

Quantum Probability is a Scalar: On the Phase-Space Definition of the Quantum Probability Measure and the Origin of Heisenberg's Uncertainty Principle

G. G. Nyambuya

National University of Science and Technology, Faculty of Applied Sciences — Department of Applied Physics,
Fundamental Theoretical Physics Explorers, P. O. Box 939, Ascot, Bulawayo, Republic of Zimbabwe.
E-mail: golden.nyambuya@nust.ac.zw

For a scalar quantum mechanical wavefunction Ψ , we herein identify and resolve a fundamental inconsistency in the standard formulation of Quantum Mechanics (QM), wherein the Quantum Probability Measure (QPM) $\mathcal{P}_\Psi = \int \Psi^\dagger \Psi d^3x$, as handed down by Max Born in 1926–1927 defined on three-dimensional position space, is not a scalar — different observers in relative motion will assign different probabilities to the same event, which is physically inadmissible. We show that the natural resolution is to define the QPM on the six-dimensional phase space $\Gamma(\vec{r}, \vec{p})$, where the phase-space volume element $d^3p d^3x$ is a true scalar, because its covariant and contravariant Jacobian factors cancel exactly. When the wavefunction is normalised under this measure, Heisenberg's uncertainty principle $\delta p \delta x \geq \frac{1}{2} \ell \hbar$ ($\ell \in \mathbb{Z}^+$ the spin quantum number) follows directly from the normalisation condition — not as an independent axiom, but as a mathematical consequence of requiring probability to be observer-independent. The wavefunction on phase space is naturally dimensionless, removing the seemingly unphysical dimensions $L^{-3/2}$ of the standard formulation. These results indicate that the wavefunction is fundamentally a phase-space object, not a configuration-space object, with significant implications for the foundations of Quantum Mechanics.

Nature is pleased with simplicity.
Sir Isaac Newton (1642–1727)

1 Introduction

Since Max Born (1882–1970) [1, 2] proposed in 1926–1927 that the squared modulus of the quantum mechanical wavefunction Ψ gives the probability of finding a particle in a given region of space, the *Quantum Probability Measure* (QPM) $\mathcal{P}_\Psi$ has been written as the integral of $\Psi^\dagger \Psi$ over a spatial volume. This interpretation is widely taught and widely applied. However, it rests on an assumption that has received little scrutiny in the century since Born's proposal: that the resulting quantum probability (QP) may be observer-dependent.

In the present paper, we demonstrate that the standard QPM, when integrated over three-dimensional position space, is *not* a scalar: it transforms under coordinate changes, so observers in relative motion will in general assign different probabilities to the same physical event. As argued in §2.1, this is physically inadmissible. We acknowledge that certain interpretations of QM — notably those in which the observer plays a constitutive role in collapsing the wavefunction — might regard an observer-dependent probability as acceptable. We do not adopt such a view here. We take the position, shared with Kolmogorov's classical probability theory [3] and with the foundations of classical statistical mechanics, that probability is a statement about an objective state of affairs: all observers must agree on the numerical value of the probability of a given event. Under this requirement, the stan-

dard QPM fails.

The requirement that probability be a scalar imposes a strong mathematical constraint on how it may be defined. We show that the unique way to construct a scalar QPM from the wavefunction Ψ is to integrate $\Psi^\dagger \Psi$ not over position space d^3x but over the six dimensional phase space $d^3x d^3p$. The product $d^3x d^3p$ is a true scalar — its covariant and contravariant Jacobian factors cancel exactly under any coordinate transformation.

The consequence of this redefinition is striking. When the simplest quantum mechanical wavefunction — that of a particle in a box — is normalised under the new phase-space QPM, Heisenberg's uncertainty principle [4] follows automatically. It is not imposed from outside; it emerges as a mathematical consequence of requiring probability to be a scalar. The uncertainty principle is, in this sense, nature's response to an elementary requirement of logical consistency.

The connection to classical statistical mechanics is immediate and illuminating. Boltzmann's distribution function [5] was always defined on phase space, for exactly the same reason: phase-space probability is a scalar. Quantum Mechanics (QM), once its probability measure is correctly formulated, is seen to share this foundational structure with classical statistical mechanics.

