Galileo's Principle and the Origin of Gravitation According to General Relativity

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Using the chronometrically invariant notation of General Relativity (chronometric invariants are the physically observable projections of four-dimensional quantities onto the time line and the three-dimensional space of an observer), we deduce Galileo's principle and Newton's law of gravitation as a particular case of the chr.inv.-formula for the gravitational inertial force acting in the four-dimensional pseudo-Riemannian space (space-time of General Relativity). This is a "mathematical bridge", connecting the empirical laws of Newton's theory of gravitation with the purely geometric laws of General Relativity. We also show that the origin of the gravitational field in the space of the Schwarzschild mass-point metric is a spherical surface that surrounds any mass-point at a very small radius, equal to the gravitational radius calculated for the mass. There, on the spherical surface, a breaking of the three-dimensional observable space takes place, and the observer's physical observable time stops. It is not possible to get these results using the general covariant notation of General Relativity, because physically observable quantities in the general covariant notation are not mathematically defined.

We dedicate this article to Prof. Kyril Stanyukovich (1916–1989), our long friendly conversations with whom in the 1980s formed the basis of this study 40 years later and prompted us to write this article.

1 Problem statement

Our closest colleague, patron and friend over decades was Prof. Kyril Stanyukovich (1916–1989). In addition to his groundbreaking works on gas dynamics and super-powerful non-nuclear ammunition, he was also a prominent researcher in the field of General Relativity; see [1–4] and References therein. Over many years in the 1980s, he repeatedly focused our attention onto a still unsolved problem: in the framework of Riemannian geometry (which is the basis of General Relativity), the fundamental laws of Newtonian classical mechanics have not yet been mathematically deduced as an unambiguous special case of the purely geometric laws of General Relativity.

This problem was also pointed out earlier by Alexei Petrov (1910–1972), the outstanding scientist in the field of General Relativity, who in 1950 introduced an algebraic classification of the spaces (and the gravitational fields) known in the framework of General Relativity [5–8]. This classification is called the Petrov classification of Einstein spaces thanks to his monograph *Einstein Spaces* [7], first published in 1961.

In our personal opinion, the fundamental laws of Newtonian classical mechanics have not yet been deduced as a special case of the geometric laws of General Relativity only because the researchers, who worked on this problem earlier, used the *general covariant notation* of General Relativity. In the framework of the general covariant notation, physically observable quantities are not mathematically determined. As a result, there is no clear mathematical transition from the four-dimensional quantities of General Relativity to the three-dimensional quantities of Newton's theory, which are measurable in experiment.

In this paper, we will solve the mentioned problem using the *chronometrically invariant notation* of General Relativity, i.e., the mathematical apparatus of chronometric invariants, which are mathematically determined as physically observable quantities in the four-dimensional pseudo-Riemannian space (space-time). To do this, we compare the mathematical basis of Newton's theory of gravitation with the mathematical basis of General Relativity. This comparison will allow us to consider the fundamental laws of Newton's theory as the three-dimensional spatial projections of the four-dimensional (space-time) laws of General Relativity.

2 The mathematical basis of Newton's theory of gravitation and that of Einstein's theory of relativity

It is well known that Newton's theory of gravitation and Einstein's theory of relativity are based on different mathematical foundations. The bases of both theories are sets, each of which has its own method of measuring infinitely small distances ds between its elements (points). Such sets are called *metric spaces*, and the quantity ds^2 is called the *space metric*. Metric spaces play a huge rôle in topology, geometry, and in the sections of theoretical physics where we study the structure of space and time.

Newton's fundamental laws, including the Law of Universal Gravitation, are formulated in the framework of the three-dimensional flat, homogeneous and isotropic (Euclidean) space E_3. Such a space allows the existence of *inertial reference frames*: in an inertial reference frame, free bodies either travel uniformly and rectilinearly or are at rest relative to the observer. In any inertial reference frame, time is homogeneous, and space is homogeneous and isotropic. The homogeneity of time means uniformity of its pace. The homogeneity of a space means the equality of all its points, and the isotropy of a space means the equality of all directions in it. The homogeneity and isotropy of space follow from Newton's first law (the law of inertia), which says: "if in the region, where inertial reference frames exist, no forces act on a body, or all forces acting on the body balance each other, then the body is either at rest or travels rectilinearly and uniformly".

In a three-dimensional flat space E_3 (Euclidean space), the square of the length of an elementary three-dimensional interval ds, characterizing the distance between two infinitely close points of the space, in the Cartesian coordinates x, y, z has the form

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1}$$

where the numerical value of ds^2 can only be *positive* and, hence, the three-dimensional interval ds is always a *substantional quantity*. The metric (1) is called *positive definite*, and the space E_3 described by it is *properly Euclidean*. Here the word "properly" means that all basis vectors of E_3 have substantional lengths. The three-dimensional curvature of the space E_3 is zero. For this reason, the space E_3 is flat. The condition $ds^2 = 0$ is satisfied only in the coordinate origin $x = y = z = 0$.

The laws of Newtonian classical mechanics, including Newton's law of gravitation, are formulated in the framework of a flat three-dimensional (Euclidean) space E_3.

