Exact and Invariant Representation of Quantum Spectral Deviations in Giant Atoms Via the Iso-Bohr Atoms of Hadronic Mechanics

Ruggero Maria Santilli

The Institute for Basic Research, 35246 U.S. 19N, Suite 215, Palm Harbor, FL 34684, U.S.A.
E-mail: research@i-b-r.org

In this paper, we recall the historical achievements of Bohr's Hydrogen atom; we identify a number of experimental deviations from quantum mechanical spectral predictions in giant atoms; and we show that a most probable origin of the deviations is given by the collective entanglement of orbiting electrons with ensuing interactions expected to be nonlocal, nonlinear, and not derivable from a potential. We then review the representation of collective non-Hamiltonian entanglement interactions via the isounit of the Lie-isotopic branch of hadronic mechanics and present, apparently for the first time: 1) The reduction of unsolvable many-body quantum mechanical problems into solvable two-body Lie-isotopic problems. 2) The representation of giant atoms via the Einstein-Podolsky-Rosen Lie-isotopic completion of Bohr's Hydrogen atom under the suggested name of *iso-Bohr atoms.* 3) The numerically exact and time invariant representation via the iso-Bohr atoms of anomalous quantum mechanical spectral deviations for giant atoms.

TABLE OF CONTENT

1 Experimental evidence of deviations from quantum spectral lines in giant atoms

1.1 Quantum-mechanical model of giant atoms

As it is well known, the 1913 historical structure model of the Hydrogen atom by Niels Bohr [1,2] provided the first numerically exact and time invariant representation of the spectral emissions or absorptions of the Hydrogen atom whose accuracy was such to trigger the wide acceptance of the local-differential mathematical and physical laws of quantum mechanics (for additional historical notes see, e.g. [3]).

Let us also recall that, in the extension to atoms with large atomic number $Z \gg 1$, known as giant atoms, Bohr's fundamental quantization postulate for the orbital angular momentum is retained,

$$mvr = n\hbar\,, \qquad n = 1, 2, 3, \dots\,, \tag{1}$$

where n is the principal quantum number.

The balance between the centripetal force and the Coulomb attraction yields

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\,\frac{Z_{eff}\,e^2}{r^2}\,, \tag{2}$$

where m and v denote the electron mass and orbital velocity, respectively, and Z_{eff} includes the Coulomb interaction of the selected electron with the remaining electrons.

From (1) and (2), one obtains the radius of the allowed electronic orbits,

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{me^2}\,\frac{n^2}{Z_{eff}} = a_0\,\frac{n^2}{Z_{eff}}\,, \tag{3}$$

where a_0 is the Bohr radius

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = 5.291\,772\,109 \times 10^{-9}\ \text{cm}\,. \tag{4}$$

The total mechanical energy of the electron in the n-orbit is evidently given by the sum of kinetic and potential energies,

$$E_n = K_n + V_n = \frac{1}{2}\,mv_n^2 - \frac{1}{4\pi\varepsilon_0}\,\frac{Z_{eff}\,e^2}{r_n}\,. \tag{5}$$

By using the preceding equations, this expression reduces to the well-known Hydrogenic energy levels,

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{Z_{eff}^2}{n^2} = -Ry\,\frac{Z_{eff}^2}{n^2}\,, \tag{6}$$

where $Ry \approx 13.6\,\text{eV}$ is the Rydberg energy.

Consequently, the quantum-mechanical energy difference associated with a transition between two stationary states n_i and n_f ($n_i > n_f$) is

$$\Delta E = E_{n_f} - E_{n_i} = -Ry\,Z_{eff}^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), \tag{7}$$

which are scaled up to large Z without structural changes.

1.2 Experimental evidence of deviations from quantum-mechanical predictions of giant atoms

Systematic and reproducible deviations from quantum mechanical predictions have been obtained in [4–8] via X-ray emission lines, core-electron binding energies, and spin orbit splittings, which deviations increase monotonically with the atomic number Z and can be summarized as follows.

1.2-A. Deviations in X-ray spectral lines
Inner-shell X-ray emission lines in heavy atoms exhibit systematic shifts relative to predictions based on relativistic Dirac-Fock theory with local exchanges. These discrepancies exceed experimental uncertainties and increase with Z, indicating a structural rather than instrumental origin as illustrated by the example:
Atom/system: $^{238}U_{92}$.
Observable: L_α X-ray emission line.
Quantum-mechanical prediction: Relativistic atomic structure calculation with local interactions [8],

$$E_{L_\alpha}^{QM} \approx 13.61\,\text{keV}\,. \tag{8}$$

Experimental value: Precision X-ray spectroscopy [8],

$$E_{L_\alpha}^{exp} \approx 13.63\,\text{keV}\,. \tag{9}$$

Deviation:

$$\Delta E_{L_\alpha} \approx +20\,\text{eV}\,. \tag{10}$$

Such deviations persist despite refined relativistic treatments and signal the insufficiency of strictly local electron-electron interactions in giant atoms as predicted by A. Einstein, B. Podolsky and N. Rosen [10] (see recent experimental confirmations [11–13]).

1.2-B. Deviations in core-electron binding energies
Systematic discrepancies are also observed in experimentally determined core-electron binding energies of heavy atoms. Local and semi-local relativistic quantum mechanical models consistently underestimate the measured energies, with deviations that grow with atomic number Z [4], as shown in example:
Atom/system: $^{198}Po_{84}$.
Observable: K-shell core-electron binding energy.
Quantum–mechanical prediction: Relativistic Dirac-Fock calculation with local exchange [4],

$$E_K^{QM} \approx 93.1\,\text{keV}\,. \tag{11}$$

Experimental value: High-resolution X-ray spectroscopy [4],

$$E_K^{exp} \approx 93.25\,\text{keV}\,. \tag{12}$$

Deviation:

$$\Delta E_K \approx +150\,\text{eV}\,. \tag{13}$$

A similar effect is observed in uranium:
Atom/system: $^{238}U_{92}$.
Observable: K-shell core-electron binding energy.
Quantum-mechanical prediction: Relativistic Dirac-Fock calculation [4],

$$E_K^{QM} \approx 115.6\,\text{keV}\,. \tag{14}$$

Experimental value: X-ray absorption spectroscopy [4],

$$E_K^{exp} \approx 115.8\,\text{keV}\,. \tag{15}$$

Deviation:

$$\Delta E_K \approx +200\,\text{eV}. \tag{16}$$

Accurate representations of the above deviations appear to require explicit nonlocal exchange and correlation functionals, contradicting the foundational locality assumption of quantum mechanics [4, 5].

1.2-C. Deviations in spin-orbit splittings
Additionally, as interested readers can verify, observed spin-orbit splittings in giant atoms and correlated electron systems consistently exceed predictions based on quantum mechanical Hamiltonian.

In summary, extensive experimental and theoretical analyses have established clear quantum mechanical deviations in spectral predictions for giant atoms that cannot be eliminated via relativistic corrections, higher-order perturbative expansions, or improved local exchange-correlation potentials, by therefore supporting the EPR completions of quantum methods.

1.3 Expected nonlocal origin of spectral deviations

The inability by local-differential models of representing the anomalous spectral lines of giant atoms has triggered the study of their nonlocal origin in diversified fields, including: condensed matter and bound-electron systems [5, 6]; giant nonlocal responses near Dirac points [7]; nonlocal spin transport [9]; spatially extended spectral effects for bound electrons in external fields [8]; and other studies.

In this paper, we study the representation of spectral deviations via the nonlocal completion of quantum mechanics suggested by A. Einstein, B. Podolsky and N. Rosen (EPR) in their historical 1935 paper [10] (see recent experimental confirmations [11–13]) according to axiom-preserving, thus isotopic methods initiated in 1978 by R. M. Santilli [18,19] at Harvard University under DOE support with the inclusion of:

1.3.1. Lagrangian/Hamiltonian interactions, here referred to long range interactions that are *local* (solely defined at isolated points), *linear* (solely valid for the first power of the wave function) and *variationally selfadjoint* (SA) [18] (necessary and sufficient conditions for the existence of a Lagrangian or a Hamiltonian in the reference frame of the experimenter).

1.3.2. Non-Hamiltonian/non-Lagrangian interactions, here referred to interactions that are *nonlocal* (defined over volumes not reducible to isolated points), *nonlinear* (depending on arbitrary powers of the wave function) and *variationally nonselfadjoint* (NSA) [18], thus not being derivable from a potential.

Nonpotential interactions were introduced by I. Newton [15] with his velocity-dependent resistive forces experienced by *extended* bodies moving within a resistive medium. Nonpotential interactions were then represented by J. L. Lagrange [16] and V. R. Hamilton [17] with the external terms in their celebrated analytic equations. A systematic study of nonpotential interactions was conducted by R. M. Santilli [18] via the conditions of variational selfadjointness.

Note that NSA interactions are solely possible for extended particles/wave packets in conditions of deep mutual penetration, as occurring in nuclear structures since nuclear volumes are generally *smaller* than the sum of the volumes of their protons and neutrons [14]. *Vice versa,* in view of their extension in space, wave packets generally imply nonlocal, nonlinear and nonpotential interactions caused by mutual overlapping.

Similarly, by recalling the historical geometric difficulties in admitting that a dimensionless point-electron can rotate, the admission of the extended character of the electron wave packets attracts the attention on the possibility for a physically well defined intrinsic rotation of the electron studied in the 1998 work [103].

The above insufficiencies motivated R. M. Santilli to construct the axiom-preserving EPR completion of quantum mechanics into *hadronic mechanics* initiated in the 1983 volume [19] for the time-invariant representation of extended particles or wave packets under Hamiltonian/SA and non-Hamiltonian/NSA interactions. Thanks to contributions by a number of mathematicians, theoreticians and experimentalists, hadronic mechanics reached maturity for scientific and industrial applications reported in the 1996 volumes [20–22] (see also works [23–35]) with recent upgrade [44] published by Cambridge Scholars Publishing.

Hereon, we assume the use of hadronic mechanics for our study of giant atoms due to the preservation by the iso-Bohr models of the axioms of the Bohr model and the mere use of broader realizations, to the extent that, at the abstract, realization-free level, the iso-Bohr models coincide with the historical Bohr's model.

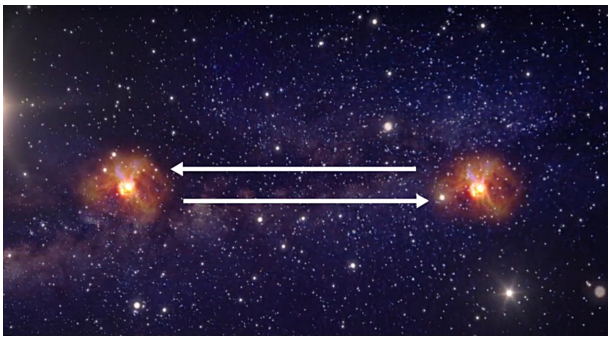

Fig. 1: We illustrate the experimental evidence called by Albert Einstein a "spooky action at a distance" according to which two particles/wave packets that were initially bound together and then separated, can instantaneously influence each other at arbitrary distances without any visible energy source.

1.4 Expected implications for Pauli Exclusion Principle

As studied in details in the recent monograph [44], the exact validity of the Pauli Exclusion Principle (PEP) is based on the exact validity of its axiomatic foundations, including: (i) Exact validity of quantum mechanics beginning with its strict locality; (ii) Quantized local orbits; (iii) Geometrically consistent particles with quantized spin $J = 1/2$, and other requirements.