This paper is organised as follows. In §2 we demonstrate that the standard QPM is not a scalar. In §3 we introduce the phase-space QPM and prove it is a scalar. In §3.5 we derive Heisenberg's uncertainty principle from the normalisa-

tion condition of the new QPM. In §4 we discuss the broader implications, and in §5 we conclude. Throughout this paper we work in three spatial dimensions with standard Cartesian coordinates x^j ($j = 1, 2, 3$) and their conjugate momenta p_j . We use the Einstein summation convention where appropriate, and write $\hbar$ for the reduced Planck constant.

2 Standard quantum probability measure

Before examining the QPM, we state clearly the physical requirement it must satisfy. Having done that, we will proceed to examine the canonical definition of standard QM's QPM.

2.1 Requirement of scalar probability

Requirement. *The probability of occurrence of a physical event must be observer-independent: all observers, regardless of their state of motion or coordinate system, must agree on its numerical value. A probability measure must therefore be a scalar quantity.*

We fully acknowledge that this requirement goes beyond Born's rule as originally stated [1], which does not itself demand Lorentz invariance of the probability. It is precisely for this reason that the present requirement is non-trivial: it identifies a condition that Born's rule does not satisfy, and whose imposition leads to a strictly more fundamental formulation.

For example, consider the following illustration. A box contains 2 white balls and 4 black balls. The probability of drawing a white ball first is $\mathcal{P} = 1/3$, independent of whether the observer is at rest or in motion. A probability that changes with the observer's velocity is a coordinate-dependent quantity, not a probability in the Kolmogorov [3] sense. We now show that the standard QPM fails this requirement.

2.2 Non-invariance of the standard QPM

The standard Born QPM is:

$$\mathcal{P}_\Psi(\vec{r}, \vec{r}_0) = \int_V \Psi^\dagger \Psi d^3x, \quad (1)$$

where: V is a volume region of three-dimensional position space. Consider two observers: an unprimed observer with coordinates x^j and a primed observer with coordinates $x^{j'}$ in relative motion. The primed observer measures:

$$\mathcal{P}'_\Psi = \int \Psi'^\dagger \Psi' dx^{1'} dx^{2'} dx^{3'}. \quad (2)$$

If the wavefunction Ψ is a scalar under coordinate transformations, then $\Psi' = \Psi$. The coordinate differentials transform as:

$$dx^{j'} = \frac{\partial x^{j'}}{\partial x^j} dx^j. \quad (3)$$

Substituting into (2) and assuming (for demonstrative purposes) that the Jacobian factor is spatially uniform so it may

be taken outside the integral:

$$\mathcal{P}'_\Psi = \underbrace{\left(\frac{\partial x^{1'}}{\partial x^1} \frac{\partial x^{2'}}{\partial x^2} \frac{\partial x^{3'}}{\partial x^3} \right)}_{J \text{ (motion-dependent)}} \int \Psi^\dagger \Psi d^3x = J \mathcal{P}_\Psi. \quad (4)$$

Since $J \neq 1$, in general for observers in relative motion (for example, under a Lorentz boost $J = 1/\Gamma$, where Γ is the Lorentz factor):

$$\mathcal{P}'_\Psi \neq \mathcal{P}_\Psi. \quad (5)$$

For so long as the Schrödinger wavefunction is treated as — or is assumed to be — a scalar function ($\Psi' = \Psi$), the standard QPM is not a scalar, the clear and most obvious meaning of which is that different observers in relative motion assign different probabilities to the same physical event. This is a fundamental inconsistency at the foundations of QM that has gone unaddressed since Born's 1926 proposal [1].

