Yang-Mills Theory in the Framework of General Relativity

Patrick Marquet

Calais, France. E-mail: patrick.marquet6@wanadoo.fr

In our recent publication, we derived a solution that allows the coupling between the Yang-Mills theory and the space-time curvature; Progr. Phys., 2021, v.18, 97–102 [1]. This result was achieved by considering a specific manifold which we named the Weyl-Einstein manifold spanned by the connection coefficients displaying a 4-vector. We then deduced a Weyl-Einstein tensor, which was found to be conserved. The Weyl-Einstein 4-vector was directly identified with the Yang-Mills gauge field vectors as described in the Minkowski space tangent to the Weyl-Einstein manifold. In the present work, we investigate further this topic, and we examine how this coupling fits into the field equations.

Notations

Throughout this text, we assume the Einstein summation, whereby a repeated index implies summation over all values of this index.
4-tensor or 4-vector: small Latin indices $a, b, \ldots = 0, 2, 3, 4$.
3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$.
Signature of the space-time metric: $(+---)$.
Ordinary derivative: $\partial_a U$.
Riemannian covariant derivative on $(\mathrm{M}, \boldsymbol{g})$: ∇_a or (;).

1 The Weyl-Einstein field equations

1.1 The Weyl-Einstein tensor

Following Lichnerowicz [2] we defined the semi-metric manifold $(\mathrm{M_w}, \boldsymbol{g})$ spanned by the Weyl-Einstein connexion coefficients expressed here with the metric connexion

$$W^c_{ab} = \frac{1}{2}\, g^{cd}\left(\partial_b\, g_{da} + \partial_a\, g_{db} - \partial_d\, g_{ab}\right) - \\ - \frac{1}{2}\, g^{cd}\left(J_b\, g_{da} + J_a\, g_{db} - J_d\, g_{ab}\right), \tag{1.1}$$

$$W^c_{ac} = \frac{1}{2}\, g^{cd}\left(\partial_a\, g_{cd} - J_a\, g_{cd}\right), \tag{1.2}$$

where J_a is referred to as the Weyl-Einstein 4-vector.

The Einstein-Weyl-curvature tensor is assumed to keep its original form

$$(R^c{}_{adb})_{\mathrm{w}} = \partial_b\, W^c_{ad} - \partial_d\, W^c_{ab} + W^c_{eb} W^e_{ad} - W^c_{ed} W^e_{ab}\,. \tag{1.3}$$

Setting

$$(\Gamma^c_{ab})_J = \frac{1}{2}\, g^{cd}\left(J_b\, g_{da} + J_a\, g_{db} - J_d\, g_{ab}\right) \tag{1.4}$$

and using the Riemannian covariant derivatives, we found

$$(R_{ab})_{\mathrm{w}} = R_{ab} + \nabla_c(\Gamma^c_{ab})_J - \nabla_b(\Gamma^c_{ac})_J + \\ + (\Gamma^d_{ab})_J(\Gamma^c_{dc})_J - (\Gamma^d_{ae})_J(\Gamma^e_{db})_J\,, \tag{1.5}$$

$$R_{\mathrm{w}} = R - \left(\nabla_a\, J^a + \frac{1}{2}\, J^2\right). \tag{1.6}$$

With these, we derived the Weyl-Einstein tensor as

$$(G_{ab})_{\mathrm{w}} = (R_{ab})_{\mathrm{w}} - \frac{1}{2}\, \left(g_{ab} R_{\mathrm{w}} - 2\, J_{ab}\right), \tag{1.7}$$

where

$$J_{ab} = (\Gamma^d_{ab})_J(\Gamma^c_{dc})_J - (\Gamma^d_{ae})_J(\Gamma^e_{db})_J\,.$$

The Weyl-Einstein tensor was shown to be conserved.

1.2 Massive source

The Weyl-Einstein field equations are now expressed by

$$(G_{ab})_{\mathrm{w}} = \varkappa\, T_{ab}\,. \tag{1.8}$$

Using the Riemannian covariant derivatives, the Weyl-Einstein tensor conservation law reads

$$\nabla_a (G^a_b)_{\mathrm{w}} = 0\,. \tag{1.9}$$

The right hand side of (1.8) should also verify

$$\nabla_a\, T^a_b = 0$$

or

$$\partial_a\, \boldsymbol{T}^a_b = 0 \tag{1.10}$$

with the tensor density $\boldsymbol{T}^a_b = \sqrt{-g}\, T^a_b$.

