the shell-model mass extrapolation, that $^{45}$Fe nuclei are unbound and emit diprotons in the decay. The half-life of $^{45}$Fe calculated using an R-matrix formula for the contribution due to the diproton decay agrees with the experimental values [8]. The diproton tunneling half-life decreases with the decay energy. First

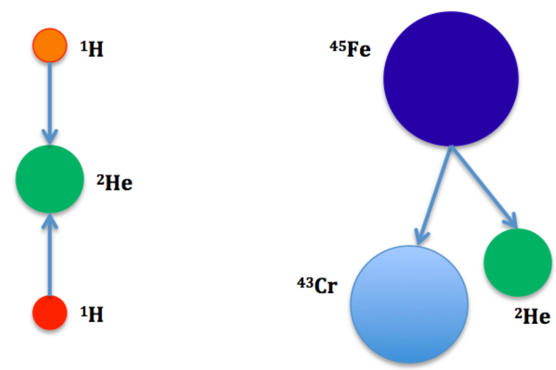


Fig. 1: Two ways of diproton formation: either formed from combination of two separate protons or emitted from decay of proton-rich radioactive heavy nuclei such as $^{15}$Ne, $^{45}$Fe, $^{67}$Kr, and so on. The right panel shows a schematic diagram for a $^{45}$Fe nucleus to decay into $^{43}$Cr after it emits a diproton $^2$He.

observations of diproton decays from $^{45}$Fe showed the half-life to be about 3.8 ms and the energy about 1.15 MeV [5, 6]. Fig. 2 shows the observations of diproton decay of $^{45}$Fe and its half-life [6, 11]. Observations also show that the unbound proton-rich nucleus $^{15}$Ne directly decays to $^{13}$O with a simultaneous diproton emission [7]. The diproton decay of $^{67}$Kr is measured to be unexpectedly fast [9].

Recently, the first author of this paper has theoretically modeled and numerically studied the transmission and proton decay of unbound diprotons, transmission and positron decay of protons, and transmission and diproton decay of unbound proton-rich heavy nucleus $^{15}$Ne in accordance with the Gamow theory for the quantum tunneling and radioactive decays [14]. It was shown that an unbound diproton is extremely unstable and quickly decays through two types of decay modes with lifetime to be extremely short down to about $10^{-21}$ seconds. A diproton mostly undergoes a proton decay to be two separate protons with a transmission probability higher that 99.99%, and rarely undergoes a $\beta^+$ decay to form a deuteron with a transmission probability lower than 0.01%. The transmission probability for the diproton decay of $^{15}$Ne increases with the energy.

In this paper, we quantum-mechanically study the transmission probability and lifetime for the diproton decays of proton-rich radioactive nuclei according to the Gamow theory that describes and models the quantum tunneling of the Coulomb barrier between the emitted diproton and the leftover nucleus. We obtain that the transmission probability and lifetime of unbound proton-rich heavy nuclei depend on the energy of the emitted or decayed diprotons. In general, the probability increases with the energy, while the lifetime decreases with the energy. With a certain probability, the heavier the nucleus is, the greater the energy of the emitted diproton is. For $^{45}$Fe nuclei, the lifetime of the diproton decay with energy about 1.1-1.2 MeV, obtained from this study, is about some milliseconds, which is consistent with the measurements [8].

## 2 Quantum theory for diproton decay of heavy nuclei

In 1928, on the basis of quantum mechanics, George Gamow proposed a theory for the $\alpha$-decay of radioactive heavy nuclei [15]. In this study, we apply the Gamow theory to describe and explain the diproton decay of proton-rich radioactive heavy nuclei. An $\alpha$ particle is a helium nucleus, $^4$He, while a diproton is an isotope of helium, $^2$He. Both are electrically charged by $2e$, where $e$ is the electric charge of proton. The Gamow theory that was developed for the $\alpha$-decay of radioactive nuclei should be applicable to the diproton decay of radioactive nuclei. During the diproton decay of a proton-rich radioactive heavy nucleus, a diproton is electrically repelled by and further escapes from the leftover nucleus. In the Gamow theory, the potential energy function is approximately modeled by a finite potential square well to represent the attractive nuclear force and joined with a Coulomb repulsive potential tail [14, 16],