With the understanding that the exact validity of the quantum mechanical form of PEP for *light* atoms is beyond scientific doubts, departures from the PEP exact validity for *giant* atoms are implicit in the 1935 EPR argument [10] because a nonlocal character of electron orbits for extended and entangled particles/wave packets imply the loss of the axiomatic foundation of the principle.

In line with the EPR argument, Santilli [23] pointed out in 1978 doubts on the geometric consistency of the 20th century assumption that dimensionless point-electrons can rotate, let alone have quantized rotations, in favor of extended, thus nonlocal representations of particles/wave packets. Additional doubts have been expressed in [45, 46] and papers quoted therein.

We should therefore note that the experimental deviations from quantum mechanical spectral predictions for giant atoms reported in §1.2 support doubts [10,23,45,46] on the exact validity of PEP for *giant atoms in vacuum* due to the *mathematical elaboration* of the extremely small limit of possible violations of PEP $1 + q < 10^{-40}$ suggested by various works [47–53]. In fact, the crucial value of the 1987 Greenberg-Mohapatra [45] "quon" $q \neq -1$ characterizing possible *phys-*

ical deviations from spin $J = 1/2$ is done via the *mathematical* algebra of ladder operators $a_i a_j^\dagger - q a_j a_i^\dagger = \delta_{ij}$ which is a particular case of *Santilli's Lie-admissible algebra* [54–56] $p a_i a_j^\dagger - q a_j a_i^\dagger = \delta_{ij}$ characterizing the time evolution of *irreversible* processes [44], rather than possible changes of a *permanently stable* quantity such as the spin $J = 1/2$ of the electron.

Experimental deviations from quantum mechanical predictions much bigger than those for giant atoms in vacuum exist for *light atoms within physical media*, such as for the Hydrogen atom within the hyperdense Solar chromosphere [57–65].

Note that systematic discrepancies between standard quantum mechanical spectral predictions and observed spectra arise in dense media due to: plasma microfields (Stark broadening and asymmetry), non-ideal plasma effects, screening, level dissolution; many-body and collective interactions, breakdown of two-body Coulomb assumptions; and other complex physical conditions. This experimental evidence establishes beyond reasonable doubts that the spectral behavior of any atom in dense media, whether light or giant, cannot be exactly represented by quantum mechanics.

It should be finally indicated that a significant covering of PEP with the inclusion of nonlocal effects is given by *Aringazin's isotopies of Pauli's principle* [66] (see also the review in Section 4.2.C, p. 140 of [21]) to be studied in a future work.

2 Collective electron entanglements in giant atoms

2.1 The mystery of particle entanglements

In this section, we study *particle entanglements* (Fig. 1) [67–74] and their mutual action (also called *steering* [70–72]) applied to the collection of orbiting electrons in giant atoms as a plausible nonlocal origin of the spectral deviations identified in the preceding section.

According to clear experimental evidence dating back to the early 2000's, particles that were initially bound together and then separated, can continuously and instantaneously influence each other at arbitrary distances. As an illustration, consider two valence electrons originally bonded together in a singlet state and then separated, one electron being kept in a physics laboratory and the second sent to the opposite side of Earth. In the event the first electron is subjected to a 180^o spin flip, the second electron continuously and instantaneously experiences exactly the same spin flip without any human intervention.

The most mysterious aspects of particle entanglements are created by their interaction, called by Albert Einstein a "spooky action at a distance", which must verify the conditions of being:

2.1.1) Continuous;

2.1.2) Instantaneous;

2.1.3) Invisible via current technologies;

2.1.4) Acting at arbitrary distances;

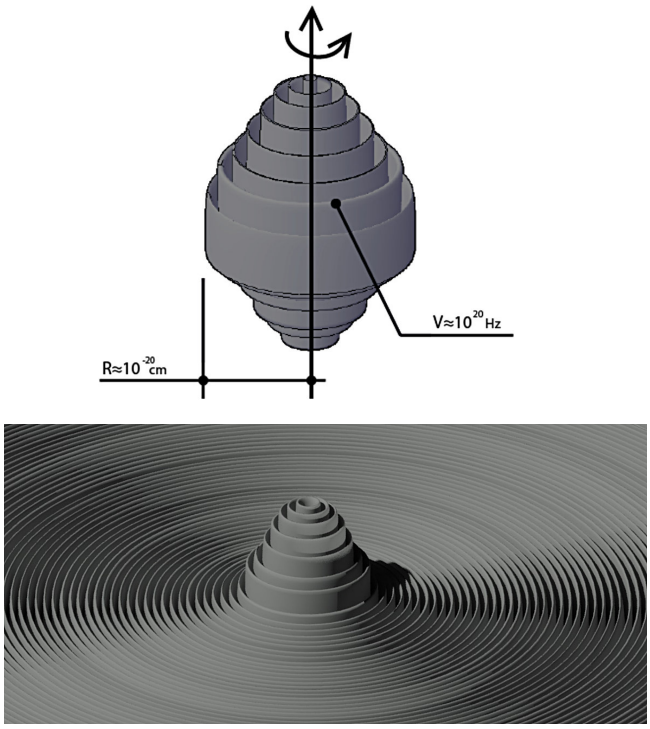

Fig. 2: We illustrate the central notion of this paper according to which, as a necessary condition to admit entanglements, atomic electrons have a very small but finite wave packet with the frequency $\nu = 10^{20}$ Hz (top view) which generates a new type of spherical "entanglement wave" (here represented in two dimensions for simplicity) with amplitude $A \approx 10^{-20}$ cm whose values are beyond current experimental detection (bottom view).

2.1.5) Occurring without any measurable energy source.

In preceding works [73, 74], we present a quantitative and invariant representation of the above features via hadronic mechanics under the assumption that entanglements are caused by the continuous and instantaneous overlapping of particle wave packets, resulting in interactions that are nonlocal, nonlinear and not derivable from a potential, by therefore verifying the indicated five conditions without any need for super-luminal speeds (as desired by Einstein [10]).

In this paper, we provide a visual illustration of the above representation of entanglements via the unavoidable addressing the problem of the structure of the electron. To begin, as a necessary condition to admit entanglements, the electron is assumed to have a finite wave packet, although with characteristics beyond current experimental capabilities, thus being compatible with ongoing assumptions in particle physics that the electron is dimensionless.

More particularly, we assume that the electron is characterized by the oscillation of one dimensionless point of the ether conceived as a universal substratum for the characterization and propagation of elementary particles and electromagnetic waves [75–78] (see also review in Chapter 3 of [79]).

The use of characterization by the ether as a medium propagating electromagnetic waves at 300,000 km/s then leads to the following main characteristics of the electron (top view of Fig. 2):

2.1.A) An extended rotating wave packet under the single valued irreducible representation of the *spinorial* covering of the Lie symmetry SO(2) with spin $J = 1/2$;

2.1.B) The frequency $\nu = E_r/h = 1.2356 \times 10^{20}$ Hz characterized by the rest energy of the electron $E_e = 0.511$ MeV;

2.1.C) A very small amplitude $A(\nu) \approx 10^{-20}$ cm.

It is evidently expected that the indicated oscillation of a point of the ether creates a spherical wave in all three dimensions of the ether (bottom view of Fig. 2). The overlapping of said waves in a pair of entangled particles then creates the entanglement interaction verifying conditions 2.1.1 to 2.1.5, as desired (Fig. 3).

It should be stressed that, even though currently invisible to both our eyes as well as to our detectors, the interactions underlying particle entanglements are real, as established by its physical actions [70–72].

2.2 Einstein's criticism of quantum entanglements

Particle entanglements are generally assumed to be represened by quantum mechanics, for which reason they are widely called *quantum entanglement.* However, Albert Einstein strongly criticized such an assumption because their experimentally established instantaneous character would imply superluminal communications that violate special relativity. This occurrence motivated the 1935 historical EPR argument according to which "*Quantum mechanics is not a complete theory*" [10].

To honor Albert Einstein, R. M. Santilli [73] recalled that quantum mechanics can only represent interactions derivable from a potential. Since there exist no known potential that can represent the entanglement steering at arbitrary distances, the Schrödinger equation can only represent entangled particles as being free due to the sole characterization of the Hamiltonian via their kinetic energy.

It should be noted that particle entanglements have been studied for a century via their statistic-probabilistic *correlations,* [80–82]. These studies are axiomatically correct for a representation of particle entanglements at large distances but, being strictly local, said correlations are unable *ab initio* to represent the indicated five basic features of entanglements with ensuing inability to generate spectral anomalies for giant atoms.

2.3 Collective entanglements of orbiting electrons

In this paper, we present, apparently for the first time, a nonlocal extension of Bohr's atom to giant atoms, herein referred to as *iso-Bohr atom,* to attempt a numerically exact and time invariant representation of spectral deviations from quantum mechanical predictions for giant atoms, where the prefix "iso" is intended in the Greek sense that the proposed model for giant atom preserves the abstract axioms of Bohr's atom.

The main conceptual assumption of the model is that all atoms on Earth originated within the high pressure of the plasma inside a primordial star, as a result of which all atomic electrons of stable atoms existing on Earth were entangled [73, 74], and under the indicated condition of stability, they have remained entangled to these days. This assumption implies that, in the transition from Bohr's atom for the atomic number $Z = 1$ to large Z values, the original, isolated, fully-Hamiltonian Bohr electron is turned into a *collection* of non-Hamiltonian entanglements with the remaining $Z - 1$ electrons.

By recalling that the exact, quantum mechanical representation of spectral emissions or absorptions for small atomic numbers Z is beyond scientific doubt, we here show for completeness that the entanglement of individual electron pairs appears to be insufficient for the generation of spectral anomalies in giant atoms, which anomalies therefore emerge as being due to the collective entanglement of all orbiting electrons.

Consider the radius $R(n)$ of a stable electron orbit for a given value of the principal quantum number n [1, 83, 84]

$$R(n) = \frac{a_0}{Z}\, n^2\,, \tag{17}$$

where a_0 is is Bohr's radius (4).

By maintaining correct spectral emissions for Pauli's electron pairs, the minimal entanglement contribution under consideration is that of the four electrons of adjacent orbits $R(n)$ and $R(n+1)$. Consequently, we consider the inter-orbital distance

$$\Delta R_n = \frac{a_0}{Z}\,[(n+1)^2 - n^2]\,, \tag{18}$$

which, for the smallest qualifying atom, the stable Beryllium isotope $^9\mathrm{Be}_4$, acquires the value

$$\Delta R^{\mathrm{Be}}_{n=1,2} \equiv R(2) - R(1) = a_0 - \frac{a_0}{4} \approx 3.96 \times 10^{-9}\,\mathrm{cm}\,, \tag{19}$$

with corresponding value for the stable Polonium isotope $^{198}\mathrm{Po}_{84}$

$$\Delta R^{\mathrm{Po}}_{n=5,6} = \frac{11}{84} \times 5.291 \times 10^{-9}\,\mathrm{cm} = 6.93 \times 10^{-10}\,\mathrm{cm}, \tag{20}$$

and for the stable Uranium isotope $^{238}\mathrm{U}_{92}$

$$\Delta R^{\mathrm{U}}_{n=6,7} = 7.48 \times 10^{-10}\,\mathrm{cm}. \tag{21}$$

By recalling that no spectral deviation has been measured with current technologies for the $^9\mathrm{Be}_4$, comparison of value (19) with values (20) and (21) indicates no appreciable increase of entanglements between inter-orbital electron pairs with the increase of the atomic number Z, thus confirming the need for studying the collective entanglement between all orbital electrons of giant atoms.