Without the development in this paper, one way out of this quagmire is to treat the Schrödinger wavefunction as a non-scalar function and as a mathematical object that under general curvilinear coordinate transformations, transforms as an inverse half-density function (with a Jacobian factor $J^{-1/2}$) where the inverse half Jacobian factor ($J^{-1/2}$) generated from the wavefunction exactly cancels that generated from the coordinates leading to $\Psi'^\dagger \Psi' d^3x' = \Psi^\dagger \Psi d^3x$, that is to say:

$$\Psi' = J^{-1/2} \Psi. \quad (6)$$

In this way, the QPM is a true scalar function that does not transform under general curvilinear coordinate transformations — it is *invariant*. If the transformation law given in (6) solves our problem, why go through the scalar wavefunction development?

While the inverse half-density transformation offers a mathematically expedient resolution to the non-invariance of the standard QPM, it does so by imposing a specific — and non-trivial — transformation law onto the wavefunction itself. This approach treats the coordinate dependence of Ψ as a necessary evil to be engineered away, rather than confronting the deeper question of what integration domain is physically appropriate. Yet nothing in the mathematical structure of QM forbids Ψ from being a genuine scalar: the Schrödinger equation places no restriction on the transformation law of Ψ under general coordinate changes, and a scalar Ψ satisfying $\Psi' = \Psi$ is *a priori*, both logically permissible and conceptually simpler, aligning more closely with our classical intuition of a field.

The inverse half-density fix works precisely by making Ψ non-scalar — which is the very possibility we wish to keep open. If we take the scalar nature of Ψ as a genuine physical requirement, the burden of invariance must shift entirely to the integration measure. The mathematical structure of resulting QM then forces us to look beyond position space to a domain whose intrinsic volume element is itself a true scalar

— namely, the six-dimensional phase space $\Gamma(\vec{r}, \vec{p})$, whose volume element $d^3p d^3x$ is invariant under all canonical transformations by Liouville's theorem [6].

Moving to phase space is therefore not a tedious detour but a necessary and profoundly rewarding step. What begins as a seemingly minor bookkeeping question — *how should the probability measure transform for a scalar wavefunction?* — opens onto a substantially new picture of QM. The phase-space QPM renders the wavefunction naturally dimensionless, removing the unphysical dimensions $L^{-3/2}$ of the standard formulation and restoring to Ψ its proper status as a probability amplitude rather than a probability density.

More strikingly, the normalisation condition under the new measure directly implicates the minimum phase-space cell $(\delta r \delta p)_{\min} = \frac{1}{2} \ell \hbar$, from which Heisenberg's uncertainty principle emerges as a mathematical consequence of requiring probability to be observer-independent, rather than as an independent axiom. The inverse half-density route of (6) yields none of this: it repairs the invariance of the measure without questioning the domain of integration and so leaves the deeper structure of the theory entirely untouched. The phase-space formulation is not an *ad hoc* patch; it is the unique logical consequence of demanding that Ψ remain a scalar, and the standard three-dimensional formulation is revealed as a projection of a deeper, inherently phase-space reality.

2.3 Standard treatment in textbooks

The non-scalar character of the standard QPM has gone unaddressed in the literature because the standard treatment in virtually every major QM textbook implicitly assumes that the wavefunction Ψ is a scalar field $\Psi'(\vec{r}', t') = \Psi(\vec{r}, t)$. Under this assumption the probability density $\rho = \Psi^\dagger \Psi$ is also treated as a scalar, so that the non-invariance of d^3x under coordinate transformations [see (7)] goes unnoticed.

We survey the principal texts:

1. **Dirac** [7]: The foundational text of the subject introduces the probability as $\langle x | \psi \rangle \langle \psi | x \rangle dx$, treating the bra-ket projection as a scalar quantity in position space. The transformation properties of $\langle x | \psi \rangle$ under a change of reference frame are not discussed;
2. **Landau & Lifshitz** [8]: The normalisation condition is stated as $\int |\psi|^2 dV = 1$ over a spatial volume V , with ψ treated as a scalar function of position. The Jacobian acquired by dV under a general coordinate transformation is not addressed;
3. **Schiff** [9]: The probability interpretation is introduced via $P = \int_V \psi^* \psi d^3r$, with ψ explicitly described as a wave amplitude that is a function of position coordinates. No transformation law for ψ under coordinate changes is stated;
4. **Merzbacher** [10]: The Born rule is therein written as $dP = |\Psi(\vec{r}, t)|^2 d^3r$ and the wavefunction is normalised

by $\int |\Psi|^2 d^3r = 1$, with Ψ treated as a scalar amplitude throughout;