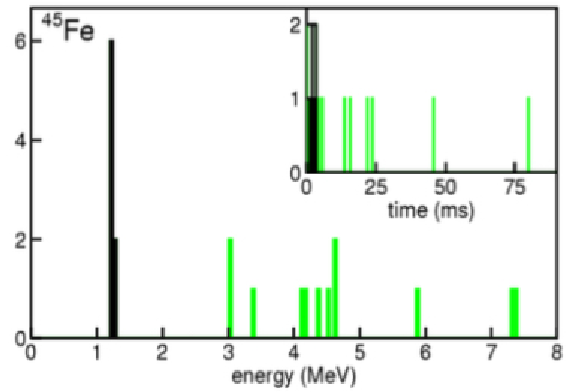


Fig. 2: The energy and time distribution of decay events from $^{45}$Fe, measured in experiments by GANIL [12] and GSI [13]. The events represented by black lines corresponding to the diproton decays, while other high frequency events represented by green lines represented the $\beta-$decays.

$$V(r) = \begin{cases} -V_0\,, & \text{if } 0 < r < r_1 \\ \dfrac{1}{4\pi\epsilon_0}\,\dfrac{Z_1 Z_2 e^2}{r}\,, & \text{if } r_1 \le r < \infty \end{cases}\,. \tag{1}$$

Here $Z_1$ and $Z_2$ are atomic numbers or charge states of the emitted particle and leftover nucleus; $\epsilon_0$ is the electric permittivity constant in free space; and $V_0$ is the depth of the potential square well. Fig. 3 is a schematic diagram for the potential energy $V(r)$ given by (1) as a function of the radial distance $r$ in all the classical and quantum regions. The width of the potential square well, denoted by $r_1$, can be determined as the radius of the nucleus, given by a constant times the cubic root of the mass number of the nucleus as

$$r_1 = r_0\,A^{1/3} \tag{2}$$

where $A$ is the mass number of the nucleus and the constant is $r_0 = 1.2 \times 10^{-15}$ m. The depth of the potential square well, $V_0$, is much greater than the maximum height of the Coulomb barrier, $U_c$, given by

$$U_c = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r_1} \ll V_0\,. \tag{3}$$

The outer turning point (i.e. at $r = r_2$) can be determined, in terms of the energy $E$ of the emitted $\alpha$ particle to be equal to the potential energy at $r_2$, by

$$r_2 = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E}\,. \tag{4}$$

In this central force problem with potential $V(r)$ given by (1) or shown in Fig. 3, the radial Schrödinger equation of the

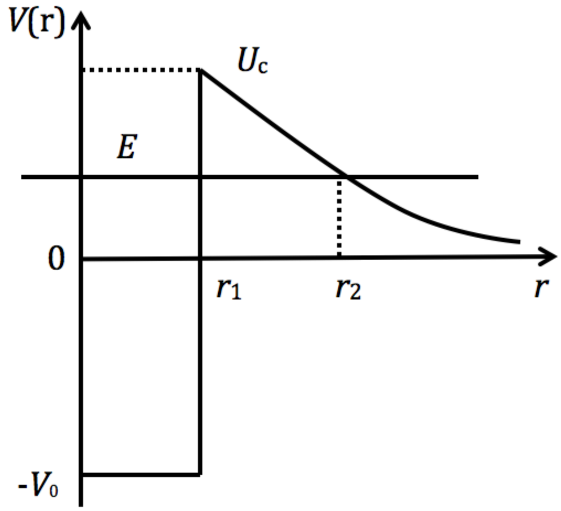


Fig. 3: The Gamow model for the potential energy of an electrically charged particle to be emitted from a radioactive nucleus. It consists of the potential energy square well for the attractive nuclear force and the Coulomb potential energy tail for the repulsive electric force between the emitted particle and the leftover nucleus of the decay.

particle wave is [14, 16],

$$\frac{d^2u(r)}{dr^2} = \frac{2\mu}{\hbar^2}[V(r) - E]u(r) + \frac{l(l+1)}{r^2}u(r)\,, \qquad (5)$$

where $u(r)$ is the radial wave function, $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass with $m_1$ the mass of the emitted particle and $m_2$ the mass of the leftover nucleus. The integer $l$ is the quantum number for the magnitude of angular momentum and $h = 2\pi\hbar$ is the Planck constant. A two-body system with a central force or potential can be generally described as a system of one body with the reduced mass.