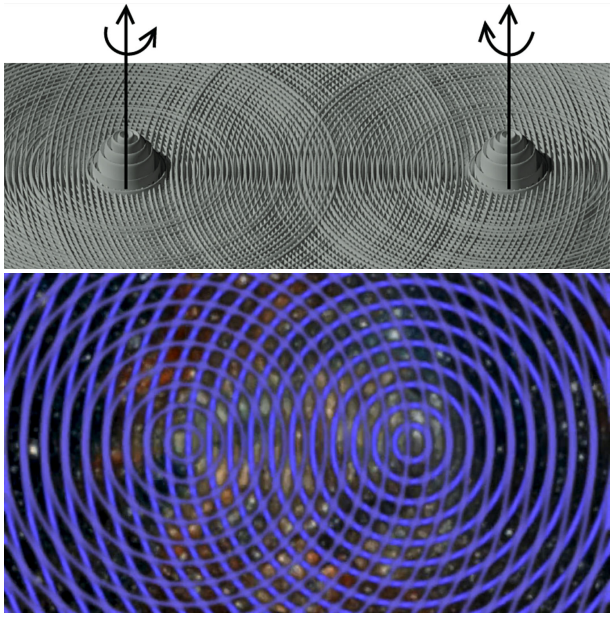

Fig. 3: We illustrate the continuous, therefore instantaneous entanglement of atomic electrons which is implied by their finite wave packets per Fig. 2, without any need for superluminal actions as desired by Albert Einstein.

3 Lie-isotopic representation of collective electron entanglements

3.1 Basic assumptions

In preparation for the extension of the Hamiltonian Bohr's atom to non-Hamiltonian iso-Bohr atom (§5), in this section we outline the representation via hadronic mechanics [18, 44] of nonlocal, nonlinear, nonpotential, and therefore non-Hamiltonian particle entanglements specialized, apparently for the first time, to the collective entanglement of peripheral electrons of giant atoms. Since we have restricted our analysis to stable giant atoms, we shall use the *Lie-isotopic branch of hadronic mechanics,* including its *isomathematics* [20] and isomechanics [21], where the prefix "iso" indicates the axiom-preserving, thus isotopic character of the representation.

The main conceptual assumption of our model is that all atoms on Earth originated within the high pressures in the interior of some primordial star, as a result of which all atomic electrons were entangled, and under the indicated condition of atomic stability, remained entangled to these days. This implies that, in the transition from Bohr's Hydrogen atom to giant atoms, the original single electron for $Z = 1$ is turned into an isoelectron entangled with all remaining $Z-1$ isoelectrons.

A main methodological insufficiency is that the resulting system is a complex, generally unsolvable many-body problem. This insufficiency is resolved by isomathematics with the embedding of all $Z-1$ non-Hamiltonian interactions in the multiplicative, positive-definite isounit $\hat{I}$ of the isotheory with consequential reduction of the original multi-body system to a two-body isoproblem axiomatically equivalent to Bohr's atom, which as we shall see, does admit exact analytic solutions while assuring the recovering of Bohr's atom uniquely and unambiguously for $\hat{I} = 1$.

3.2 Isomathematical representation

The universal enveloping associative algebra $\xi : \{A, B, \ldots; \times; -1\}$ for Hermitean operators $A, B, \ldots$ of Bohr's atom with associative product $AB = A \times B$, $A(BC) = (AB)C$ and millenary multiplicative unit $\hat{I}$, $1B = A1 = A \; \forall A \in \xi$ is lifted into the universal enveloping isoassociative algebra $\hat{\xi} : \{A, B, \ldots; A \hat{\times} B; \hat{I}\}$ of the iso-Bohr atom with isoassociative isoproduct (first introduced in Tables 5.1 to 5.4 of [19])

$$\begin{aligned} &A \hat{\times} B = A \, S \, B \,, \;\; S > 0 \,, \\ &A \hat{\times} (B \hat{\times} B) = (A \hat{\times} B) \hat{\times} C \,, \end{aligned} \tag{22}$$

and consequential lifting of the multiplicative unit 1 into the isounit

$$\begin{aligned} &\hat{I} = 1/S \,, \;\; \hat{I} > 0 \,, \\ &\hat{I} \times A = A \times \hat{I} \equiv A \; \forall \; A \in \xi \,, \end{aligned} \tag{23}$$

where the Hermitean, dimensionless, positive-definite, integrodifferential function or operator S (originally called the *isotopic element* and nowadays called the *Santillian* [85–87]) has an arbitrary dependence on local quantities, such as time t, coordinates r, velocities v, acceleration a, energy E, density d, pressure π, temperature τ, wave function ψ, or any other needed local variable $S = S(t, r, v, a, E, d, \pi, \tau, \psi, \ldots) = S^\dagger$, $S > 0$, and nonrelativistic realizations of the type [88]

$$\begin{aligned} S &= 1/\hat{I} = \Sigma_{\alpha=1,..,Z-1} \, \mathrm{Diag} \, (b^2_{x,\alpha}, b^2_{y,\alpha}, b^2_{z,\alpha}) = \\ &= \Sigma_{\alpha=1,...,Z-1} \, \mathrm{Diag} \, (1/n^2_{x,\alpha}, 1/n^2_{y,\alpha}, 1/n^2_{z,\alpha}) \times e^{-\Gamma} > 0 \,, \\ &b_{k,\alpha} > 0 \,, \;\; \Gamma = \Gamma(r, v, a, E, d, \pi, \tau, \psi, \ldots) > 0 \,, \end{aligned} \tag{24}$$

where: $n^2_{\alpha,k}$, $k = 1, 2, 3$ represent the dimension and shape of atomic orbital and the positive-definite function Γ represents all non-Hamiltonian interactions.

It should be stressed that, despite their simplicity, basic axioms (22) and (23) imply the corresponding compatible lifting of the totality of 20th century applied mathematics and related mechanics into isomathematics [20] and related isomechanics [21] with no exception known to this author. To render this paper minimally self-sufficient, in this subsection we outline a simple method for the construction of the isomathematics needed for the iso-Bohr atom, while the corresponding method for the construction of physical laws is outlined in the subsequent subsection.

In its simplest possible form, isomathematics can be constructed via the following *non-unitary transformation*

$$\begin{aligned} &\hbar \ \to \ U\hbar U^\dagger = \hat{I} > 0\,, \\ &U \times (A \times B) \times U^\dagger = A' \,\hat{\times}\, B' = A' S\, B'\,, \\ &S \ = (U \times U^\dagger)^{-1}\,, \\ &A' = U \times A \times U^\dagger\,, \ \ B' = U \times B \times U^\dagger\,, \end{aligned} \tag{25}$$

applied to the *totality* of conventional methods and their operations.

It should be stressed that, being defined on the conventional Hilbert space $\mathcal{H}$ over the numeric field of complex numbers C, transformations (25), characterize *nonunitary quantum theories*, that as such, are afflicted by a number of insufficiencies [89]. Their resolution requires the reformulation of the theories into the full isomathematical language [20], with particular reference to the reconstruction of unitarity on iso-Hilbert isospaces $\hat{\mathcal{H}}$ [90] over the isofield $\hat{C}$ [91], called *isounitarity* [92]

$$\begin{aligned} &U = \hat{U} S^{\,1/2}\,, \\ &U U^\dagger = \hat{U} \,\hat{\times}\, \hat{U}^\dagger = \hat{U}^\dagger \,\hat{\times}\, \hat{U} = \hat{I}\,, \end{aligned} \tag{26}$$

with corresponding *isolinearity* and *isolocality* (see Sec. 4.2, p. 128 on of [20]). In such a reformulation, we assume for simplicity that Hermiticity coincides with iso-Hermiticity,

$$U^\dagger = (U^t)^* \equiv (U^{\hat{t}})^{\hat{*}} = U^{\hat{\dagger}}\,. \tag{27}$$

Consider the field $R(n, \times, 1)$ of real numbers n with product nm and unit 1 on which the Bohr atom is constructed. Assumptions (22)–(25) imply that the iso-Bohr atom is constructed on Santilli isofield $\hat{\mathcal{R}}(\hat{n},\, \hat{\times}\,, \hat{I})$ of isoreal isonumbers $\hat{n} = n\hat{I}$ with isoproduct $\hat{n}\,\hat{\times}\,\hat{m} = nm\hat{I}\ \forall\ n, m \ \in \hat{\mathcal{R}}$ [91] (see also [93, 94]) which are obtained via the application of isotopic transformation (24) to all axioms of a numeric field and all their operations. Consequently, all scalars $s \in \mathcal{R}$ of Bohr's atom, such as coordinates r, velocities v, real valued functions $f(r)$, etc. are mapped into isoscalars $\hat{s} = s\hat{I} \in \hat{\mathcal{R}}$, thus including isocoordinates $\hat{r} = r\hat{I}$, isovelocities $\hat{v} = v\hat{I}$, isofunctions $\hat{f}(\hat{r}) = [f(\hat{I})]\hat{I}$, etc.

Note that, by conception and construction, Santilli numeric isofields verify all axioms of a numeric field. Nevertheless, when projected in our space, *isonumbers represent volumes* in view of realizations of the type

$$\hat{n} = n\hat{I} = \text{Diag}\,(n/n_1^2, n/n_2^2, n/n_3^2)\,, \tag{28}$$

which are necessary for the invariant representation of the extended particles [21].

Geometrically, the nonrelativistic Bohr atom is formulated on the Euclidean space $E(r, \delta, 1)$ over $\mathcal{R}$ with coordinates $r = (x, y, z) = (r^k)$, $k = 1, 2, 3$, the familiar metric $\delta = (\delta_{ij})$ and invariant $r^2 = r^i \delta_{ij} r^j = r_1^2 + r_2^2 + r_3^2 \ \in \mathcal{R}$. Consequently, our nonrelativistic iso-Bohr atom is formulated on the iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ over the isoreal isofield $\hat{\mathcal{R}}$ [20, 95] with isocoordinates $\hat{r} = r\hat{I} \ \in \ \hat{\mathcal{R}}$, isometric $\hat{\delta} = (S\,\delta)\hat{I} \ \in \ \hat{\mathcal{R}}$ and isoinvariant

$$\begin{aligned} \hat{r}^{\hat{2}} &= \hat{r}^i \,\hat{\times}\, \hat{\delta}_{ij} \,\hat{\times}\, \hat{r}^j = \left(r^i S_i^k \delta_{kj}\, r^j\right) \hat{I} = \\ &= \left(x^2 b_x^2 + y^2 b_y^2 + z^2 b_z^2\right)\, \hat{I} = \left(\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2}\right) \hat{I} \in \hat{\mathcal{R}}\,. \end{aligned} \tag{29}$$

Algebraically, the Bohr atom is elaborated via the various branches of Lie's theory with particular reference to the SU(2)-spin algebra with Hermitean operators J_k, $k = 1, 2, 3$ and commutation rules $[J_i, J_j] = J_i J_j - J_j J_i = \epsilon_{ij}^k J_k$, whose fundamental representation characterizes the spin $J = 1/2$ of the electron.