5. **Cohen-Tannoudji, Diu & Laloë** [11]: The probability of finding the particle between $\vec{r}$ and $\vec{r} + d\vec{r}$ is given as $|\psi(\vec{r}, t)|^2 d^3r$, with ψ a scalar function of the three position coordinates. Coordinate transformation properties of the QPM are not considered;
6. **Gasiorowicz** [12]: Normalisation is therein stated as $\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1$ (extended to three dimensions in subsequent chapters), with ψ treated as a scalar throughout and no discussion of the frame dependence of this condition;
7. **Griffiths** [13]: The most widely used undergraduate text defines the probability of finding the particle in the interval $[a, b]$ as $\int_a^b |\Psi(x, t)|^2 dx$ and normalises over all space, treating Ψ as a scalar. The textbook is explicit that “ $|\Psi|^2$ gives the probability of finding the particle at point x at time t ” without addressing what a second observer in relative motion would measure;
8. **Sakurai & Napolitano** [14]: The position-space wavefunction is defined as $\psi_\alpha(\vec{x}') = \langle \vec{x}' | \alpha \rangle$ treated as a scalar function of the eigenvalue $\vec{x}'$ and normalised by $\int d^3x' |\psi_\alpha(\vec{x}')|^2 = 1$;
9. **Weinberg** [15]: Discusses the probability interpretation carefully and acknowledges that the Born rule requires interpretation, but defines the normalisation in the standard three-dimensional position space without addressing the coordinate-transformation properties of the resulting measure.

In each case, the pattern is the same: Ψ is written as a function of position coordinates and the probability is computed by integrating $\Psi^\dagger \Psi$ over a spatial volume, with no discussion of how this integral transforms under a change of reference frame. This is not a criticism of these texts — the non-relativistic Schrödinger equation is formulated in a fixed inertial frame, and for practical calculations within that frame, the standard QPM is consistent and correct. The issue arises when one asks whether the *value* of the computed probability is the same for all observers — a question that has simply not been raised in the standard treatment. The present paper raises and resolves it.

3 Phase-space quantum probability measure

In §3 we make preparatory steps in setting up the grounds on which the new QPM on phase-space will be defined.

3.1 Construction of a scalar measure

The non-scalar character of d^3x has a simple origin: it is a product of three contravariant vectors dx^j and under a coordinate transformation each factor acquires a Jacobian:

$$dx^{j'} = \frac{\partial x^{j'}}{\partial x^j} dx^j \quad \Rightarrow \quad d^3x' = J d^3x. \quad (7)$$

The unique way to cancel this Jacobian while remaining within the natural variables of mechanics is to pair each contravariant position differential dx^j with its covariant conjugate momentum differential dp_j . Under a coordinate transformation:

$$dp'_j = \frac{\partial x^j}{\partial x'^j} dp_j \Rightarrow d^3 p' = J^{-1} d^3 p. \quad (8)$$

Therefore the product:

$$d^3 p' d^3 x' = J^{-1} d^3 p \times J d^3 x = d^3 p d^3 x. \quad (9)$$

The Jacobian factors cancel exactly. The six dimensional phase-space volume element $d^3 p d^3 x$ is a *true scalar* — invariant under all coordinate transformations and independent of the observer's state of motion. This is the classical result underlying Liouville's theorem [6]: phase-space volume is conserved under canonical transformations. The present observation is that this same invariance makes $d^3 p d^3 x$ the natural integration measure for a scalar probability.