According to the WKB approximation of quantum mechanics, we can approximately solve the radial Schrödinger equation and find the radial wave functions to be

$$u(r) = \frac{C}{\sqrt{|p(r)|}}\,\exp\left[\pm\frac{1}{\hbar}\int |p(r')|dr'\right], \qquad (6)$$

where the parameter $p(r)$ is defined by

$$p(r) = \sqrt{2\mu\,[V(r) - E]}\,. \qquad (7)$$

Here, it should be pointed out that the general solution of the radial Schrödinger equation should be the combination of these two corresponding to the plus sign and minus sign. We have also neglected the effect of angular motion and considered the case of $l = 0$.

Then, from the solved wave function, the transmission (or tunneling) probability for the electrically charged particle to tunnel through the Coulomb barrier is obtained as [14, 16]

$$T = e^{-2\gamma}\,, \qquad (8)$$

where the parameter $\gamma$ is determined by

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2} dr\,\sqrt{2\mu\,[V(r) - E]}$$

$$= \frac{\sqrt{2\mu E}}{\hbar}\left[r_2\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{r_1}{r_2}}\right) - \sqrt{r_1(r_2 - r_1)}\right]. \qquad (9)$$

And the lifetime of the parent nucleus or decay is given by

$$\tau = \frac{2r_1}{v}\,e^{2\gamma}\,, \qquad (10)$$

where $v = \sqrt{2E/m_1}$ is simply chosen to be the speed of the emitted (or $\alpha$) particle. It should be noted that, although being proposed for explaining the $\alpha$-decay of radioactive nuclei, the Gamow model is applicable in general for the decay or emission of any type of charged particles from a radioactive nucleus such as the $\beta^+$ decay from a proton, and emission of a proton or a diproton from a proton-rich radioactive heavy nucleus (e.g. diproton decays of $^{15}$Ne, $^{45}$Fe, $^{67}$Kr, and so on).

## 3 Probability and lifetime of diproton decay

A heavy nucleus with the elemental formula $^A_Z$X, if it is proton-rich (or $A < 2Z$), may be radioactive and decay. If the emitted particle of the decay is a diproton, we call the diproton decay. Here $X$ is the elemental symbol of the nucleus, usually called the parent nucleus, $Z$ is the atomic number of the parent nucleus, and $A$ is the mass number of the parent nucleus. In this diproton decay, we have $Z_1 = 2$, $Z_2 = Z - 2$, $m_1 = 2m_p$, $m_2 = (A - 2)\,m_p$, and $\mu = (m_1 \times m_2)/(m_1 + m_2)$, where $m_p = 1.67 \times 10^{-27}$ kg is the proton mass. We have approximately considered both proton and neutron having about the same mass. The width of the potential square well or the radius of the parent nucleus, $r_1$, can be estimated from (2) and the outer turning point, $r_2$, can be calculated from (4). With the values of these parameters and given a nucleus' $Z$ and $A$, we can calculate, from (8) to (10), the transmission probability and lifetime of the diproton decay. For the typical proton-rich radioactive heavy nuclei $^{45}$Fe and $^{67}$Kr, we have plotted the results obtained from calculations of the transmission probability and lifetimes of the diproton decay.

For the diproton decay of $^{45}$Fe, the leftover nucleus is $^{43}$Cr. In this case, we have $Z = 26$, $A = 45$, $Z_1 = 2$, $Z_2 = 24$, $m_1 = 2m_p$, $m_2 = 43m_p$, $\mu = 1.91m_p$, The width of the potential square well or the radius of $^{45}$Fe nucleus can be obtained from (2) as $r_1 \simeq 4.27 \times 10^{-15}$ m. With the values of these parameters and (8) to (10), we can plot, in Fig. 4, the transmission probability for the diproton decay from a radioactive nucleus $^{45}$Fe (red line) and the lifetime of the nucleus $^{45}$Fe via this diproton decay mode (blue line) as a function of the energy of the diproton. It is seen that the transmission probability increases with the energy. In the energy range from 1 MeV to 5 MeV, the transmission probability of the diproton