Einstein's theory of relativity was created to describe space and time as a single entity, which is "space-time". The necessary prerequisites for Einstein's theory were obtained in the works of several other scientists, mainly in the works authored by Hermann Minkowski and Henri Poincaré. The basis of the theory is the four-dimensional curved pseudo-Riemannian space V_4. The prefix "pseudo" in this case indicates the fundamental difference between the mathematical basis of Newton's theory and the mathematical basis of Einstein's theory: this prefix means that one coordinate basis vector (time basis vector) has an imaginary length, and three other three-dimensional (spatial) basis vectors have substantional lengths (or vice versa, which is the same).

Initially, Einstein created the Special Theory of Relativity, the mathematical basis of which is the flat four-dimensional pseudo-Euclidean space E_4, later called the *Minkowski space*. The Minkowski space is a simplest particular case of four-dimensional pseudo-Riemannian spaces, which is homogeneous and isotropic, while its four-dimensional curvature is zero: the three-dimensional subspace of the Minkowski space may be non-uniform and anisotropic in one reference frame, but these factors in the Minkowski space depend on the observer's reference frame and, therefore, they can be reduced to zero simply by choosing another different reference frame. Bodies that are not affected by external forces travel uniformly and rectilinearly in the Minkowski space.

The Minkowski space is described by the metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{2}$$

where $x^0 = ct$ is the time coordinate, in which c is the velocity of light, t is the ideal (uniform) coordinate time, and $x^1 = x$, $x^2 = y$, $x^3 = z$ are the Cartesian three-dimensional (spatial) coordinates. In this notation, each of the three-dimensional spatial basis vectors e^i (where $i = 1, 2, 3$) has a substantional unit length, and the time basis vector e^0 has an imaginary unit length $(e^0)^2 = -1$ or vice versa, depending on the choice for the space signature $(-{+}{+}{+})$ as in (2) above or $(+{-}{-}{-})$ as is most commonly used in General Relativity.

The basic space (space-time) of the General Theory of Relativity is the curved four-dimensional pseudo-Riemannian space V_4 — the generalization of the flat four-dimensional pseudo-Euclidean (Minkowski) space E_4, which can be inhomogeneous, anisotropic, etc. per se, i.e., independently of the choice of the observer's reference frame. The laws of General Relativity are formulated in the framework of the curved four-dimensional pseudo-Riemannian space V_4.

The square of the elementary distance ds between two infinitely close points (i.e., the space metric) in V_4 is expressed as follows

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = \\ = g_{00}\, dx^0 dx^0 + 2g_{0i}\, dx^0 dx^i + g_{ik}\, dx^i dx^k, \tag{3}$$

where $\alpha, \beta = 0, 1, 2, 3$ are the space-time (four-dimensional) indices, $i, k = 1, 2, 3$ are the spatial (three-dimensional) indices, and $g_{\alpha\beta}$ is the fundamental metric tensor of the space (it is a symmetric tensor, i.e., $g_{\alpha\beta} = g_{\beta\alpha}$). In the pseudo-Riemannian space, the time basis vector e^0 has the length dependent on the gravitational field potential, and the lengths of the three-dimensional spatial basis vectors e^i depend on the inhomogeneity and anisotropy of space, i.e., they are not unit length vectors, in contrast to the four-dimensional pseudo-Euclidean (Minkowski) space. The factors that deviate the lengths of the space basis vectors from unit are determined by the components of the fundamental metric tensor $g_{\alpha\beta}$ (while in the Minkowski space we can always find an inertial reference frame, in which the diagonal components of $g_{\alpha\beta}$ are units, and its non-diagonal components are zero). The time component g_{00} characterizes the gravitational field potential, the spatial components g_{ik} characterize the inhomogeneity and anisotropy of the observer's three-dimensional space, and the mixed (space-time) components g_{0i} characterize the angle of inclination of his three-dimensional space to the lines of time (the spaces in which this inclination takes place are called *non-holonomic spaces*; see Section 3, where we explain the basics of the theory of physically observable quantities in the space-time of General Relativity). In particular, the physically observable time of the observer depends on the magnitude of the gravitational potential at the place of observation, and also on the magnitude and direction of the rotation speed of his three-dimensional physical space (its inclination to the time line).

As a result of the aforementioned absolute factors of $g_{\alpha\beta}$, which cannot vanish by choosing an inertial reference frame in the pseudo-Riemannian space (in contrast to the pseudo-Euclidean space of Special Relativity), we formulate Newton's first law (the law of inertia) in General Relativity as follows: "a body can be at rest or travelling rectilinearly and uniformly only in the absence of the gravitational field, inhomegeneity and isotropy of space and its rotation (the latter means the absence of the inclination of the three-dimensional space to the time lines)".

3 Physically observable quantities in the space-time of General Relativity

Before comparing the mathematical foundations of Newton's theory of gravitation and Einstein's theory of relativity, we must explain the basics of the theory of physically observable quantities in the four-dimensional pseudo-Riemannian space (which are also known as the Zelmanov chronometric invariants).

This mathematical apparatus that uniquely determines physically observable quantities in the space-time of General Relativity was created in 1944 by Abraham Zelmanov [9–11], who was our teacher. In addition to Zelmanov's original publications, which were very concise, this mathematical apparatus was explained in detail by us in a special Chapter given in each of our three research monographs, originally published in 2001 [12, 13] and 2013 [14]. The most comprehensive survey of Zelmanov's mathematical apparatus was published by us in 2023, in the special paper [15], where we collected everything (or almost everything) that we know about this mathematical apparatus personally from Zelmanov and on the basis of our own research studies.