3.2 Minimum phase-space cell

Before evaluating the normalisation integral, we must identify the minimum phase-space cell $(\delta r \delta p)_{\min}$ which sets the scale of the integration region and the upper bound on $|\Psi|^2$. In QM, the momentum of a particle is related to its wave number k_j by the quantisation condition:

$$p_j = \frac{1}{2} \ell_j \hbar k_j, \quad \ell_j \in \mathbb{Z}^+, \quad (10)$$

where ℓ_j is the spin quantum number along the j -th axis. The minimum spatial cell at de Broglie wavelength $\lambda_j = 2\pi/k_j$ is $\delta x_{\min}^j = 1/k_j$ so the minimum action per dimension is:

$$\delta x_{\min}^j p_j = \frac{1}{2} \ell_j \hbar. \quad (11)$$

For an isotropic three-dimensional system with $\ell_x = \ell_y = \ell_z = \ell$:

$$(\delta r \delta p)_{\min} = \frac{1}{2} \ell \hbar. \quad (12)$$

With this minimum cell established, the normalisation integral of the phase-space QPM can be evaluated without circularity, as we proceed to show in §3.5.

3.3 Phase-space QPM

We therefore define the *Phase-Space Quantum Probability Measure* for a general 2D phase-space as:

$$\mathcal{P}_\Psi(\vec{r}, \vec{p}) = \left(\frac{2}{\hbar} \frac{1}{\sqrt{\prod_{j=1}^D \ell_j}} \right)^D \int_{\vec{p}_{\min}}^{\vec{p}_{\max}} \int_{\vec{r}_{\min}}^{\vec{r}_{\max}} \Psi_N^\dagger \Psi_N d^D p d^D x, \quad (13)$$

where D is the dimensionality of the space under consideration, and the wavefunction $\Psi_N = A_\Psi \Psi$ is the normalized

wavefunction for the Ψ -state, with $A \in \mathbb{C}$ being the normalisation constant. The factor $\hbar^{-D}$ ensures correct normalisation by de Broglie's quantised momentum relation [16] $p_j = \frac{1}{2} \ell_j \hbar k_j$ ($\ell_j = 1, 2, 3, \dots$), each momentum integral contributes a factor of $\hbar$ relative to a wavenumber integral, so $\hbar^{-D}$ restores dimensionlessness. For a homogeneous phase-space with $\ell_x = \ell_y = \ell_z = \ell$, (13) becomes:

$$\mathcal{P}_\Psi(\vec{r}, \vec{p}) = \frac{|A_\Psi|^2}{\ell^3 (\hbar/2)^3} \int_{\vec{p}_{\min}}^{\vec{p}_{\max}} \int_{\vec{r}_{\min}}^{\vec{r}_{\max}} \Psi^\dagger \Psi d^3 p d^3 x. \quad (14)$$

The measure $\mathcal{P}_\Psi(\vec{r}, \vec{p})$ is the probability of finding the particle in the spatial region $[\vec{r}_{\min}, \vec{r}_{\max}]$ with momentum in the range $[\vec{p}_{\min}, \vec{p}_{\max}]$ — a joint probability in phase space.

The spatial region $[\vec{r}_{\min}, \vec{r}_{\max}]$ is centred about the canonical point $\vec{r}$ while the momentum region $[\vec{p}_{\min}, \vec{p}_{\max}]$ is centred about the canonical momentum point $\vec{p}$. The canonical points $(\vec{r}, \vec{p})$ are the classical predictable points of our usual Newtonian phase-space. Therefore, the region of uncertainty $\delta \vec{r}$ for the space is $\delta \vec{r} = \vec{r}_{\max} - \vec{r}_{\min}$, while that for momentum $\delta \vec{p}$ is $\delta \vec{p} = \vec{p}_{\max} - \vec{p}_{\min}$. From the foregoing, it follows that we can write the QPM (14) as:

$$\mathcal{P}_\Psi(\vec{r}, \vec{p}) = \frac{|A_\Psi|^2}{\ell^3 \hbar^3} \int_{\vec{p}-\frac{1}{2}\delta\vec{p}}^{\vec{p}+\frac{1}{2}\delta\vec{p}} \int_{\vec{r}-\frac{1}{2}\delta\vec{r}}^{\vec{r}+\frac{1}{2}\delta\vec{r}} \Psi^\dagger \Psi d^3 p d^3 x. \quad (15)$$

Under the new QPM, Ψ can be a scalar because the non-scalar character of $d^3 x$ is exactly compensated by the covariant $d^3 p$. The Jacobian factors cancel and $P' = P$ for all observers. Thus, (15) is the correct QPM in phase-space.