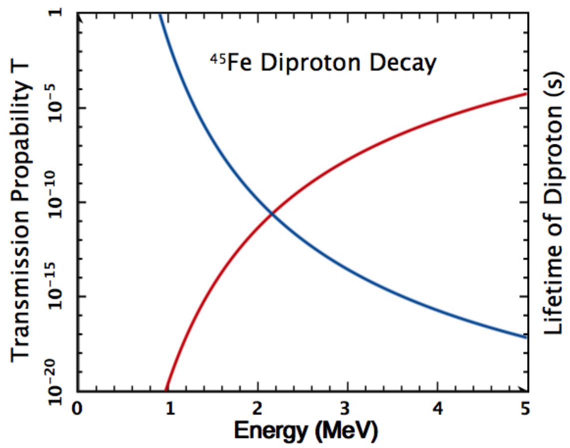


Fig. 4: Diproton decay transmission probability and lifetime of $^{45}$Fe nucleus. The red line plots the transmission probability of emitting a diproton from the radioactive nucleus $^{45}$Fe in the potential energy well to tunnel through the Coulomb barrier as a function of the energy of the diproton. The blue line plots the lifetime of the diproton decay.

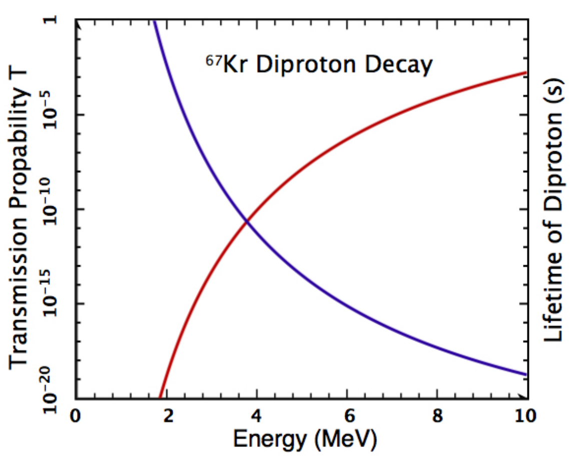


Fig. 5: Diproton decay transmission probability and lifetime of $^{67}$Kr nucleus. The red line plots the transmission probability of emitting a diproton from the radioactive nucleus $^{67}$Kr in the potential energy well to tunnel through the Coulomb barrier as a function of the energy of the diproton. The blue line plots the lifetime of diproton decay.

decay increases from $10^{-20}$ to $10^{-4}$, while the lifetime decreases from $10^{-3}$ s to $10^{-17}$ s. It is consistent with the measurement at energy 1.15 MeV with the lifetime of diproton decay from $^{45}$Fe to be of order $10^{-3}$ s [8].

For the diproton decay of $^{67}$Kr, the leftover nucleus is $^{65}$Se. In this case, we have $Z = 36$, $A = 67$, $Z_1 = 2$, $Z_2 = 34$, $m_1 = 2m_p$, $m_2 = 65m_p$, $\mu = 1.94m_p$. The width of the potential square well or the radius of $^{67}$Kr nucleus can be obtained from (2) as $r_1 \simeq 4.87 \times 10^{-15}$ m. With the values of these parameters and (8) to (10), we can plot, in Fig. 5, the transmission probability for the diproton decay from a radioactive nucleus $^{67}$Kr (red line) and the lifetime of the nucleus $^{67}$Kr via this diproton decay mode (blue line) as a function of the energy of the diproton. It is seen that the transmission probability increases with the energy. In the energy range from 2 MeV to 10 MeV, the transmission probability of the diproton decay increases from $10^{-20}$ to $10^{-3}$, while the lifetime decreases from $10^{-4}$ s to $10^{-19}$ s.

## 4 Discussions and conclusions

The fact or observations that diprotons are emitted from the decays of proton-rich radioactive nuclei implies that diprotons might have bound states with a positive binding energy. Even if a diproton is only weakly bound with an extremely small but positive biding energy, e.g. 0.384 MeV [17], a star such as the Sun would fuse its protons to deuterons at a rate many orders faster (i.e. $\gg 10^{39}$ protons/s) so that the star becomes much brighter in luminosity (i.e. $\gg 10^{26}$ W). This will result in a universe to fail the life support [18, 19]. This diproton disaster can be overcome by plasma oscillations or waves, which have been shown recently by the first author of this paper to be extremely efficient in inhibiting the nuclear reaction [3, 4], to have the observed luminosity without need to adjust the stars' central temperature, density, and initial number of deuterons. We will study in more details the transmission probability of bound diprotons for the fusion reaction in future.