However, inspection shows that

$$\partial_a\, \boldsymbol{T}^a_b = \frac{1}{2}\, \boldsymbol{T}^{ca}\, \partial_b\, g_{ca} \tag{1.11}$$

or equivalently

$$\partial_a\, \boldsymbol{T}^a_b = \frac{1}{2}\, \boldsymbol{T}^{ca}\left(\partial_b\, g_{ca} - J_b\, g_{ca}\right).$$

Thus the condition (1.10) is never satisfied in a general coordinates system. This circumstance results from the fact that the *global* conservation should hold for the 4-momentum of *both* the matter and its gravitational field.

To keep the equation (1.10) consistent with (1.9), we must look for a solution of the form

$$\partial_a \left(\boldsymbol{T}^a_b + \boldsymbol{t}^a_b \right) = 0\,, \tag{1.12}$$

where $\boldsymbol{t}^{ab}$ is the given tensor's density.

Let us compute

$$d\boldsymbol{g}^{ab} = d\left(\sqrt{-g}\, g^{ab}\right) = \sqrt{-g}\left(dg^{ab} + \frac{1}{2}\, g^{ab} g^{ed}\right) dg_{ed} = $$

$$= \sqrt{-g}\left(-g^{ae} g^{bd} + \frac{1}{2}\, g^{ab} g^{ed}\right) dg_{ed}\,,$$

therefore

$$(R_{ab})_{\mathrm{w}}\, d\boldsymbol{g}^{ab} = \sqrt{-g}\left(-R^{ce}_{\mathrm{w}} + \frac{1}{2}\, g^{ce} R_{\mathrm{w}}\right) dg_{ce} = -\varkappa\, \boldsymbol{T}^{ce} dg_{ce}\,.$$

Taking into account the Lagrangian form of the Weyl-Einstein Ricci tensor

$$(R_{ab})_{\mathrm{w}} = \partial^e \left[\frac{\boldsymbol{L}_{\mathrm{w}}}{\partial\left(\partial_e\, \boldsymbol{g}^{ab}\right)} \right] - \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial \boldsymbol{g}^{ab}}\,,$$

where the effective Weyl-Einstein Lagrangian is now

$$\boldsymbol{L}_{\mathrm{w}} = g^{ab} \sqrt{-g}\left(W^e_{ab} W^d_{de} - W^d_{ae} W^e_{bd}\right) \tag{1.13}$$

one obtains

$$-\varkappa\, \boldsymbol{T}^{ab} dg_{ab} = \left(\partial_c \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial_c\, \boldsymbol{g}^{ab}} - \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial \boldsymbol{g}^{ab}}\right) d\boldsymbol{g}^{ab} = $$

$$= \partial_c \left(d\boldsymbol{g}^{ab} \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial_c\, \boldsymbol{g}^{ab}} \right) - d\boldsymbol{L}_{\mathrm{w}}\,,$$

$$-\varkappa\, \boldsymbol{T}^{ab}\, \partial_d\, g_{ab} = \partial_c \left(\partial_d\, \boldsymbol{g}^{ab} \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial\left(\partial_c\, \boldsymbol{g}^{ab}\right)} - \delta^c_d \boldsymbol{L}_{\mathrm{w}} \right) = 2\varkappa\, \partial_c\, \boldsymbol{t}^c_d\,.$$

From the last equation we find

$$\partial_c\, \boldsymbol{T}^c_a = \frac{1}{2}\, \boldsymbol{T}^{dc}\, \partial_a\, g_{dc} = -\partial_c\, \boldsymbol{t}^c_a\,.$$

In order to satisfy the conservation law (1.12), one clearly sees that the gravitational field energy-momentum tensor density should be described by the Weyl-Einstein extension of the *Einstein-Dirac pseudo-tensor* [3, p.61]

$$\boldsymbol{t}^c_d = \frac{1}{2}\, \varkappa \left[\partial_d\, \boldsymbol{g}^{ab} \frac{\partial \boldsymbol{L}_{\mathrm{w}}}{\partial(\partial_c\, \boldsymbol{g}^{ab})} - \delta^c_d \boldsymbol{L}_{\mathrm{w}} \right] \tag{1.14}$$

the quantities $\boldsymbol{t}^{ab}$ are called "pseudo-tensor density" since they can be transformed away by a suitable choice of the reference frame and they are not irreducible [4]. This is why the classical theory stipulates that a (free) gravitational energy cannot be *localizable*.

In the classical General Relativity, the non symmetric tensor $\boldsymbol{t}_{ab}/\sqrt{-g}$ is symmetrized through the Belinfante procedure [5] to suit the standard symmetric Einstein tensor. The relevant symmetric tensor is denoted t_{ab}.