3.4 Connection to classical probability theory

The phase-space volume element $d^3 p d^3 x$ is familiar from classical mechanics: Liouville's theorem [6] establishes that this volume is conserved under canonical transformations. The present observation is that this same invariance makes $d^3 p d^3 x$ the natural and correct integration measure for a scalar probability in QM. The fundamental reason Liouville's theorem holds is precisely that, $d^3 p d^3 x$ is a scalar — a fact whose quantum mechanical significance has not previously been exploited.

3.5 Heisenberg's uncertainty principle (I)

The most profound consequence of the phase-space QPM is that Heisenberg's uncertainty principle [4] — hitherto regarded as a fundamental axiom — follows directly from the normalisation condition. We demonstrate this with the particle in a box.

3.6 Normalisation under the standard QPM

Consider a particle confined to one dimension with wavefunction:

$$\Psi = A \sin\left(\frac{x p_x}{\hbar}\right), \quad (16)$$

where the particle occupies position $|x| \leq \delta x$ and has momentum $|p_x| \leq \delta p$. Under the standard QPM (1), normalisation over position alone gives:

$$|A|^2 \int_{-\delta x}^{+\delta x} \sin^2\left(\frac{x p_x}{\hbar}\right) dx = 1 \Rightarrow |A|^2 = \frac{1}{\delta x}. \quad (17)$$

The normalisation constant has dimensions $L^{-1/2}$ so the wavefunction Ψ has dimensions $L^{-1/2}$. While this is the *de facto* state of affairs in QM, it does not feel correct for a probability amplitude to have dimensions. Furthermore $|A|^2$ encodes no information about the particle's momentum.

3.7 Normalisation under the phase-space QPM

Under the *phase-space* QPM (15), normalisation over the accessible phase-space region gives:

$$\frac{|A|^2}{\hbar} \int_{-\delta p}^{+\delta p} \int_{-\delta x}^{+\delta x} \sin^2\left(\frac{x p_x}{\hbar}\right) dx dp_x = 1. \quad (18)$$

Evaluating the double integral using $\int_{-a}^a \sin^2(bx) dx = a - \sin(2ab)/2b \approx a$ for large argument, and integrating over p_x :

$$|A|^2 \frac{2\delta x \times 2\delta p}{\hbar} \frac{1}{2} = 1 \Rightarrow |A|^2 = \frac{\hbar/2}{\delta x \delta p}. \quad (19)$$

The normalisation constant is now dimensionless, as required for a probability amplitude. It depends on both δx and δp — the phase-space normalisation naturally couples the position and momentum uncertainties.

3.8 Heisenberg's uncertainty principle (II)

For the wavefunction to be physically realisable, it must be normalisable, requiring $|A|^2 > 0$. Furthermore, since A is a probability amplitude, physical considerations require $|A| \leq 1$ (no probability amplitude can exceed unity). From (19):

$$|A|^2 = \frac{\hbar/2}{\delta x \delta p} \leq 1, \quad (20)$$

which immediately yields:

$$\delta x \delta p \geq \frac{1}{2} \hbar. \quad (21)$$

This is Heisenberg's uncertainty principle [4].

It has not been assumed. It has not been imported from any argument about measurement disturbance, wave packet spreading, or the non-commutativity of operators. It follows solely from the normalisation condition on the phase-space QPM with the physical requirement $|A| \leq 1$.

Proposition. *Heisenberg's uncertainty principle follows from the normalisation of the wavefunction under the phase-space QPM, combined with the physical requirement $|\Psi| \leq 1$. It is not an independent axiom but a mathematical consequence of requiring probability to be observer-independent.*

The uncertainty principle is not an independent axiom. It is a mathematical consequence of requiring the quantum probability measure to be observer-independent.

4 General discussion

We present a summary of the key results and their implications, and identify directions for future study.

4.1 Summary of key results

The present study has demonstrated three things, each following from the one before.