As a consequence of this study, we have investigated diproton decays of two typical proton-rich (or neutron-rare) radioactive heavy nuclei $^{45}$Fe and $^{67}$Kr. First, we have applied the Gamow theory for the $\alpha$-decay of radioactive heavy nuclei to quantum-mechanically model the diproton decay of proton-rich radioactive heavy nuclei. We have derived expressions for the transmission probability and lifetime of diproton decay. Then, for the two typical proton-rich radioactive nuclei, we have numerically calculated the transmission probabilities and lifetimes of the diproton decays. We have found that the transmission probabilities rapidly increase with the energy of the emitted diprotons, and the lifetimes for the diproton decay decrease with the energy of the emitted particle. And finally, we have compared our obtained results with laboratory measurements. At the energy of 1.15 MeV, the lifetime for the diproton decay of $^{45}$Fe is observed to be about the order of milliseconds, which is consistent with the results of Gamow modeling obtained from this study.


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## References


1. Pauli W. Exclusion Principle and Quantum Mechanics. *Writings on Physics and Philosophy*, 1946, 165–198.
2. Bertulani C. A. Nuclear Physics in a Nutshell. Princeton University Press, 2007, ISBN 978-0-691-12505-3.
3. Zhang T. X. The Role of Plasma Oscillation Played in Solar Nuclear Fusion. *Bulletin of American Astronomical Society*, 2020, v. 52, 106.05.
4. Zhang T. X. The Role of Plasma Oscillation in Solar Nuclear Fusion. *Progress in Physics*, 2021, v. 17, 67–71.
5. Blank B. *et al*. First Observation of Two-Proton Decay from Radioactivity Nucleus. *Comptes Rendus Physique*, 2003, v. 4, 521–527.
6. Blank B., Ploszajczak M. Tow-Proton Radioactivity. *Reports on Progress in Physics*, 2008, v. 71, 046301.
7. Wamers F. *et al*. First Observation of the Unbound Nucleus $^{15}$Ne. *Physical Review letters*, 2014, v. 112, 132502.
8. Brown B. A. and Barker F. C. Diproton Decay of $^{45}$Fe. *Phys. Rev. C*, 2003, v. 67, 041304.
9. Wang S. M. Puzzling Two Proton Decay of $^{67}$Kr. *Phys. Rev. Lett.*, 2008, v. 120, 212502.
10. Brown B. A. Diproton Decay of Nuclei on the Drip Line. *Phys. Rev. C*, 1991, v. 43, R1513.
11. Giovinazzo J. *et al*. Two-proton radioactivity: 10 years of experimental progresses. *Journal of Physics: Conference Series*, 2013, v. 436, 012057.
12. Giovinazzo J. *et al*. Two-proton radioactivity of $^{45}$Fe. *Phys. Rev. Lett.*, 2002, v. 89, 102501.
13. Pfützner M. *et al*. First Evidence for the two-proton decay of $^{45}$Fe. *The European Physical Journal A*, 2002, v. 14, 279–285.
14. Zhang T. X. Gamow Theory for Transmission Probability and Decay of Unbound Diprotons. *Progress in Physics*, 2021, v. 17, 185–188.
15. Gamow G. Zur Quantentheorie des Atomkernes. *Z. Physik*, 1928, v. 51, 204–212.
16. Griffiths D. J. Introduction to Quantum Mechanics, 2nd Edition. Person Prentice Hall, 2005.
17. Kadenko I. M. *et al*. Bound Diproton: An "Illusive" Particle or Exotic Nucleus? *Acta Physica Polonica A*, 2022, v. 142, 337–341.
18. Bradford R. A. W. The Effect of Hypothetical Diproton Stability on the Universe. *J. of Astrophys. Astron.*, 2009, v. 30, 119–131.
19. Barnes L. A. Binding the Diproton in Stars: Anthropic Limits on the Strength of Gravity. *JCAP*, 2015, No. 12, 050.