Thanks to the majestic axiomatic consistency of quantum mechanics, *the numeric predictions of the Bohr atom are invariant over time* in view of the invariance of its nonrelativistic (relativistic) dynamical equations under the Galilean G(3.1) (Poincaré SO(3.1)) symmetry.

By comparison, the iso-Bohr atom is elaborated via the Lie-Santilli isotheory [19] (see also [96, 97]) for the construction of the isotopic space-time isosymmetries [98–108] with generators J_k, where $k = 1, 2, \ldots, N$ (which are the same as those in quantum mechanics because they are observable) with isocommutation rules

$$[J_i, J_j]^* = J_i \,\hat{\times}\, J_j - J_j \times J_i = J_i S\, J_j - J_j S\, J_i = C_{ij}^k J_k\,, \tag{30}$$

where the Santillian S (24) represents *extended* particle/wave packets and their consequential *contact*, thus *zero-range* non-Hamiltonian interactions.

The first role of the Lie-Santilli isotheory for the study of giant atoms is that of introducing the notion of *isoparticles* at large and *isoelectrons* in particular (called *eletons* in the original 1978 proposal [23]) as isounitary isoirreducible isorepresentations of the *iso-Poincaré isosymmetry* $\mathcal{P}$ (3.1) [107, 108].

In turn, by remembering that it is geometrically impossible for a dimensionless point to have rotations, the second role of the Lie-Santilli isotheory is that of introducing the notion of *hadronic spin* [101, 103] with axiomatically consistent rotations due to the extended character of particles represneted by the Santillian (24), e.g. a rotating prolate ellipsoid with NSA interactions due to the environment represented by the exponential term. Said hadronic spin does admit the spin $\hat{J} = 1/2$ for the electron as a particular case when the particle is isolated in vacuum, although the numeric value of $\hat{J}$ generally depends on the environmental conditions. As an example, due to the dependence of the Santillian on local densities, the spin of an electron under the immense pressures in the core of a star is expected to be different than the spin of the same electron when orbiting in vacuum around a nucleus.

The third role of the Lie-Santilli isotheory is that of assuring the invariance over time of the numeric predictions of the theory at the nonrelativistic (relativistic) level because the dynamical equations isoequations are invariant under the iso-Galilean isosymmetry $\hat{\mathrm{G}}(3.1)$ [36–38] (iso-Poincaré $\hat{\mathrm{P}}(3.1)$ [39–41]) isosymmetry.

We should finally indicate that, under the assumption that the electron is actually dimensionless, no entanglement is conceivably possible, in which case the author is aware of no actual physical interaction capable of a *numeric* representation of the quantum mechanical spectral anomalies reported in §1.

We also recall that, from a differential viewpoint, the Bohr atom is elaborated via the historical Newton-Leibnitz differential calculus with basic differential of the local coordinates dr and derivative $\partial_r f(r)/\partial_r$ from which we derive the operator realization of the linear momentum (for $\hbar = 1$) $p_k|\psi(r)\rangle = -i\partial_r|\psi(r)\rangle$.

The iso-Bohr atom cannot be elaborated via the Newton-Leibnitz differential calculus because said calculus can be solely used over a numeric field whose multiplicative unit is the constant number 1, thus being inapplicable for multiplicative units with a dependence on the local coordinates $\hat{I} = \hat{I}\,v\,(t, r, v, a, E, d, \pi, \tau, \psi, \ldots)$.

Following an extensive search, Santilli [109] proposed the generalization of the Newton-Leibnitz differential calculus applicable to all possible multiplicative units admitted by the axioms of a numeric field, with basic isodifferential and isoderivatives

$$\begin{aligned} &\hat{d}\hat{r} = \hat{d}\,[r\hat{I}(r,\ldots)] = S\,d\,[r\hat{I}(r,\ldots)] = dr + rS\,d\hat{I}\,,\\ &\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\,\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}\,, \end{aligned} \tag{31}$$

which finally allowed the axiomatically consistent identification of the operator form of the isomomentum

$$\hat{p}_k\,\hat{\times}\,|\hat{\psi}(\hat{r})\rangle = -i\,\hat{I}\,\partial_{\hat{r}}|\hat{\psi}(\hat{r})\rangle\,, \tag{32}$$

needed for the elaboration of the isodynamical isoequations of hadronic mechanics.

It should be noted that two decades following proposal [109], the mathematician S. Georgiev discovered Santilli's isodifferential calculus by initiating its systematic mathematical study reported in the various monographs [110] and papers quoted therein.

3.3 Isomechanical representation

We finally recall that, on operator grounds, the Bohr model is formulated on a Hilbert space $\mathcal{H}$ with: states $|\psi\rangle$; normalization $\langle\psi||\psi\rangle = 1$; expectation values $< A >= \langle\rho|A|\psi\rangle$; Hermiticity condition $\langle\rho|(A|\psi\rangle) = (\langle\rho|A)|\psi\rangle\ \forall\rho,\psi\ \in \mathcal{H}i$; as well as the familiar Schrödinger equation for a *stable* bound state of two particles with masses $m_\alpha,\ \alpha = 1, 2$, relative coordinate $r = (r^k), k = 1, 2, 3$, Coulomb potential $V_c(r)$, and $\hbar = 1$

$$\begin{aligned} &H(r,p)|\,\psi(r)\rangle =\\ &= \left[\Sigma_{k,\alpha}\,\frac{1}{2m_{k,\alpha}}\,p_{k,\alpha}p_{k,\alpha} + V_c(r)\right]|\psi(r)\rangle = E\,|\psi(r)\rangle\,. \end{aligned} \tag{33}$$

The iso-Bohr atom is formulated on the infinite S-class of Hilbert-Myung-Santilli isospace [90] $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\hat{C}$ with: isostates $|\hat{\psi}\rangle$; isonormalization

$$\langle\hat{\psi}|\,\hat{\times}\,|\hat{\psi}\rangle = \langle\hat{\psi}|S\,|\hat{\psi}\rangle = S\ ; \tag{34}$$

isoexpectation values

$$\hat{<}A\hat{>} = \langle\hat{\psi}|\hat{\times}A\hat{\times}|\hat{\psi}\rangle = \langle\psi|S\,A\,S\,|\psi\rangle\,; \tag{35}$$

and iso-Hermiticity condition (36), i.e.

$$\begin{aligned} \langle\hat{\rho}|\hat{\times}(A\,\hat{\times}\,|\psi\rangle) &= [\langle\hat{\rho}|\hat{\times}(A\,\hat{\times}\,|\psi\rangle)^{\hat{\dagger}}]^{\hat{*}} =\\ &= (\langle\hat{\rho}|\hat{\times}A)\hat{\times}|\hat{\psi}\rangle\ \forall\hat{\rho},\hat{\psi}\ \in\hat{\mathcal{H}}\,, \end{aligned} \tag{36}$$

where the first equation is formulated on $\hat{\mathcal{H}}$ over $\hat{C}$ while the second equation is the projection of the former onto $\mathcal{H}$ over C.

The main dynamical equation of the *stable* iso-Bohr atom is given by the Schrödinger-Santilli isoequation [19] (see also [85, 86]),

$$\begin{aligned} &H(\hat{r},\hat{p})\,\hat{\times}\,|\hat{\psi}(\hat{r})\rangle =\\ &= \left[\Sigma_{k=1,2,3}\,\frac{\hat{1}}{2\hat{m}_k}\,\hat{\times}\,\hat{p}_k\,\hat{\times}\,\hat{p}_k + \hat{V}_c(\hat{r})\right]\hat{\times}\,|\hat{\psi}(\hat{r})\rangle =\\ &= \left[\Sigma_{k=1,2,3}\,\frac{1}{2m_k}\,(-i\hat{I}\hat{\partial}_{k,\hat{r}})(-i\hat{I}\hat{\partial}_{k,\hat{r}}) + V_c(\hat{r})\right]|\hat{\psi}(\hat{r})\rangle =\\ &= HS|\hat{\psi}(\hat{r})\rangle = \hat{E}\,\hat{\times}\,|\hat{\psi}(\hat{r})\rangle = E|\hat{\psi}(\hat{r})\rangle\,, \end{aligned} \tag{37}$$

where: $\hat{V}_c(\hat{r}) = [V_c(\hat{r})]\hat{I}$; we have used (47) for the isolinear isomomentum; and we have introduced the notion of isotime $\hat{t} = t\hat{I}_t$, related isounit $\hat{I}_t$ and Santillian $S_t = 1/\hat{I}_t$.

The above isomechanical equation illustrates the complementary of the Santillian with respect to the Hamiltonian, since the latter (former) represents all potential (nonpotential) interactions. Note in (24) that measured quantities are represented by conventional numbers in view of cancellations of the type $\hat{E}\,\hat{\times}\,|\hat{\psi}(\hat{r})\rangle = E|\hat{\psi}(\hat{r})\rangle$.

As one can see, the iso-Hilbert formalism is considerably broader than the conventional Hilbert formalism, as illustrated by the fact that the former holds for the infinite family of Santillians (24). For this reason, to avoid mathematical complexities not used for measurements, in this paper we have assumed that a quantity should be observable independently from the methods used for its treatment, by therefore assuming that the definition of Hermiticity coincides with the conventional notion of Hermiticity, in which case we have the following [21]:

THEOREM 4-1. *All well behaved operators that are Hermitean (observable) in quantum mechanics remain Hermitean (observable) in the Lie-isotopic branch of hadronic mechanics.*

PROOF: *Condition (36) of iso-Hermiticity (iso-observability) of an operator A is reducible to the identity* $SA = (AS)^\dagger = S^\dagger A^\dagger = SA$ *which is assured by the Hermiticity of the Hamiltonian H and of the Santillian S*. Q.E.D.

Following the achievement in 1995 of sufficient maturity [20–22], isomathematics allowed a number of advances in physics and other fields that are generally impossible for 20th century applied mathematics, among which we mention the following:

3.1. The invariant representation of the dimension, shape and density of extended particles/wave packets and their nonlocal, nonlinear and non-Hamiltonian interactions via the Santillian S in isoassociative product (22) [19, 20];

3.2. The completion of weakly convergent or divergent perturbative series into strongly convergent isoperturbative series with consequential faster computations (see Chapter 11 of [21]);

3.3. The improvement of cybersecurity via the isomathematical formulation of cryptograms, e.g. via sufficient numerical isoproducts (22) whose solution cannot be identified in a finite period of time under a computer change of the Santillian every nanosecond (see Appendix 2.C of [20] and paper [102]);

3.4. The achievement of explicit and concrete realizations of Bohm's variables as being hidden in the associativity axiom [21, 73];

3.5. The inapplicability of Bell's inequalities for extended particles under non-Hamiltonian interactions [103];

3.6. The progressive recovering of Einstein's determinism with the increase of the strength of the interactions with consequential full recovering of Einstein's determinism at the Schwartzschild horizon [111–114];

3.7. The exact and invariant representation of the synthesis of the neutron from a proton and an electron in the core of stars [115], its experimental verifications [116] and industrial applications [117];

3.8. The exact and invariant representation of the experimental data of the Deuteron and heavier nuclei [118–120];

3.9. The exact and invariant representation of the anomalous magnetic moment of the muons [121, 122];

3.10. The theoretical prediction [123] and experimental verification [124] of the local character of the speed and frequency of electromagnetic waves propagating within physical media;

plus applications in chemistry and biology here ignored for brevity [43].