Over the past decades, the following problems of General Relativity have been solved using the mathematical apparatus of chronometric invariants:

— The theory of non-quantum teleportation in the space-time of General Relativity [12, 16, 17], the basics of which were first outlined in 2001 in our book [12] and then developed in all necessary details in 2022 [17];

— The theory of the direct and opposite flow of time, and also the three kinds of particles in the space-time of General Relativity, published in 2001, in the book [12];

— The theory of frozen/stopped light according to General Relativity, which explained the frozen light experiment (2000). This theory was first drafted in 2001, in the 1st edition of our book [12], then in 2011 published in all necessary details in our paper [18] and since 2012 added to all subsequent editions of the book [12];

— The cosmological mass-defect — a new effect of General Relativity, predicted in 2011 [19], according to which the observed masses of cosmic bodies depend on their distance from the observer if they are at cosmological large-scale distances from him (depending on the specific metric of space);

— The non-linear cosmological redshift, deduced in 2012 [20] for various space metrics, including the Friedmann expanding universe and the de Sitter static universe. Three short papers [21–23] were then focused on specific aspects of the obtained solutions, and a final analysis of those of them that are most suitable for explaining the non-linear cosmological redshift observed by astronomers was given in 2013, in the paper [24];

— The deflection of light rays and mass-bearing particles, and also the length stretching and time dilation in the field of a rotating body — these are three new effects of General Relativity, deduced in 2023 [25, 26];

— The condensed matter model of the Sun, created in the framework of General Relativity, according to which the space breaking in the gravitational field of the Sun meets the maximum concentration of the asteroids in the Asteroid belt. This study was first published in 2009–2010 [27, 28];

— The theory of the internal constitution of stars and the sources of stellar energy according to General Relativity, which was first published in 2013, in the book [14];

— The exact solutions, obtained in 2005 to the equations of deviating geodesics for solid-body and free-mass gravitational wave detectors [29, 30] (different from the approximate solutions presumed in 1961 by Joseph Weber). Since 2008, this study was added to all subsequent editions of our book [12]. The obtained solutions are based on the comprehensive theoretical study of gravitational waves performed during a decade in 1968–1978 [31–33];

— "Zitterbewegung" of travelling electrons, explained in 2023 by Pierre Millette [34] on the basis of the theory of spin-particles in General Relativity, published in 2001 [13, Chapter 4].

For a complete list of the published research studies performed using the mathematical apparatus of chronometric invariants as of January 2023, see Bibliography in our comprehensive paper on this subject [15].

In short, the essence of Zelmanov's mathematical apparatus of chronometric invariants (known also as the chronometrically invariant formalism) is as follows. Zelmanov unambiguously determined physically observable quantities in the space-time of General Relativity as the projections of four-dimensional tensor quantities onto the time line and the three-dimensional spatial section of the space-time, which are associated with an observer. Such projections remain invariant throughout the three-dimensional spatial section associated with the observer (his observable three-dimensional physical space), i.e., they are "chrono-metric invariants" in the physical reference frame of the observer and depend on the physical and geometric properties of his space, such as the gravitational potential, rotation, curvature, etc., which are determined by the respective components of the fundamental metric tensor $g_{\alpha\beta}$ and their derivatives.

The "chronometrically invariant" projections of any four-dimensional tensor quantity onto the time line and the three-dimensional spatial section associated with an observer are calculated using the Zelmanov operators of projection, which take the physical and geometric properties of the observer's space into account (see our comprehensive survey [15] of the chronomerically invariant formalism for detail).

As a result, the square of the four-dimensional (space-time) interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$, expressed with chronometrically invariant (physically observable) quantities, has the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{4}$$

where $d\tau$ is the chr.inv.-time interval (physically observable time interval), obtained as the chr.inv.-projection of the four-dimensional displacement vector dx^α onto the observer's time line

$$d\tau = \sqrt{g_{00}}\, dt - \frac{1}{c^2}\, v_i\, dx^i, \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{5}$$

and $d\sigma^2$ is the square of the chr.inv.-spatial interval (physically observable three-dimensional spatial interval)

$$d\sigma^2 = h_{ik}\, dx^i dx^k, \tag{6}$$

created using the chr.inv.-metric three-dimensional tensor h_{ik}

$$h_{ik} = -g_{ik} + \frac{1}{c^2}\, v_i v_k\,, \quad h^{ik} = -g^{ik}, \quad h^i_k = \delta^i_k\,, \tag{7}$$

which is the chr.inv.-projection of the fundamental metric tensor $g_{\alpha\beta}$ onto the observer's three-dimensional space (the spatial section of the space-time, which is associated with him). So forth, w is the physically observable chr.inv.-potential of the gravitational field that fills the observer's space, and v_i is the three-dimensional vector of the linear velocity of rotation of the observer's space

$$\mathrm{w} = c^2 \left(1 - \sqrt{g_{00}}\right), \qquad v_i = -\frac{c\, g_{0i}}{\sqrt{g_{00}}}\,, \tag{8}$$

dx^i is the elementary interval of the three-dimensional spatial coordinates ($i = 1, 2, 3$), and $\mathrm{v}^i = dx^i/d\tau$ is the chr.inv.-velocity vector (physically observable three-dimensional velocity), which is different from the three-dimensional coordinate velocity vector $u^i = dx^i/dt$.