Unfortunately, the Einstein field equations whatever their transcriptions, are yet unbalanced since they do not exhibit a full real tensor as a source. To remedy this problem, we showed that a slightly variable cosmological term Λ-term induces a stress-energy tensor of vacuum, which restores a true gravitational tensor on the right-hand side of equation (1.6) as it should be [6, 7].

This real tensor is given by

$$(t_{ab})_{\mathrm{vac}} = -\frac{1}{2\varkappa}\, \Lambda\, g_{ab}\,. \tag{1.15}$$

The Λ-term was found to be [8]

$$\Lambda = \nabla_a K^a = \theta^2, \tag{1.16}$$

where K^a is a 4-vector and

$$\theta = X^a_{;a} \tag{1.17}$$

is the space-time volume scalar expansion characterizing the vacuum stress-energy tensor $(t_{ab})_{\mathrm{vac}}$, and X^a is a congruence of non intersecting unit time lines $X^a X_a = 1$

$$X^a_{;a} = h^{ab}\theta_{ab}\,, \tag{1.18}$$

while θ_{ab} stands for the expansion tensor and $h_{ab} = g_{ab} - X_a X_b$ is the projection tensor.

Due to the form of (1.16), the Lagrangian (1.13) differs only from a divergence and varying its action generates the same field equations. The real tensor $(t_{ab})_{\mathrm{vac}}$ which corresponds to the vacuum stress-energy tensor can be added to t_{ab} without affecting the Weyl-Einstein Lagrangian.

With this definition the Weyl-Einstein field equations can be finally written as

$$\begin{aligned}(G_{ab})_{\mathrm{w}} &= (R_{ab})_{\mathrm{w}} - \frac{1}{2}\left(g_{ab} R_{\mathrm{w}} - 2\, J_{ab}\right) = \\ &= \varkappa \left[\rho\, c^2 u_a u_b + \frac{\boldsymbol{t}_{ab}}{\sqrt{-g}} + (t_{ab})_{\mathrm{vac}} \right].\end{aligned} \tag{1.19}$$

Here the symmetrization procedure is evaded, because the quantity $\boldsymbol{t}_{ab}/\sqrt{-g}$ is genuinely antisymmetric.

When gravity is weak and velocities are low compared to c, we have the Newtonian approximation where the massive tensor in (1.19) reduces to

$$T^0_0 = \rho\, c^2.$$

Inspection then shows that

$$(R^0_0)_{\mathrm{w}} = R^0_0 = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial^2_\beta}$$

with $g_{00} = 1 + \varphi/c^2$, from which we find the well-known Poisson equation

$$\Delta\varphi = 4\pi G\rho\,,$$

where G is Newton's constant.

1.3 Electromagnetic contribution

The field equations are expressed by

$$(R_{ab})_{\mathrm{w}} - \frac{1}{2}\left(g_{ab} R_{\mathrm{w}} - 2 J_{ab}\right) = \\ = \varkappa \frac{1}{4\pi}\left(-\partial_a A^c F_{bc} + \frac{1}{4}\, g_{ab} F_{cd} F^{cd}\right), \tag{1.20}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a\,.$$

The source tensor is antisymmetric. Its form is derived from the canonical equation

$$(t^b_a)_{\mathrm{elec}} = \frac{\partial_a A_c\, \partial L}{\partial(\partial_b A_c)} - \delta^b_a L\,,$$

where $L = -\frac{1}{16\pi} F_{bc} F^{bc}$.

If the Weyl part is neglected, the term $\frac{1}{4\pi}\partial_c A^c F_{bc}$ is classically added so that when charge is absent, holds the relation

$$\frac{1}{4\pi}\,\partial_c A_a F^c_b = \frac{1}{4\pi}\,\partial_c\left(A_a F^c_b\right).$$

This eventually yields the well-known symmetric energy-momentum tensor of the electromagnetic field

$$\tau_{ab} = \frac{1}{4\pi}\left(-F^c_a F_{bc} + \frac{1}{4}\, g_{ab} F^{cd} F_{cd}\right).$$

1.4 Charged matter

The Weyl-Einstein field equations are

$$(R_{ab})_{\mathrm{w}} - \frac{1}{2}\left(g_{ab} R_{\mathrm{w}} - 2 J_{ab}\right) = \\ = \varkappa\left[\rho c^2 u_a u_b + \frac{\boldsymbol{t}_{ab}}{\sqrt{-g}} + (t_{ab})_{\mathrm{vac}} + \right. \\ \left. + \frac{1}{4\pi}\left(-\partial_a A^c F_{bc} + \frac{1}{4}\, g_{ab} F_{cd} F^{cd}\right)\right]. \tag{1.21}$$

We easily check that the right hand side of the equations is conserved.