4 Nonlocal Lie-isotopic Bohr atoms

4.1 Foreword

In this section we present apparently for the first time:

1) The reduction of unsolvable many-body quantum mechanical problems to solvable two-body Lie-isotopic problems.

2) The Lie-isotopic reduction of Giant atoms with $Z \gg 1$ to Bohr's Hydrogen atom with $Z = 1$ under the suggested name of *iso-Bohr atom.*

3) The numerically exact and time invariant representation via the iso-Bohr atom of anomalous quantum mechanical spectra of giant atoms.

The iso-Bohr atom has been conceived and it is constructed as being axiom-preserving, i.e. as being a two-body isoproblem composed by a central isonucleus and a simgle orbiting isoelectron as a isounitary isoirreducible isorepresentation of the Lie-Santilli iso-Poincaré symmetry $\hat{\mathcal{P}}_S$ (3.1) with Santillian (24). The entanglement interactions responsible for the contributions by all remaining orbiting electrons are represented by the isorenormalization of the characteristics of the electron, beginning with its charge, which are generated by non-Hamiltonian effects.

Besides the achievement of exact solutions for many-body problems, the Lie-isotopic reformulation converts divergent perturbative series into strongly convergent isotopic series (see Chap. 11 of [21]) and having other computational advantages.

4.2 Isodynamical representation

Let us assume that: 1) $\hat{m}_e$ represents the isomass of the isoelectron; 2) $\hat{m}_n$ represents the isomass of the nucleus; 3) $\hat{r}_k : \{\hat{x}_k, \hat{y}_k, z_k\}$ $k = e, n$ represents the iso-Euclidean coordinates; 4) $\hat{p}_k$, $k = 1, 2$ represents the isolinear isomomenta of the isoparticles with realization (32); 5) $\hat{t}$ represents the isotime hereon assumed to be, for simplicity, identical to the conventional time, $\hat{t} = t$; 6) $\hat{r} = \{\hat{x}_1 - \hat{x}_2,\ \hat{y}_1 - \hat{y}_2,\ \hat{z}_1 - \hat{z}_2\}$ represents the relative iso-Euclidean isocoordinates between the isoelectron and the nucleus with isomomentum $\hat{p}$; 7) $\hat{R} = \{\hat{X}, \hat{Y}, \hat{Z}\}$ represents the isocoordinates of the center of mass with isomomentum $\hat{P}$; 8) $\hat{M} = \hat{m}_e + \hat{m}_n$ represents the total isomass of the considered two-body isoproblem with isomass $\mu = \hat{m}_e \hat{\times} \hat{m}_n/(\hat{m}_e + \hat{m}_n)$; and 9) Assume the expressions $\hat{M} \hat{\times} \hat{R} = \{\hat{m}_e \hat{\times} \hat{x}_1 - \hat{m}_n \hat{\times} \hat{x}_2;\ \hat{m}_e \hat{\times} \hat{y}_1 - \hat{m}_n \hat{\times} \hat{y}_2;\ \hat{m}_e \hat{\times} \hat{z}_1 - \hat{m}_n \hat{\times} \hat{z}_2\}$.

Then the Schrödinger-Santilli isoequation for stable iso-Bohr atom can be written

$$\begin{aligned}
&\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{t}} |\hat{\Psi}(\hat{r})\rangle = i \frac{\partial}{\partial t} |\hat{\Psi}(\hat{r})\rangle = \\
&= [\Sigma_{k=e,n} (\hat{2} \hat{\times} \hat{m}_k)^{-1} \hat{\times} \hat{p}_k \hat{\times} p_k + \hat{V}_c(\hat{r}_k)] \hat{\times} |\hat{\Psi}(\hat{r})\rangle = \\
&= [\Sigma_{k=e,n} 2 {m_k}^{-1} \hat{p}_k S \hat{p}_k + V_c(\hat{r}_k)] S |\hat{\Psi}(\hat{r})\rangle = \\
&= \hat{E}_{tot} \hat{\times} |\hat{\psi}_k(\hat{r})\rangle = E_{tot} |\hat{\Psi}(\hat{r})\rangle \,,
\end{aligned} \tag{38}$$

and reduces via (32) to

$$\begin{aligned}
&\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\Psi}(\hat{r},\hat{R})\rangle = i\frac{\partial}{\partial t}|\hat{\Psi}(\hat{r},\hat{R})\rangle = \\
&= \{(\hat{2}\hat{\times}\hat{M})^{-1}[\hat{P}_x^{\hat{2}}+\hat{P}_y^{\hat{2}}+\hat{P}_z^{\hat{2}}]+ \\
&+(\hat{2}\hat{\times}\hat{\mu})^{-1}[\hat{p}_x^{\hat{2}}+\hat{p}_y^{\hat{2}}+\hat{p}_z^{\hat{2}}]+V_c(\hat{r})]\}\hat{\times}|\hat{\Psi}(\hat{r},\hat{R})\rangle = \\
&= \hat{E}_{tot}\hat{\times}|\hat{\psi}_k(\hat{r}_k)\rangle = E_{tot}|\hat{\Psi}(\hat{r}_k)\rangle\,.
\end{aligned} \tag{39}$$

Since hadronic mechanics verifies the isosuperposition principle thanks to its isolinearity [21] and the motion of the center of isomass is free, we can use the decomposition

$$|\hat{\Psi}(\hat{r},\hat{R})\rangle = |\hat{\Phi}(\hat{R})\rangle\hat{\times}|\hat{\psi}(\hat{r})\rangle\,, \tag{40}$$

which reduces (39) to the equation for the center of isomass

$$\begin{aligned}
&\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\Phi}(\hat{R})\rangle = \\
&= [(\hat{2}\hat{\times}\hat{M})^{-1}\hat{\times}(\hat{P}_x^{\hat{2}}+\hat{P}_y^{\hat{2}}+\hat{P}_z^{\hat{2}})]\hat{\times}|\hat{\Phi}(\hat{R})\rangle = \hat{E}_R\hat{\times}|\hat{\Phi}(\hat{R})\rangle\,,
\end{aligned} \tag{41}$$

and to the dynamical isoequation for the iso-Bohr atom

$$\begin{aligned}
&\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\psi}(\hat{r})\rangle = \\
&= [(\hat{2}\hat{\times}\hat{\mu})^{-1}\hat{\times}(\hat{p}_x^{\hat{2}}+\hat{p}_y^{\hat{2}}+\hat{p}_z^{\hat{2}})+V_c(\hat{r})]\hat{\times}|\hat{\psi}(\hat{r})\rangle = \\
&= \hat{E}\hat{\times}|\hat{\psi}(\hat{r})\rangle\,.
\end{aligned} \tag{42}$$

To reach an analytic solution of (42), we now introduce the isospherical isocoordinates (Sect. 5.5 of [21])

$$\begin{aligned}
&x = rb_1^{-1}\sin(\hat{\theta})\cos(\hat{\phi})\,,\quad y = rb_2^{-1}\sin(\hat{\theta})\cos(\hat{\phi})\,, \\
&z = rb_3^{-1}\cos(b_3\theta)\,,\quad \hat{\theta} = b_3\theta\,,\quad \hat{\phi} = b_1b_2\phi\,,
\end{aligned} \tag{43}$$

under which (42) reduces to the *hadronic radial equation* of the iso-Bohr atom

$$\Big[-\frac{1}{2\mu}\frac{1}{\hat{r}^{\hat{2}}}\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{r}^{\hat{2}}\frac{\hat{\partial}}{\hat{\partial}\hat{r}}+\rho+V_c(\hat{r})\Big]|\hat{\psi}(\hat{r})\rangle = E\,|\hat{\psi}(\hat{r})\rangle\,, \tag{44}$$

where ρ represents the contribution from the orbital angular momentum treated below, plus the expression for the *hadronic angular momentum* and its eigenvalues (Sect. 6.6 of [21]):

$$\begin{aligned}
&\hat{L}_x = \hat{x}\hat{\times}\hat{p}_x\,,\quad \hat{L}_y = \hat{y}\hat{\times}\hat{p}_y\,,\quad \hat{L}_z = \hat{z}\hat{\times}\hat{p}_z\,,\quad \text{and} \\
&\hat{L}^{\hat{2}}\hat{\times}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle = \\
&= \Big(\hat{L}_x\hat{\times}\hat{L}_x+\hat{L}_y\hat{\times}\hat{L}_y+\hat{L}_z\hat{\times}\hat{L}_z\Big)\hat{\times}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle = \\
&= \left(\frac{1}{\sin\hat{\theta}}\frac{\hat{\partial}}{\hat{\partial}\hat{\theta}}+\frac{1}{\sin^{\hat{2}}\hat{\theta}}\frac{\partial^{\hat{2}}}{\partial\hat{\psi}^{\hat{2}}}\right)|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle = \\
&= \hat{I}_{phi}\hat{\ell}(\hat{\ell}+1)\hat{\times}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle\,,\quad \text{and} \\
&\hat{L}_z\hat{\times}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle = -i\frac{\hat{\partial}}{\hat{\partial}\hat{\phi}}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle = i\hat{I}_\phi\hat{m}|\hat{Y}_{\hat{m}}^{\hat{\ell}}(\hat{\theta},\hat{\psi})\rangle\,, \\
&\hat{\ell} = \ell b_3 = 0,1,2,3,\ldots \\
&\hat{m} = m\,b_1b_2 = \hat{\ell},\hat{\ell}-1,\hat{\ell}-2,\ldots,-\hat{\ell}\,.
\end{aligned} \tag{45}$$

To appraise the physical value of the above iso-Bohr atom we now assume in first approximation: 1) Hadronic ground state for which $\hat{\ell} = 0$; 2) The consequential values $\hat{m} = 0$ and $\rho = 0$ in (44); 3) The spherical symmetry of the isoorbits for which $b_1 = b_2 = b_3 = b > 0$; 4) The value of the isounit $\hat{I}_\phi = 1$ (Bohm's hidden variable, p. 245 of [21]); 5) the average of the resulting Santillian (24)

$$S = \text{Diag}\,(b_1^2,b_3^2,b_3^2) = b^2\text{Diag}\,(1,1,1)\,,\quad b^2 = \text{constant}\,, \tag{46}$$

resulting in the simplified expression for the hadronic radial equation (44) (see Ch. 5, p. 182 of [21] for the general case)

$$\begin{aligned}
&\Big[\frac{1}{2\mu}\hat{I}^2\frac{1}{\hat{r}^{\hat{2}}}\frac{\partial}{\hat{r}}\hat{r}^{\hat{2}}\frac{\partial}{\partial\hat{r}}+V_c(\hat{r})-E\Big]|\hat{\psi}(\hat{r})\rangle = \\
&= \Big[\frac{1}{2\mu}\frac{1}{\hat{r}^{\hat{2}}}\frac{\partial}{\hat{r}}\hat{r}^{\hat{2}}\frac{\partial}{\partial\hat{r}}+S^2V_c(\hat{r})-S^2E\Big]|\hat{\psi}(\hat{r})\rangle = 0\,,
\end{aligned} \tag{47}$$

that, by recalling (2), is reduced to the isorenormalization of the effective nuclear charge

$$\tilde{Z}_{eff}\,e = S^2Z_{eff}\,e\,, \tag{48}$$

with corresponding isonormalization of the eigenvalues

$$\tilde{E} = S^2E\,, \tag{49}$$

from which we can write

$$S^2 = \frac{E_{\text{exp}}}{E_{\text{QM}}} = 1+\frac{\Delta E}{E_{\text{QM}}}\,. \tag{50}$$

By using the spectral deviations of (10), (13), and (16), we then obtain the values

$$\begin{aligned}
&\text{Eq. (10)}: S^2_{(\Delta E\approx 20\text{ eV})} = \frac{13.63\text{ keV}}{13.61\text{ keV}} = 1.0014\,, \\
&\text{Eq. (13)}: S^2_{(\Delta E\approx 150\text{ eV})} = \frac{93.25\text{ keV}}{93.10\text{ keV}} = 1.0016\,, \\
&\text{Eq. (16)}: S^2_{(\Delta E\approx 200\text{ eV})} = \frac{115.8\text{ keV}}{115.6\text{ keV}} = 1.0017\,,
\end{aligned} \tag{51}$$

which constitute numerically exact representations of spectral experimental data for giant atoms via hadronic mechanics [19, 21] that are invariant over time due to the invariance of the basic dynamical isoequations (38) under the Galileo-Santilli isosymmetry $\hat{\text{G}}(3.1)$ and related isorelativity [36] to [38].