If all g_{0i} of a four-dimensional (space-time) metric ds^2 are zero, then such space-time is *holonomic*. In this case the three-dimensional spatial section associated with the observer (his observed three-dimensional space) is everywhere orthogonal to the time lines $x^0 = ct = const$ that pierce it. If at least one of the components g_{0i} of the four-dimensional metric is different from zero, then such space-time is *non-holonomic*. In such a (non-holonomic) space-time, the observer's three-dimensional spatial section $x^0 = const$ is inclined to the time lines. In this case, at different points the observed three-dimensional space can be inclined to the time lines at different angles depending on the local geometric structure of the particular four-dimensional space-time.

The formula for the physically observable time interval $d\tau$ (5) can therefore be re-written as

$$d\tau = \left(1 - \frac{\mathrm{w} + v_i u^i}{c^2}\right) dt\,, \tag{9}$$

where $v_i u^i$ is the scalar product of the linear rotational velocity of the observer's space v_i and the three-dimensional coordinate velocity vector u^i

$$v_i u^i = |v_i||u^i| \cos\left(v_i u^i\right), \tag{10}$$

which means that if the vectors v_i and u^i are orthogonal to each other, then their scalar product $v_i u^i = 0$. In this case, the rotation of the three-dimensional reference space does not contribute to the change in the observer's physically observable time τ. If the vectors v_i and u^i are inclined to each other, then their mutual orientation in space affects the physically observable time τ, as well as its direction to the future or to the past: in the case, where the vector of the linear rotational velocity of the observer's reference space v_i is inclined in the same direction as the velocity motion vector of his reference body u^i (i.e., $v_i u^i > 0$), the observer's physical time τ flows faster; in the case, where the vectors v_i and u^i are inclined in opposite directions ($v_i u^i < 0$), the observed time τ flows slower. This purely theoretical conclusion was confirmed by the Hafele and Keating experiment (1971, repeated in 2005), in which they compared the readings of atomic clocks installed on board a jet airplane flying along a parallel around the globe with the readings of atomic clocks left on the surface of the Earth [35–39]. Thus, it was proven that the observed time on our planet depends on the following physical factors: 1) the magnitude of the gravitational field potential at the place of observation; 2) the speed of the Earth's rotation around its own axis (diurnal rotation); 3) the speed of the observer's motion relative to the Earth's rotation. For detail, see our recent publication on this subject [26].

In the theory of chronometric invariants, there are physically observable (chronometrically invariant) analogues of the quantities known in Newtonian classical mechanics. This fact will help us to find a connexion between Newton's theory of gravitation and General Relativity.

So, the physical reference space of a real observer (which is his physical reference frame) is characterized by the following physically observable chr.inv.-quantities. These are the chr.inv.-vector of the physically observable gravitational inertial force F_i acting in the observer's space, the first (gravitational) term of which is created by the gradient of the gravitational potential w and the second (inertial) term is created by the centrifugal force of inertia

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{11}$$

the antisymmetric chr.inv.-tensor A_{ik} of the physically observable three-dimensional angular velocity of rotation of the observer's space

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{12}$$

the symmetric chr.inv.-tensor D_{ik} of the physically observable deformation rate of the space

$$\left.\begin{aligned} D_{ik} &= \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, & D^{ik} &= -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} \\ D = h^{ik}D_{ik} &= \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, & h &= \det\|\,h_{ik}\,\| \end{aligned}\right\}, \tag{13}$$

the chr.inv.-Christoffel symbols of the 1st rank $\Delta_{jk,m}$ and the 2nd rank Δ^i_{nk} (they are the coefficients of the physically observable inhomogeneity of the observer's space)

$$\Delta^i_{nk} = h^{im}\Delta_{nk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{nm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^n} - \frac{{}^*\partial h_{nk}}{\partial x^m}\right), \tag{14}$$

and the physically observable chr.inv.-curvature of the observer's space, which is expressed with the chr.inv.-curvature tensor C_{lkij} that has all properties of the Riemann-Christoffel curvature tensor throughout the entire three-dimensional spatial section associated with the observer, whereas its subsequent contractions produce the chr.inv.-curvature scalar C

$$\begin{aligned} C_{lkij} &= \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right) = \\ &= H_{lkij} - \frac{1}{2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + \right. \\ &\qquad\qquad \left. + A_{kl}D_{ij} + A_{li}D_{jk}\right), \end{aligned} \tag{15}$$

$$C_{lk} = C_{lki\cdot}^{\cdots i} = H_{lk} - \frac{1}{2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right), \tag{16}$$

$$C = h^{lk}C_{lk} = h^{lk}H_{lk}\,, \tag{17}$$

where it is denoted, for brevity and a better association with the Riemann-Christoffel curvature tensor,

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial \Delta^j_{il}}{\partial x^k} - \frac{{}^*\partial \Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\,, \tag{18}$$

and the operators

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{v_i}{c^2}\frac{{}^*\partial}{\partial t} \tag{19}$$

are the chr.inv.-operators of derivation with respect to time t and the spatial coordinates x^i.

It should be noted that the physically observable chr.inv.-curvature of the observer's space is depended on not only the space inhomogeneity (Christoffel symbols), but also on the rotation A_{ik} and deformation D_{ik} of the space, and, therefore, does not vanish in the absence of the gravitational field.