2 Relation to the Yang-Mills gauge fields

We first write the Minkowskian line element ds and the Weyl-Einstein line element ds_{w}, then we set

$$dJ = dA\left(1 + \mathrm{Log}\,\frac{ds_{\mathrm{w}}}{ds}\right) \tag{1.22}$$

with the following one-forms

$$dJ = J_a\, dx^a,$$
$$dA = A_a\, dx^a.$$

The above 4-vector A_a is a generic gauge vector of the Yang-Mills field defined in the flat space tangent to the Weyl-Einstein manifold.

2.1 Weak interaction SU(2) symmetry

Let us now examine the rôle of the Weyl-Einstein tensor in the field equations. We write the group element of SU(2) as

$$\mathrm{U} = \exp\left[-i\,\mathrm{T}^\beta \mathrm{k}_\beta\right],$$

where k is the group parameter with the generators

$$\mathrm{T}^\beta = \frac{1}{2}\,\sigma^\beta,$$

(here σ^β are the 2×2 Pauli spin matrices) with the coupling constant h, the gauge field transforms as

$$B_a \to B_a - \mathrm{T}^\beta\,\partial_a \mathrm{k}^a(x) + i\,\mathrm{h}\,\mathrm{k}^a(x)\left[\mathrm{T}^\beta, B_a(x)\right].$$

Here, the Weyl-Einstein field equations (1.19) apply with the correspondence

$$J_a \to B_a\,.$$

2.2 The electromagnetic symmetry U(1)

This symmetry group is the abelian group U(1) with a single commuting generator $\mathrm{T}_1 = \mathrm{Q}$ satisfying

$$[\mathrm{T}_1, \mathrm{T}_1] = 0\,,$$

where Q is the quantity of the charges of the field $\Phi(x)$ proportional to the fundamental charge unit e. Under the phase rotation

$$\Phi(x) \to \Phi(x)\exp\left[-i\,\mathrm{k}\,\mathrm{Q}(x)\right]$$

the vector field $A_a(x)$ transforms as

$$A_a(x) \to A_a(x) + \partial_a \mathrm{k}\,.$$

Within the Weyl-Einstein field equations (1.20), we have the correspondence

$$J_a \to A_a\,.$$

2.3 Combined symmetry U(1) × SU(2)

Here the Weyl-Einstein field equations for charged matter (1.21) are used, where we simply have

$$J_a \to A_a + B_a\,,$$

where A_a is the electromagnetic vector field gauge field and B_a is the gauge vector field of the weak interaction.

Other combinations implying for example strong interaction SU(3) could be derived in the same way.

3 Conclusion

What have we achieved? Our theory relies on the specific form of the connexion coefficients which displays a 4-vector. This connexion form was first considered by H. Weyl by relating this vector to the "segment curvature" next to the Riemann curvature and zero torsion, with the aim to unify electricity and gravitation in a non trivial way [9]. Although we kept the name Weyl-Einstein connexion, the extra segment curvature is not introduced here. On the contrary, we have exploited the Weyl-Einstein 4-vector to connect the Yang-Mills gauge fields through an extended field equations set where both the left and right sides are still conserved. In doing so, such field equations can now display the type of interactions that is considered thus informing us between either electromagnetic field or weak and strong interactions of matter which was basically impossible with the standard field equations.

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References

1. Marquet P. How to couple the space-time curvature with the Yang-Mills Theory. *Progress in Physics*, 2022, v.18, no.2, 97–102.
2. Lichnerowicz A. Les espaces variationnels généralisés. *Annales scientifiques de l'Ecole Normale Supérieure*, *série 3*, 1945, t.62, 339–384.
3. Dirac P.A.M. General Theory of Relativity. Princeton University Press, 2nd edition, 1975.
4. Landau L. et Lifchitz E. Théorie des champs. Éditions MIR, Moscou, 1964.
5. Rosenfeld L. Sur le tenseur d'impulsion-énergie. *Acad. Roy. de Belgique*, *Mémoires de Classes de Sciences*, t.18, 1940.
6. Marquet P. The gravitational field: a new approach. *Progress in Physics*, 2013, v.9, no.3, 62–66.
7. Marquet P. Vacuum background field in General Relativity. *Progress in Physics*, 2016, v.12, no.4, 314–316.
8. Marquet P. Some insights on the nature of the vacuum background field in General Relativity. *Progress in Physics*, 2016, v.12, no.4, 366–367.
9. Weyl H. Gravitation und Elektrizität. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 1918, 465–480.