Representative mean values of the Santillian and of the isounit are, therefore, given by

$$\begin{aligned}
&\bar{S} = \hat{I}^{-1} = 1.0008014756\,, \\
&\bar{I} = \bar{S}^{-1} = 0.9991991669\,,
\end{aligned} \tag{52}$$

and, according to our basic assumptions, they represent *all* quantum mechanical spectral deviations of giant atoms as a

necessary condition to reduce unsolvable many-body problems to solvable two-body isoproblems.

Note that approximation (46) of Santillian (24) is merely intended to show the capability by the iso-Bohr atoms to achieve an exact solution for giant atoms, the complete solution being that for the full expression (24).

By keeping in mind that, according to Refs. [111–114], the uncertainties under strong interactions are predicted to be *smaller* than the corresponding values under electromagnetic interactions (evidently due to the former interactions being dramatically stronger than the latter), an intriguing implication of exact representations (51), and related mean values (52), is that the uncertainties δr and Δp of the orbiting electrons in giant atoms are predicted to be *bigger* than the corresponding quantum mechanical uncertainties for light atoms, as one can see from the nonunitary transformations (25) and (26) and values (52) of Heisenberg's uncertainty principle

$$\begin{aligned} \Delta r \Delta p &= \frac{1}{2} \left| U \langle \phi | [r, p] | \psi \rangle U^\dagger \right| = \\ &= \frac{1}{2} \left| \langle \hat{\phi} | \hat{\times} [\hat{r}, \hat{p}] \hat{\times} | \hat{\psi} \rangle \right| \geqslant \frac{1}{2} \hbar S > \frac{1}{2} \hbar \,. \end{aligned} \tag{53}$$

This result was expected due to an apparent shielding of said electrons by the entanglement interactions to be studied in a subsequent work.

Note that, as indicated in §1, by conception and construction the iso-Bohr model coincides with the Bohr model at the abstract realization-free level to such an extent that they could be expressed via the same symbols, all differences occurring in the preferred *realization* of their universal enveloping associative algebra.

5 Concluding remarks

In this paper, we have:

5.i) Identified the experimental values of the spectral deviations from quantum mechanical predictions for giant atoms by therefore providing an additional confirmation of the EPR argument on the lack of completeness of quantum mechanics [10–13].

5.ii) Indicated the most plausible origin of said deviations in the nonlocal, nonlinear and non-Hamiltonian interactions of the entanglement of all orbital electrons.

5.iii) Represented the non-Hamiltonian/non-Lagrangian interactions of particle entanglements via the Santillian S in all isoassociative products $A \hat{\times} B = AS B$ of hadronic mechanics.

5.iv) Introduced the notion of *extended* isoparticles at large and isoelectrons, in particular, for a geometrically consistent admission of intrinsic rotations with the spin value $J = 1/2$.

5.v) Introduced, apparently for the first time, the axiom-preserving completion of Bohr's Hydrogen atom into an exactly solvable form applicable to giant atoms and shown that the resulting *iso-Bohr atoms* provide a numerically exact and time invariant representation of quantum mechanical spectral deviations.

In preceding works outlined in comments 3.1–3.10 we have shown that isomathematics offers significant computational advantages over 20th century applied mathematics, while in this paper we have shown that isomathematics can map unsolvable many-body problems into solvable two-body isotopic problems, by therefore validating the power of isomathematics as a nonlocal, nonlinear and non-Hamiltonian EPR completion of local, linear and Hamiltonian applied mathematics for novel advances in all branches of physics, chemistry and biology.

Acknowledgements

Thanks for invaluable comments are due to Professors A. K. Aringazin from Kazakhstan, S. Beghella Bartoli from Italy, S. Georgiev from France, A. Muktibodh from India, J. Dunning Davies from England, F. Inglese from Italy, M. Kashchenko from Russia, J. Lissner from Israel, E. Velardo from Italy and Th. Vougiouklis from Greece. Special thanks are due to my Artificial Intelligence for outlines of old papers and for technical controls. Thanks are finally due to Mrs. Sherri Stone, of the Hadronic Press, for a linguistic control of the paper.

Received on April 16, 2026

References

1. Bohr N. On the constitution of atoms and molecules, Part I. *Philosophical Magazine*, 1913, v. 26, 1–25.
2. Bohr N. On the constitution of atoms and molecules, Part II. *Philosophical Magazine*, 1913, v. 26, 476–502.
3. Condon E. U. and Shortley C. H. The Theory of Atomic Spectra. Cambridge University Press, 1935.
4. Das M. P. Core electron binding energies in heavy atoms. *Nucl. Part. Phys.*, 1983, v. 21, 103–110.
5. Kapetanovi'c E., *et al.* Nonlocal exchange interactions in strongly correlated electron systems. *Phys. Rev. B*, 2020, v. 101, 235165.
6. Tusche C., *et al.* Nonlocal electron correlations in an itinerant ferromagnet. *Nat. Commun.*, 2018, v. 9, 3727.
7. Abanin D. A., *et al.* Giant nonlocality near the Dirac point in graphene. *Science*, 2011, v. 332, 328–342.
8. Christov I. Effects of spatial nonlocality versus nonlocal causality for bound electrons in external fields. *Entropy*, 2022, v. 24, 840.
9. Jeon K. T., *et al.* Giant transition-state quasiparticle spin-Hall effect detected by nonlocal spin transport. *ACS Nano*, 2020, v. 14, 15874–15883.
10. Einstein A., Podolsky B., and Rosen N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 1935, v. 47, 777–780. www.eprdebates.org/docs/epr-argument.pdf.
11. Fadel M., *et al.* Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates. *Science*, 2018, v. 360, 409–415. www.santilli-foundation.org/Basel-paper.pdf.
12. Colciaghi P., *et al.* Einstein-Podolsky-Rosen experiment with two Bose-Einstein condensates. *Phys. Rev. X*, 2023, v. 13, 021031–1/021031–10. journals.aps.org/prx/pdf/10.1103/PhysRevX.13.021031.