Since the task here is to find a connexion between Einstein's theory of relativity and Newton's theory of gravitation, in which space-time is static (non-deforming) and flat, we will not consider the deformation and curvature of space (i.e., we will assume that $D_{ik} = 0$ and $C_{lkij} = 0$). In addition, if in this particular case the three-dimensional observable space does not rotate or if its rotation velocity does not depend on time, then the gravitational inertial force F_i depends only on the numerical value of the gravitational potential w and its spatial derivatives. We will therefore consider this particular case in the next Section to deduce Galileo's principle and Newton's law of gravitation as consequences of the purely geometric laws of General Relativity.

4 Galileo's principle and Newton's law of gravitation in the framework of General Relativity

According to the biography of Galileo, in 1589 he conducted his famous experiments with bodies falling from the Leaning Tower of Pisa to the surface of the Earth. Galileo wanted to prove his case in a correspondence dispute with Aristotle, who, in turn, about 2000 years before Galileo, in 360–330 B.C., argued that the motion speed of falling bodies depends on the magnitude of their masses: he argued the greater the mass of a falling body, the faster it falls down.

In contrast to Aristotle, Galileo made a supposition that the fall time of bodies does not depend on their masses. In support of his hypothesis, Galileo dropped down balls of different masses from the Leaning Tower of Pisa. With this experiment, Galileo established that bodies of different masses, dropped down to the surface of the Earth simultaneously from the same altitude above the Earth's surface, access the ground simultaneously. Since the Tower's height h is much less than the radius of the Earth ($h \ll R_\oplus$), it can be assumed that any body located at a small altitude above the Earth's surface is attracted to the centre of the Earth with a force proportional to the numerical value of the body's mass. In fact, Galileo had discovered that the fall time of the body does not depend on the numerical value of its mass. Therefore, he had arrived at the conclusion that is now called *Galileo's principle*:

> All bodies, regardless of the numerical values of their masses, fall to the surface of the Earth with the same acceleration, called the *free-fall acceleration*.

Later, in 1666, Isaac Newton formulated the Law of Universal Gravitation. According to this law, the force of attraction F between two material points with masses m_1 and m_2, located at a distance r from each other, acts along the line connecting their centres. This force is formulated as

$$F = -\frac{Gm_1 m_2}{r^2}\,, \tag{20}$$

where $G = 6.67 \times 10^{-8}$ cm^3/gram sec^2 is the Newton gravitational constant. From the above formula (20) it follows that in a flat (Euclidean) space E_3 the gravitational force of attraction F is determined only by the numerical values of the interacting masses and the distance between them, and does not depend on the size of the bodies. Such an interaction is called *point interaction*. Thus, in Newton's theory of gravitation, the gravitational interaction between two bodies is "point-like", i.e., it is carried out between the gravitating centres of these bodies (material points).

Applying (20) to the gravitational interaction between the Earth and a body of mass m falling to the Earth's surface, we obtain

$$F = -\frac{GmM_\oplus}{R_\oplus^2} = -mg\,, \tag{21}$$

where $g = GM_\oplus/R_\oplus^2$ is the free-fall acceleration due to the Earth's gravitation, $M_\oplus = 5.97 \times 10^{27}$ gram is the mass of the Earth, $R_\oplus = 6.37 \times 10^8$ cm is the radius of the Earth, and m is the mass of the body falling down to the Earth's surface. Formula (4.2) explains the results of Galileo's experiments under the condition that the bodies fall on the surface of the Earth from a small altitude $h \ll R_\oplus$. In this case, it is easy to calculate the magnitude of the free-fall acceleration on the Earth's surface: $g = 981$ cm/sec^2.

Formula (21) is the mathematical expression of Galileo's principle in the framework of Newton's theory of gravitation.

Let us now deduce Galileo's principle and Newton's law of gravitation in the framework of the four-dimensional space (space-time) of General Relativity. To do this, we consider Schwarzschild's mass-point metric. This metric is an exact solution of Einstein's field equations, which describes a spherically symmetric gravitational field created in an empty space (space-time) by a spherical island of substance, the mass of which is M, and which is approximated by a mass-point. The Schwarzschild mass-point metric in the spherical coordinates r, θ, φ has the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{22}$$

where $r_g = 2GM/c^2$ is the so-called *gravitational radius* calculated here for a spherical body of the mass M (which we approximate by a mass-point). The polar coordinate angle θ is measured from the North pole to the equator.

Since, according to the chronometrcally invariant formalism, the component g_{00} in a general case is expressed with the gravitational field potential w as

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \tag{23}$$

and according to the Schwarzschild mass-point metric (22) we have

$$g_{00} = 1 - \frac{r_g}{r}\,, \tag{24}$$

then in the space of the Schwarzschild mass-point metric the gravitational field potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ has the form

$$\mathrm{w} = c^2 \left(1 - \sqrt{1 - \frac{r_g}{r}}\right) = c^2 \left(1 - \sqrt{1 - \frac{2GM}{c^2 r}}\right), \tag{25}$$

which in the quasi-Newtonian approximation ($r_g \ll r$), where the ratio r_g/r takes small numerical values and, therefore,

$$\sqrt{1 - \frac{2GM}{c^2 r}} \simeq 1 - \frac{GM}{c^2 r}\,, \tag{26}$$

takes the form

$$\mathrm{w} = c^2 \left(1 - \sqrt{1 - \frac{2GM}{c^2 r}}\right) \simeq \frac{GM}{r}\,, \tag{27}$$

which coincides with the gravitational field potential according to Newton's theory of gravitation.