13. Aspect A., *et al.* Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedanken experiment: A new violation of Bell's inequalities. *Phys. Rev. Lett.*, 1982, v. 49, 91–94. ui.adsabs.harvard.edu/abs/1982PhR vL..49...91A.
14. Santilli R. M. Lie-isotopic representation of stable nuclei, II: Exact and time invariant representation of the Deuteron data. *Ratio Mathematica*, 2024, v. 52, 64–130. www.santilli-foundation.org/docs/4-RM-SI-Vol-52-2024-fin.pdf.
15. Newton I. Principia, Book 2: De motu corporum. 1687. archive.org/details/newtonspmathema00newtrich/page/n5/mode/2up.
16. Lagrange J. L. Méchanique Analytique. 1788. Reprinted by Gauthier-Villars, Paris, 1888.
17. Hamilton V. R. On a General method in Dynamics. 1834. Reprinted in Hamilton's Collected Papers. Cambridge University Press, Cambridge, UK, 1940.
18. Santilli R. M. Foundation of Theoretical Mechanics, Vol. I: The Inverse Problem in Newtonian Mechanics. Springer-Verlag, Heidelberg, Germany, 1978. www.santilli-foundation.org/docs/Santilli-209.pdf.
19. Santilli R. M. Foundation of Theoretical Mechanics, Vol. II: Birkhoffian Generalization of Hamiltonian Mechanics. Springer-Verlag, Heidelberg, Germany, 1983. www.santilli-foundation.org/docs/santilli-69.pdf.
20. Santilli R. M. Elements of Hadronic Mechanics, Vol. I: Mathematical Foundations. Ukraine Academy of Sciences, Kiev, 1995. www.santilli-foundation.org/docs/Santilli-300.pdf.
21. Santilli R. M. Elements of Hadronic Mechanics, Vol. II: Theoretical Foundations. Ukraine Academy of Sciences, Kiev, 1995. www.santilli-foundation.org/docs/Santilli-301.pdf.
22. Santilli R. M. Elements of Hadronic Mechanics, Vol. III: Experimental verifications. Ukraine Academy of Sciences, Kiev, 2016. www.santilli-foundation.org/docs/elements-hadronic-mechanics-iii.compressed.pdf.
23. Santilli R. M. Need of subjecting to an experimental verification the validity within a hadron of Einstein special relativity and Pauli exclusion principle. *Hadronic J.*, 1978, v. 1, 574–901. www.santilli-foundation.org/docs/santilli-73.pdf.
24. Santilli R. M. On a possible Lie-admissible covering of Galilei's relativity in Newtonian mechanics for nonconservative and Galilei form-non-invariant systems. *Hadronic Journal*, 1978, v. 1, 223–423. www.santilli-foundation.org/docs/Santilli-58.pdf. Addendum, *Hadronic Journal*, 1978, v. 1, 1279–1342. www.santilli-foundation.org/docs/Santilli-58-Addendum.pdf.
25. Santilli R. M. Initiation of the representation theory of Lie-admissible algebras of operators on bimodular Hilbert spaces. *Hadronic J.*, 1979, v. 3, 440–467. www.santilli-foundation.org/docs/santilli-1978-paper.pdf.
26. Fronteau J., Tellez-Arenas A. and Santilli R. M. Lie-admissible structure of statistical mechanics. *Hadronic Journal*, 1979, v. 3, 130–176. www.santilli-foundation.org/docs/arenas-fronteau-santilli-1981.pdf.
27. Ktorides C. N., Myung H. C. and Santilli R. M. Elaboration of the recently proposed test of Pauli's principle under strong interactions. *Physical Review D*, 1980, v. 22, 892–907. www.santilli-foundation.org/docs/santilli-527.pdf.
28. Myung H. C. and Santilli R. M., Editors. Proceedings of the Second Workshop on Lie-Admissible formulations. Part A: Review Papers. *Hadronic Journal*, 1979, v. 2 (6). www.santilli-foundation.org/docs/hj-2-6-1979.pdf. Part B: Research Papers, *Hadronic Journal*, 1979, v. 3 (1). www.santilli-foundation.org/docs/hj-3-1-1979.pdf.
29. Santilli R. M. Lie-Admissible Approach to the Hadronic Structure. International Academic Press, Vol. I, 1978. www.santilli-foundation.org/docs/Santilli-71.pdf. Vol. II, 1982. www.santilli-foundation.org/docs/Santilli-72.pdf.
30. Myung H. C. and Santilli R. M., Editors. Proceedings of the Third Workshop on Lie-Admissible Formulations. Part A: Mathematics. *Hadronic Journal*, 1981, v. 4 (2). www.santilli-foundation.org/docs/hj-4-2-1981.pdf. Part B: Theoretical Physics. *Hadronic Journal*, 1981, v. 4 (3). www.santilli-foundation.org/docs/hj-4-3-1981.pdf. Part C: Experimental Physics, and Bibliography. *Hadronic Journal*, 1981, v. 4 (4). www.santilli-foundation.org/docs/hj-4-4-1981.pdf.
31. Arenas T., Fronteau J. and Santilli R. M., Editors. Proceedings of the First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment. Part A: Invited Papers. *Hadronic Journal*, 1982, v. 5 (2). www.santilli-foundation.org/docs/hj-5-2-1982.pdf. Part B: Invited Papers. *Hadronic Journal*, 1982, v. 5 (3). www.santilli-foundation.org/docs/hj-5-3-1982.pdf. Part C: Contributed Papers. *Hadronic Journal*, 1982, v. 5 (4). www.santilli-foundation.org/docs/hj-8-4-1982.pdf. Part D: Contributed Papers. *Hadronic Journal*, 1982, v. 5 (5). www.santilli-foundation.org/docs/hj-5-5-1982.pdf.
32. Myung H. C. and Santilli R. M., Editors. Proceedings of the First Workshop on Hadronic Mechanics. *Hadronic Journal*, 1983, v. 6 (6). www.santilli-foundation.org/docs/hj-6-6-1983.pdf.
33. Myung H. C. and Santilli R. M., Editors. Proceedings of the Second Workshop on Hadronic Mechanics. Vol. I. *Hadronic Journal*, 1984, v. 7 (5). www.santilli-foundation.org/ docs/hj-7-5-1984.pdf. Vol. II. *Hadronic Journal*, 1984, v. 7 (6). www.santilli-foundation.org/ docs/hj-7-6-1984.pdf.
34. Schoeber A., Editor. Irreversibility and Non-potentiality in Statistical Mechanics. Hadronic Press, 1984. www.santilli-foundation.org/ docs/Santilli-110.pdf.
35. Anderson R. Collected OA Papers on Hadronic Mathematics, Mechanics and Chemistry. www.santilli-foundation.org/docs/HMMC.pdf.
36. Santilli R. M. Isotopic Generalizations of Galilei and Einstein Relativities, Vol. I: Mathematical Foundations. International Academic Press, 1991. www.santilli-foundation.org/docs/Santilli-01.pdf.
37. Santilli R. M. Isotopic Generalizations of Galilei and Einstein Relativities, Vol. II: Classical Formulations. International Academic Press, 1991. www.santilli-foundation.org/docs/Santilli-61.pdf .
38. Aringazin A. K., Jannussis A., Lopez F., Nishioka M. and Veljanosky B. Santilli's Lie-Isotopic Generalization of Galilei and Einstein Relativities. Kostakaris Publishers, Athens, Greece, 1991. www.santilli-foundation.org/docs/Santilli-108.pdf.
39. Santilli R. M. Lie-isotopic Lifting of Special Relativity for Extended Deformable Particles. *Lettere Nuovo Cimento*, 1983, v. 37, 545–555. www.santilli-foundation.org/docs/Santilli-50.pdf.
40. Santilli R. M. Lie-isotopic generalization of the Poincaré symmetry, classical formulation. ICTP communication No. IC/91/45, 1991. www.santilli-foundation.org/docs/Santilli-140.pdf.
41. Santilli R. M. Overview of historical and recent verifications of the Einstein-Podolsky-Rosen argument and their applications to physics, chemistry and biology. APAV — Academia Piceno Aprutina dei Velati, Pescara, Italy, 2021. www.santilli-foundation.org/epr-overview-2021.pdf.
42. Anderson R. Outline of hadronic mathematics, mechanics and chemistry as conceived by R. M. Santilli. *American Journal of Modern Physics*, 2016, v. 6, 1–16. www.santilli-foundation.org/docs/HMMC-2017.pdf.
43. Velardo E. An introduction to iso-, geno- and hyper-mathematics and their applications in physics, chemistry and biology. *Ratio Mathematica*, 2025, v. 54, 118–166. www.santilli-foundation.org/docs/Velardo-2025.pdf.

44. Santilli R. M. An Introduction to Hadronic Mechanics, Volume I: Application to Nuclear Physics. Cambridge Scholars Publishing, 2026. cambridgescholars.com/product/978-1-0364-7004-3
45. Greenberg O. W. and Mohapatra R. N. Local quantum field theory of possible violation of the Pauli principle. *Physical Review Letters*, 1987, v. 59, 2507–2510.
46. Okun' L. B. On the possibility of Pauli principle violation in atoms. *JETP Letters*, 1987, v. 46, 529–532.
47. Ramberg E. and Snow G. A. Experimental limit on a small violation of the Pauli principle. *Physics Letters B*, 1990, v. 238, 438–441.
48. Bartalucci S., *et al.* New experimental limit on the Pauli exclusion principle violation by electrons. *Physics Letters B*, 2006, v. 641, 18–22.
49. Curceanu C., *et al.* New experimental limit on the Pauli Exclusion Principle violation by electrons. *International Journal of Quantum Information*, 2009, v. 7, 115–123. arXiv: quant-ph/0803.0870.
50. Shi H., *et al.* Experimental search for the violation of Pauli Exclusion Principle. *Journal of Physics: Conference Series*, 2018, v. 1138, 012003. arXiv: physics.atom-ph/1804.04446.
51. Elliott S. R., *et al.* An improved limit on Pauli-exclusion-principle forbidden atomic transitions. *Foundations of Physics*, 2012, v. 42, 1015–1030. arXiv: nucl-ex/1107.3118.
52. Back H. O., *et al.* New experimental limits on violations of the Pauli exclusion principle obtained with the Borexino Counting Test Facility. *European Physical Journal C*, 2004, v. 37, 421–431. arXiv: hep-ph/0406252.
53. Messiah A. M. L., *et al.* Testing the Pauli Exclusion Principle with the VIP-2 experiment. *Symmetry*, 2020.
54. Santilli R. M. Embedding of Lie-algebras into Lie-admissible algebras. *N*uovo Cimento, 1967, v. 51, 570–585. www.santilli-foundation.org/docs/Santilli-54.pdf.
55. Santilli R. M. Dissipativity and Lie-admissible algebras. *Meccanica*, 1969, v. 4, 3–11. www.santilli-foundation.org/docs/meccanica-1969.pdf.
56. Santilli R. M. An introduction to Lie-admissible algebras. *Suppl. Nuovo Cimento*, 1968, v. 6, 1225–1249. www.santilli-foundation.org/docs/ Santilli-501.pdf.
57. Griem H. R. Stark broadening of hydrogen lines in a plasma. *Physical Review*, 1959, v. 116, 4–15.
58. Wiese W. L., Paquette D. R. and Solarski J. E. Profiles of Stark-broadened Balmer lines in a hydrogen plasma. *Physical Review*, 1963, v. 129, 1225–1236.
59. Konjević N., *et al.* Experimental Stark widths and shifts for spectral lines of neutral and ionized atoms: A critical review of selected data. *Journal of Physical and Chemical Reference Data*, 2002, v. 31, 819–927.
60. Stambulchik E. and Maron Y. Stark broadening of hydrogen lines for high principal quantum numbers. *Physical Review E*, 2007, v. 75, 016401.
61. Fortmann C., *et al.* On the occurrence of Balmer spectra in expanding microplasmas from laser irradiated liquid hydrogen. *Physical Review E*, 2007, v. 75, 026403.
62. Tremblay P.-E. and Bergeron P. Spectroscopic analysis of DA white dwarfs: Stark broadening of hydrogen lines including non-ideal effects. *The Astrophysical Journal*, 2009, v. 696, 1755–1770.
63. Oks E. Review of recent advances in the analytical theory of Stark broadening of hydrogenic spectral lines in plasmas: Applications to laboratory discharges and astrophysical objects. *Atoms*, 2018, v. 6.
64. Thouin J., *et al.* Interpretation of Stark broadening measurements on a spatially integrated plasma spectral line. arXiv: physics.plasm-ph/2210.02244.
65. Wu C., *et al.* Plasma screening effects in Stark broadening: A fully relativistic close-coupling approach. arXiv: physics.plasm-ph/2603.05865.
66. Aringazin, A. K. Isotopies of Pauli's principle. *Hadronic J.*, 1990, v. 13, 183–195.
67. Yakaboylu E. Entanglement in relativistic quantum mechanics. arXiv: quant-ph/1005.0846.
68. Georgiev D. and Cohen E. Entanglement measures for two-particle quantum histories. *Phys. Rev. A*, 2022, v. 106, 062437.
69. Shi Y. Historical origins of quantum entanglement in particle physics. arXiv: physics.hist-ph/2507.13582.
70. Wiseman H., *et al.* Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox. *Physical Review Letters*, 2007, v. 98, 140–402.
71. Huan Yang H., *et al.* Experimental observation of Einstein-Podolsky-Rosen steering via entanglement detection. *Phys. Rev. A*, 2020, v. 101, 042115.
72. Ravindranath W., *et al.* Entanglement steering in adaptive circuits with feedback. *Phys. Rev. B*, 2023, v. 108, L041103.
73. Santilli R. M. A quantitative representation of particle entanglements via Bohm's hidden variables according to hadronic mechanics. *Progress in Physics*, 2022, v. 18, 131–137. www.santilli-foundation.org/docs/pip-entanglement-2022.pdf.
74. Inglese F., Velardo E., Santilli R. M., Vougiouklis Th. Lie-admissible irreversible biological entanglements and their apparent initiation of hadronic medicine. *Ratio Mathematica*, 2025, v. 54, 1–32. www.santilli-foundation.org/docs/hadronic-medicine.pdf.
75. Santilli R. M. Perché lo spazio é rigido (Why space is rigid). Il Pungolo Verde, Campobasso, Italy, 1956. www.santilli-foundation.org/docs/rms-56.pdf (*in Italian*), www.santilli-foundation.org/docs/rms-56-english.pdf (*in English*).
76. Santilli R. M. Fondamenti per una teoria unitaria sulla struttura dell' elettrone (in Italian). undergraduate doctoral thesis, Istituto di Fisica Nucleare, Mostra d'Oltrenare, Napoli, 1959. www.santilli-foundation.org/docs/santilli-78.pdf.
77. Santilli R. M. A structure model of the elementary charge. *Hadronic Journal*, 1981, v. 4, 770–783. www.santilli-foundation.org/docs/Santilli-03.pdf.
78. Santilli R. M. and Parr E. A conceivable lattice structure of the elementary charge. *Lettere Nuovo Cimento*, 1983, v. 37, 505–568. www.santilli-foundation.org/docs/Santilli-56.pdf.
79. Gandzha I. and Kadeisvili J. V. New Sciences for a New Era: Mathematical, Physical and Chemical Discoveries by Ruggero Maria Santilli. Sankata Printing Press, Nepal, 2011. www.santilli-foundation.org/docs/RMS.pdf.
80. Wyderka N., *et al.* Complete characterization of quantum correlations by randomized measurements. *Phys. Rev. Lett.*, 2023, v. 131, 090201.
81. Suleymanov M., *et al.* Relativity of quantum correlations: Invariant quantities and frame-dependent measures. arXiv: quant-ph/2503. 20090v2.
82. Tendick L. Quantum correlations cannot be reproduced with a finite number of measurements in any no-signaling theory. *Phys. Rev. Lett.*, 2025, v. 135, 030201.
83. Bransden B. H. and Joachain C. J. Physics of Atoms and Molecules, 2nd ed. Prentice Hall, 2003.
84. Griffiths, D. J. and Schroeter D. F. Introduction to Quantum Mechanics, 3rd ed. Cambridge University Press, 2018.
85. Deeprose C. Completing quantum mechanics: Hadronic mechanics and its potential for clean energy. *Scientia*, 2024.
86. Deeprose C. Progressive recovering of Einstein's determinism under strong interactions. *Scientia*, 2024.