So forth, looking at the Schwarzschild mass-point metric (22), we realize that it is static, since all components of its fundamental metric tensor $g_{\alpha\beta}$ do not depend on the time coordinate $x^0 = ct$. This means that the space of the Schwarzschild mass-point metric does not deform ($D_{ik} = 0$). In addition, since all space-time components of the fundamental metric tensor of the metric are zero ($g_{0i} = 0$), such a space does not rotate ($v_i = 0$, $A_{ik} = 0$). As a result of the above, the physically observable time interval $d\tau$ in the Schwarzschild mass-point field has the form

$$d\tau = \sqrt{g_{00}}\, dt - \frac{1}{c^2} v_i\, dx^i = \sqrt{g_{00}}\, dt = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt = \\ = \sqrt{1 - \frac{r_g}{r}}\, dt = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt\,, \tag{28}$$

which means that the flow of the physically observable time τ in the Schwarzschild mass-point field is determined only by the numerical value of the gravitational field potential w.

Since the space of the Schwarzschild mass-point metric is static ($D_{ik} = 0$) and does not rotate ($v_i = 0$, $A_{ik} = 0$), the components of the chr.inv.-vector of the physically observable gravitational inertial force F_i (11) that acts on a unit mass in such a space take the form

$$F_1 = \frac{1}{\sqrt{g_{00}}} \frac{\partial \mathrm{w}}{\partial r}\,, \quad F_2 = 0\,, \quad F_3 = 0\,, \tag{29}$$

where $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ is the gravitational field potential. Therefore, in terms of the gravitational radius $r_g = 2GM/c^2$ calculated for the mass M, the solely non-zero component of the physically observable gravitational inertial force acting in the space of the Schwarzschild mass-point metric is

$$F_1 = -\frac{c^2}{2g_{00}}\frac{\partial g_{00}}{\partial r} = -\frac{c^2}{2\left(1-\frac{r_g}{r}\right)}\frac{r_g}{r^2}\,. \tag{30}$$

Apply the obtained formula (30) to a body having a mass m (different from unit mass) and located on the Earth's surface ($r = R_\oplus$) or at a small altitude h above it ($h \ll R_\oplus$). Since the radius of the Earth is $R_\oplus = 6.37\times10^8$ cm, and its gravitational radius is $r_g = 0.89$ cm, the ratio $r_g/R_\oplus$ on the Earth's surface takes a very small numerical value $r_g/R_\oplus = 1.4\times10^{-9}$ that can be neglected. In this case, the formula for the gravitational force F_1 (30), which we have obtained in the framework of General Relativity, takes the following form

$$\Phi_1 = mF_1 = -\frac{c^2}{2\left(1-\frac{r_g}{r}\right)}\frac{m r_g}{r^2} = -\frac{GmM_\oplus}{R_\oplus^2} = -mg\,, \tag{31}$$

which coincides with the formula (21), which, in turn, is the mathematical expression of Galileo's principle in the framework of Newton's theory of gravitation.

This means that, according to General Relativity, all bodies located on the surface of the Earth or at a small altitude above it are attracted to the centre of the Earth with the same acceleration, equal to the free-fall acceleration $g = GM_\oplus/R_\oplus^2 = 981$ cm/sec^2 (which is a conclusion, analogous to Galileo's principle in Newton's theory of gravitation).

In fact, using the chronometrically invariant notation of General Relativity, we have just deduced the following:

> Both Galileo's principle and Newton's law of gravitation (empirical laws of classical mechanics) are direct consequences of the geometric structure of the four-dimensional pseudo-Riemannian space (space-time of General Relativity), since the force of gravity, which attracts material bodies to the Earth, is the chr.inv.-vector of the physically observable gravitational inertial force acting in the space (gravitational field) of the Schwarzschild mass-point metric.

This cannot be shown using the general covariant notation of General Relativity, because it does not include physical observable quantities. This is why there is no unambiguous mathematical transition from General Relativity to Newton's theory of gravitation in the framework of the general covariant notation of General Relativity.

5 The origin of the gravitational field according to General Relativity

Let us now consider the origin of gravitation using the chronometrically invariant notation of General Relativity.

In the space of the Schwarzschild mass-point metric, on a spherical surface of the radius $r = r_g$ from the coordinate origin (which is the centre of the gravitating body approximated by a mass-point), the time component g_{00} of the fundamental metric tensor is zero ($g_{00} = 0$), and the radial component g_{11} becomes infinitely large ($g_{11} \to \infty$)

$$r = r_g\,, \quad g_{00} = 1-\frac{r_g}{r} = 0\,, \quad g_{11} = \frac{1}{1-\frac{r_g}{r}} \to \infty\,, \tag{32}$$

and, since the Schwarzschild space does not rotate ($v_i = 0$), hence the radial component h_{11} of the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}v_i v_k$ becomes also infinite ($h_{11} \to -\infty$).

This means that on the spherical surface $r = r_g$ that surrounds any mass-point (located at the coordinate origin in the space of the Schwarzschild mass-point metric) the following conditions take place:

1) The three-dimensional observable space (and the gravitational field of the mass-point, which fills the space) has a space breaking ($g_{11} \to \infty$, $h_{11} \to -\infty$);
2) The physically observable time τ of the observer stops ($d\tau = 0$) on this surface

$$d\tau = \sqrt{g_{00}}\,dt - \frac{1}{c^2}v_i dx^i = \sqrt{g_{00}}\,dt = 0\,. \tag{33}$$

That is, there on the surface of the gravitational radius $r = r_g$, which surrounds the centre of gravity inside any material body, the physically observable time stops ($d\tau = 0$), and the observable three-dimensional space is expanded infinitely in the radial direction $x^1 = r$ since the three-dimensional physically observable chr.inv.-interval $d\sigma$ that is determined as $d\sigma^2 = h_{ik}\,dx^i dx^k$ (6) on such a surface is

$$d\sigma = \sqrt{h_{11}\,x^1 x^1} = \frac{dr}{\sqrt{1-\frac{r_g}{r}}} \to \infty\,. \tag{34}$$

Equating $d\tau$ in the Schwarzschild mass-point field, which is $d\tau = \left(1-\frac{\mathrm{w}}{c^2}\right)dt$ (28), to zero (since $d\tau = 0$ on the surface of the gravitational radius), we obtain

$$E = M\mathrm{w} = Mc^2, \tag{35}$$

i.e., the energy $E = M\mathrm{w}$ of the gravitational field, created by a body having a non-unit mass M, on the surface of the gravitational radius $r = r_g$ (which surrounds the centre of gravity inside any material body) is the same as the total energy of the body $E = Mc^2$.

We therefore arrive at the following conclusion:

> The gravitational field of any body is originated in the surface of the gravitational radius $r = r_g$, which is surrounding the centre of gravity inside the body.

This is the origin of the gravitational field according to General Relativity. Since the gravitational radius of an ordinary body is incomparably smaller than its physical radius, the conclusion we have obtained in the framework of General Relativity is completely consistent with Newton's theory of gravitation, according to which the gravitational field of any body is originated in its center of gravity (which coincides with its geometric center in the case, where the body has a spherically symmetrical shape).

For example, the surface of the gravitational radius, which is surrounding the centre of gravity of the planet Earth, is the origin of the Earth's gravitational field attracting to this surface near the centre of the planet everything that is underground, grows on the Earth's surface, moves along it and above it (in the Earth's atmosphere and in the cosmos). Trees indicate this fact: their trunks are always directed from the centre of the Earth, and not at an angle to this direction. This is especially clearly seen in cases, where the ground on which the tree grows lies at an angle to a flat surface, for example, on a mountain slope: in this case, the tree does not grow perpendicular to the slope, but its trunk is oriented strictly in the direction from the centre of the Earth.

From the above conclusion about the origin of the gravitational field it also follows that a *collapse surface* (in terms of General Relativity, this is a surface on which $g_{00}=0$ and, as a result, the physically observable time stops $d\tau=0$) is not exclusively the surface of a black hole (gravitational collapsar) — a body, the substance of which is compressed to such a super-dense state that it is concentrated under its gravitational radius. Indeed, ordinary bodies are not in the state of gravitational collapse, since almost all mass of an ordinary body is located above its gravitational radius (which is very small compared to its physical radius). However, the tiny sphere of the gravitational radius that takes place at the centre of every ordinary body is also a *collapse surface*, because the physically observable time stops and the spatial metric has a breaking on this tiny sphere, just like on the surface of a black hole (gravitational collapsar).

The same conclusion about the origin of the gravitational field follows from the geodesic equations (equations of motion of free particles) in the space of the Schwarzschild mass-point metric. "Free" here means that the moving particle is affected only by the forces, the source of which is the geometric structure of the space itself (i.e., in the absence of extraneous fields).

The geodesic equations in the chronometrically invariant notation are a system of the chr.inv.-projections onto the time line (the chr.inv.-scalar projection) and onto the three-dimensional space (the chr.inv.-vector projection) associated with a particular observer. They have the following form (see References to the Zelmanov chronometric invariants)

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \tag{36}$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} - mF^i + 2m\,(D^i_k + A^{\cdot i}_{k\cdot})\,\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{37}$$

where m is the relativistic mass of the particle, τ is the physically observable time of its motion, $\mathrm{v}^i = dx^i/d\tau$ is its physically observable chr.inv.-velocity, F_i is the chr.inv.-vector of the gravitational inertial force, A_{ik} is the chr.inv.-tensor of the angular velocity of rotation of the observer's space, D_{ik} is the chr.inv.-tensor of the rate of its deformation, and Δ^i_{nk} are the chr.inv.-Christoffel symbols of the 2nd rank (which are the coefficients of the physically observable inhomogeneity of the observer's space).

The chr.inv.-geodesic equations (36, 37) are simplified in the space of the Schwarzschild mass-point metric

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i = 0\,, \tag{38}$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{39}$$

since such a space does not rotate or deform (see above). Here $\mathrm{v}^1 = dr/d\tau$, $\mathrm{v}^2 = d\theta/d\tau$, $\mathrm{v}^3 = d\varphi/d\tau$. In addition, only the radial component F_1 of the gravitational inertial force F_i is non-zero. According to (29), it is

$$F_1 = \frac{1}{\sqrt{g_{00}}}\frac{\partial \mathrm{w}}{\partial r} = \frac{c^2}{c^2-\mathrm{w}}\frac{\partial \mathrm{w}}{\partial r}\,, \tag{40}$$

where $\mathrm{w} = c^2(1-\sqrt{g_{00}})$ is the potential of the gravitational field (created by a massive body, approximated by a mass-point), in which the particle travels. Therefore, the scalar geodesic equation (38) takes the form

$$\frac{dm}{m} = \frac{1}{c^2}\,F_1\,dr\,, \tag{41}$$

which can be re-written as

$$\frac{dm}{m} = -\,\frac{d\,(c^2-\mathrm{w})}{c^2-\mathrm{w}}\,, \tag{42}$$

which is the same as

$$d\,(\ln m) = -\,d\left[\ln\,(c^2-\mathrm{w})\right]. \tag{43}$$

Integrating (43), we obtain the solution

$$mc^2 - m\mathrm{w} = C\,, \tag{44}$$

where C in the integration constant. Since $\mathrm{w} = c^2(1-\sqrt{g_{00}})$, $g_{00} = 1 - r_g/r$, and $r_g = 2GM/c^2$, then $C=0$ under the condition $g_{00} = 0$, which satisfies at the spherical surface of the gravitational radius $r = r_g$ (where $\mathrm{w} = c^2$).