87. Muktibodh A. Santilli's recovering of Einstein's determinism. *Progress in Physics*, 2024, v. 20, 26–34. www.eprdebates.org/docs/muktibodh-2024.pdf.
88. Santilli R. M. Overview of historical and recent verifications of the Einstein-Podolsky-Rosen argument and their applications to physics, chemistry and biology. APAV — Accademia Piceno Aprutina dei Velati, Pescara, Italy, 2021. www.santilli-foundation.org/epr-overview-2021.pdf.
89. Beghella-Bartoli S., Muktibodh A. and Santilli R. M. Problematic aspects of generalized quantum theories and their apparent resolution via hadronic mechanics. *Ratio Mathematica*, 2024, v. 52, 21–42. www.santilli-foundation.org/docs/2-RM-SI-Vol-52-2024-fin.pdf.
90. Myung H. C. and Santilli R. M. Modular-isotopic Hilbert space formulation of the exterior strong problem. *Hadronic Journal*, 1982, v. 5, 1277–1366. www.santilli-foundation.org/docs/Santilli-201.pdf.
91. Santilli R. M. Isonumbers and genonumbers of dimensions 1, 2, 4, 8, their isoduals and pseudoduals, and "hidden numbers" of dimension 3, 5, 6, 7. *Algebras, Groups and Geometries*, 1993, v. 10, 273–295. www.santilli-foundation.org/docs/Santilli-34.pdf.
92. Santilli R. M. Invariant Lie-isotopic and Lie-admissible formulation of quantum deformations. *Found. Phys.*, 1997, v. 27, 1159–1177. www.santilli-foundation.org/docs/Santilli-06.pdf.
93. Jiang C.-X. Foundations of Santilli Isonumber Theory. International Academic Press, 2001. www.i-b-r.org/docs/jiang.pdf.
94. Cord C. Introduction to Santilli's Isonumbers. *AIP Conf. Proceed.*, 2012, v. 1479, 1013–1020. www.santilli-foundation.org/docs/Corda-2012.pdf.
95. Georgiev S. Introduction to iso-Euclidean Geometry. Eliva Press, 2024.
96. Sourlas D. S. and Tsagas Gr. T. Mathematical Foundation of the Lie-Santilli Theory. Ukraine Academy of Sciences, 1993. www.santilli-foundation.org/docs/santilli-70.pdf.
97. Muktibodh A. S. and Santilli R. M. Studies of the regular and irregular isorepresentations of the Lie-Santilli isotheory. *Journal of Generalized Lie Theories*, 2007, v. 11, 1–7. www.santilli-foundation.org/docs/isorep-Lie-Santilli-2017.pdf.
98. Santilli R. M. Isotopies of Lie symmetries, I: Basic theory. *Hadronic Journal*, 1985, v. 8, 8–35. www.santilli-foundation.org/docs/santilli-65.pdf.
99. Santilli R. M. Isotopies of Lie symmetries, II: Isotopies of the rotational symmetry. *Hadronic Journal*, 1985, v. 8, 36–52. www.santilli-foundation.org/docs/santilli-65.pdf.
100. Santilli R. M. Rotational isotopic symmetries. ICTP communication, No. IC-91-261, 1991. www.santilli-foundation.org/docs/Santilli-148.pdf.
101. Santilli R. M. Isotopic lifting of the SU(2) symmetry with applications to nuclear physics. *JINR Rapid Comm.*, 1993, v. 6, 24–38. www.santilli-foundation.org/docs/Santilli-19.pdf.
102. Wade M. I. and Gill T. L. Isonumbers and RGB image encryption. *Algebras, Groups and Geometries*, 2021, v. 37, 103–119. www.santilli-foundation.org/docs/wade-gill-2021.pdf.
103. Santilli R. M. Isorepresentation of the Lie-isotopic SU(2) algebra with application to nuclear physics and local realism. *Acta Applicandae Mathematicae*, 1998, v. 50, 177–190. www.santilli-foundation.org/docs/Santilli-27.pdf.
104. Santilli R. M. Lie-isotopic lifting of special relativity for extended deformable particles. *Lettere Nuovo Cimento*, 1983, v. 37, 545–555. www.santilli-foundation.org/docs/Santilli-50.pdf.
105. Santilli R. M. Lie-isotopic lifting of unitary symmetries and of Wigner's theorem for extended and deformable particles. *Lettere Nuovo Cimento*, 1983, v. 38, 509–521. www.santilli-foundation.org/docs/Santilli-51.pdf.
106. Santilli R. M. Lie-isotopic generalization of the Poincaré symmetry, classical formulation. ICTP communication No. IC/91/45, 1991. www.santilli-foundation.org/docs/Santilli-140.pdf.
107. Santilli R. M. Nonlinear, nonlocal and noncanonical isotopies of the Poincaré symmetry. *Moscow Phys. Soc.*, 1993, v. 3, 255–239. www.santilli-foundation.org/docs/Santilli-40.pdf.
108. Santilli R. M. Isotopies of the spinorial covering of the Poincaré symmetry. *Chinese J. System Eng. and Electr.*, 1995, v. 6, 177–199. www.santilli-foundation.org/docs/Santilli-18.pdf.
109. Santilli R. M. Nonlocal-integral isotopies of differential calculus, mechanics and geometries. *Circolo Matematico Palermo, Suppl.*, 1996, v. 42, 7–82. www.santilli-foundation.org/docs/Santilli-37.pdf.
110. Georgiev S. Foundations of IsoDifferential Calculus. Nova Publishers, New York.
Vol. 1: Iso-Differential and Iso-Integral Calculus for Iso-Functions in One Variable. 2014.
Vol. 2: Iso-Differential and Iso-Integral Calculus for Iso-Functions in Several Variables. 2014.
Vol. 3: Iso-Ordinary Iso-Differential Equations. 2014.
Vol. 4: Iso-Difference Equations. 2015.
Vol. 5: Iso-Stochastic Iso-Differential Equations. 2015.
Vol. 6: Theory of Iso-Measurable Iso-Functions. 2016.
New Edition of Vol. 1: Iso-Differential and Iso-Integral Calculus for Iso-Functions in One Variable. 2022.
Iso-Mathematics. Lambert Academic Publishing, 2022.
111. Santilli R. M. Generalization of Heisenberg's uncertainty principle for strong interactions. *Hadronic Journal*, 1981, v. 4, 642. www.santilli-foundation.org/docs/generalized-uncertainties-1981.pdf.
112. Santilli R. M. Isotopic lifting of Heisenberg's uncertainties for strong interactions. *Communications in Theoretical physics*, 1994, v. 3, 47–66. www.santilli-foundation.org/docs/santilli-511.pdf.
113. Santilli R. M. Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen. *Ratio Mathematica*, 2019, v. 37, 5–23. www.eprdebates.org/docs/epr-paper-ii.pdf.
114. Santilli R. M. Studies on A. Einstein, B. Podolsky and N. Rosen prediction that quantum mechanics is not a complete theory, II: Apparent proof of the EPR argument. *Ratio Mathematica*, 2020, v. 38, 71–138. eprdebates.org/docs/epr-review-ii.pdf.
115. Santilli R. M. Reduction of matter in the Universe to protons and electrons via the Lie-isotopic branch of Hadronic mechanic. *Progress in Physics*, 2023, v. 19, 73–99. www.santilli-foundation.org/docs/pip-6.pdf.
116. Norman R., Beghella Bartoli S., Buckley B,. Dunning-Davies J., Rak J. and Santilli R. M. Experimental confirmation of the synthesis of neutrons and neutroids from a hydrogen gas. *American Journal of Modern Physics*, 2017, v. 6, 85–104. www.santilli-foundation.org/docs/confirmation-neutron-synthesis-2017.pdf.
117. Thunder Energies Corporation, Specifications of the Directional Neutron Source model DNS-3B-5, 2019. www.i-b-r.org/docs/DNS-brochure-9-12-19.pdf.
Short movie on the operation of the DNS: www.i-b-r.org/docs/ MagnaPower.mp4.
Neutron counts by the Ludlum detector 375: www.i-b-r.org/docs/ Ludlum-Alarms.mp4.
Confirmation by the Berkeley Nucleonics neutron detector SAM 940: www.i-b-r.org/docs/Sam-Alarms.mp4.
118. Santilli R. M. Iso-representation of the deuteron spin and magnetic moment via Bohm's hidden variables. *Progress in Physics*, 2022, v. 18, 74–81. www.santilli-foundation.org/docs/PiP-paper-3-22.pdf.
119. Santilli R. M. and Sobczyk G. Representation of nuclear magnetic moments via a Clifford algebra formulation of Bohm's hidden vari-

ables. *Scientific Reports*, 2022, v. 12, 1–10.
www.santilli-foundation.org/Santilli-Sobczyk.pdf.

120. Santilli R. M., Guest Editor. Lie-isotopic representation of nuclear data. Special Issue of *Ratio Mathematica*, 2024, v. 52.
www.santilli-foundation.org/docs/RM-Special-Issue-.Vol.-52-2024-compressed.pdf.

121. Santilli R. M. Representation of the anomalous magnetic moment of the muons via the Einstein-Podolsky-Rosen completion of quantum into hadronic mechanics. *Progress in Physics*, 2021, v. 17, 210–215.
www.santilli-foundation.org/muon-anomaly-pp.pdf.

122. Santilli R. M. Representation of the anomalous magnetic moment of the muons via the novel Einstein-Podolsky-Rosen entanglement. In: Garcia H., Guzman J., Kauffman L. and Makaruk H., Editors. Scientific Legacy of Professor Zbigniew Oziewicz. World Scientific Publishing Company, *Series on Knots and Everything*, 2023, v. 75, 611–637. www.santilli-foundation.org/docs/Santilli-501.pdf.

123. Santilli R. M. Compatibility of arbitrary speeds with special relativity axioms for interior dynamical problems. *American Journal of Modern Physics*, 2016, v. 5, 143.
www.santilli-foundation.org/docs/ArbitrarySpeeds.pdf.

124. Ahmar H., Amato G., Kadeisvili J. V., Manuel J., West G. and Zogorodnia O. Additional experimental confirmations of Santilli's IsoRedShift and the consequential lack of expansion of the universe. *Journal of Computational Methods in Sciences and Engineering*, 2013, v. 13, 321–349.
www.santilli-foundation.org/docs/IRS-confirmations-212.pdf.