From the obtained solution (44) we see that a particle of mass m, which travels in the gravitational field of a mass M, has a maximum energy $m\mathrm{w} = mc^2$ under the condition $g_{00} = 0$, which satisfies on the surface of the gravitational radius $r = r_g = 2GM/c^2$ from this mass-point (on which the physically observable time stops $d\tau = 0$, and the space and the gravitational field have a breaking $g_{11} = -h_{11} \rightarrow \infty$).

In particular, the above solution is applicable to the Earth, planets, the Sun, stars, galaxies and generally any bodies in the Universe.

In conclusion, we note that Riemannian spaces are non-degenerate by definition: the determinant $g = \det \| g_{\alpha\beta} \|$ of the fundamental metric tensor satisfies the condition $g < 0$. In addition, Zelmanov had obtained a relation connecting the determinants of the four-dimensional metric tensor $g_{\alpha\beta}$ and the three-dimensional chr.inv.-metric tensor h_{ik}

$$h = -\frac{g}{g_{00}}\,, \tag{45}$$

where $h = \det \| h_{ik} \|$, $g = \det \| g_{\alpha\beta} \|$, and g_{00} is the time component of the four-dimensional Riemannian metric.

These quantities in the space of the Schwarzschild mass-point metric (22) are

$$h = \frac{r^4 \sin^2\theta}{1 - \frac{r_g}{r}}\,, \quad g_{00} = 1 - \frac{r_g}{r}\,, \quad g = -\,r^4 \sin^2\theta\,. \tag{46}$$

From this we see that the numerical values of h and g depend on the location of the observer with respect to the polar coordinate θ (which is opposite to the geographic latitude, because it is measured from the North pole to the equator). At the North and South poles, where $\theta = 0°$ and $180°$, respectively, the space-time of the Schwarzschild mass-point metric is *completely degenerate*, since in this case $g = 0$. The observable three-dimensional space is also degenerate ($h = 0$) at the North and South poles. In addition, the radial component h_{11} becomes infinite over the entire surface of the gravitational radius $r = r_g$ that means a breaking in the space (and the gravitational field) on this surface.

It should be noted that the complete degeneration of the four-dimensional space-time and the three-dimensional observable space takes place in the Schwarzschild mass-point field not only on the spherical surface of the gravitational radius $r = r_g$ (around the centre of gravity of the mass-point), but also everywhere along the radial coordinate r directed to North and South. But even with a tiny deviation from the polar direction $\theta = 0°$ or $\theta = 180°$ (i.e., from the polar axis of the coordinate frame) the space is already non-degenerate.

The above conclusion means that the surface of the gravitational radius $r = r_g$ is not only the origin of the gravitational field of any body, which spreads outside and inside the surface, but is also the special space-time "membrane" separating the external space (gravitational field) of the body, where $r > r_g$, from its internal space (gravitational field), where $r < r_g$. Since both the space-time metric and the spatial metric are degenerate inside the "membrane", the space (space-time) inside the "membrane" is different from the ordinary pseudo-Riemannian space (space-time) and is a completely degenerate space-time.

6 Conclusion

So, using the chronometrically invariant notation of General Relativity (chronometric invariants are the physically observable projections of four-dimensional quantities onto the time line and the three-dimensional space of an observer), we have deduced Galileo's principle and Newton's law of gravitation as a particular case of the chr.inv.-formula for the gravitational inertial force acting in the four-dimensional pseudo-Riemannian space (space-time of General Relativity).

In fact, by doing this, we have created a "mathematical bridge", connecting Newton's theory of gravitation with General Relativity. This "mathematical bridge" is important for theoretical physics, since no one earlier than us had derived the empirical laws of Newton's theory of gravitation as a particular case of the purely geometric laws of General Relativity.

We have also showed that on the spherical surface that surrounds any mass-point at a very small radius, equal to the gravitational radius calculated for the mass, a space breaking takes place in the gravitational field of the mass-point (and in its three-dimensional observable space), and the observer's physical observable time stops. That is, the gravitational field of any mass-point extends both inward from the mentioned spherical surface to the coordinate origin (which coincides with the mass-point), and outward from the mentioned surface into the surrounding space to infinity, but is absent on the surface itself. This theoretical result leads us to the conclusion that the origin of the gravitational field in the space of the Schwarzschild mass-point metric is a spherical surface that surrounds any mass-point at the gravitational radius calculated for the mass.

The above results were obtained only thanks to the chronometrically invariant notation of General Relativity, which provides an unambiguous mathematical definition of physically observable quantities in the four-dimensional pseudo-Riemannian space (space-time). It would be impossible to get these results using the conventional general covariant notation of General Relativity, because physically observable quantities in the general covariant notation are not mathematically defined.

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