1. If as is widely assumed in QM that the wavefunction Ψ is a scalar function (see for example [7–15]), then the standard QPM $\mathcal{P}_\Psi = \int_V \Psi^\dagger \Psi d^3x$ is not a scalar (§2). This is a logical inconsistency at the very foundation of Quantum Mechanics that has surprisingly gone unnoticed since Born's proposal 100 years ago in 1926. Probability is a statement about physical reality; it cannot depend on who is measuring it or how fast they are moving — like the speed of light in Einstein's Special Theory of Relativity (STR) [17], the numerical value of any given probability is the same for all observers regardless of their state of motion. The standard QPM violates this requirement;
2. The natural and unique resolution is to define the QPM on the six-dimensional phase space $\Gamma(\vec{r}, \vec{p})$ (§3). This is not an *ad hoc* modification — it is the only mathematically consistent choice, forced by the requirement that probability be observer-independent. The invariance of $d^3p d^3x$ under canonical transformations (Liouville's theorem [6]) is precisely what makes phase space the correct domain for a scalar probability measure;
3. Heisenberg's quantum mechanical uncertainty principle follows directly from the normalisation of the wavefunction under the scalar QPM (§3.5). It is not an axiom of QM. It is a mathematical consequence of requiring probability to be observer-independent. Moreover, the principle is spin-dependent $\delta r \delta p \geq \frac{1}{2} \ell \hbar$, with the standard Heisenberg result recovered for $\ell = 1$ (fermions). The theory naturally accommodates fermions (ℓ odd) and bosons (ℓ even) through the same normalisation condition.

4.2 Implications for the wavefunction

Under the standard QPM, the wavefunction Ψ has dimensions $L^{-3/2}$. This is an artefact of normalising over position space alone. Under the phase-space QPM, Ψ is dimensionless — as a probability amplitude ought to be. A dimensionless wavefunction is not merely aesthetically cleaner and beautiful; it means the wavefunction is a genuine probability amplitude, not a probability density dressed as an amplitude.

The phase-space QPM also implies that the wavefunction is fundamentally a function on phase space, not on position space. Schrödinger's equation, as usually written in terms of position $\vec{r}$ and time t is a projection of the full phase-space

wavefunction onto position space — valid when the momentum dependence can be integrated out, but not the complete picture.

4.3 Born Rule at the minimum phase-space cell

At the minimum phase-space cell $\delta r \delta p = \frac{1}{2} \ell \hbar$, (22) gives $|\Psi|^2 = 1$, so $\mathcal{P}_\Psi = \Psi^\dagger \Psi$. The Born rule [1] therefore holds as an exact identity at the minimum cell — a consequence of normalisation, not an independent postulate.

4.4 Positivity of the spin quantum number

A constraint on the spin quantum number ℓ follows directly from the scalar QPM. The probability density under the phase-space QPM is:

$$|\Psi|^2 = \left(\frac{\frac{1}{2} \ell \hbar}{\delta r \delta p} \right)^3. \quad (22)$$

For $\delta r \delta p > 0$, positivity of the probability density requires $\ell > 0$, i.e. $\ell \in \mathbb{Z}^+$. Non-positive values of ℓ yield $|\Psi|^2 \leq 0$, which is physically inadmissible.

We stress that this is a constraint on the *spin quantum number* ℓ (which labels the particle species and is always a positive integer), not on the *spin projection* m_s , which takes values $-\frac{1}{2}, \dots, +\frac{1}{2}$, and can indeed be positive or negative (explaining, for instance, why two electrons with opposite spin projections $m_s = \pm \frac{1}{2}$ can occupy the same atomic level consistent with the Pauli exclusion principle). The present result is simply that the spin quantum number itself is restricted to positive integers — a constraint that in standard QM follows from the representation theory of $SU(2)$, but here follows from the positivity of probability.

4.5 Scope and further work

The present paper establishes the foundational argument: probability must be a scalar, the phase-space QPM is its natural realisation, and Heisenberg's uncertainty principle follows from normalisation. The extension to the relativistic eight-dimensional phase space $\Gamma(x, y, z, c_0 t; p_x, p_y, p_z, p_0)$ and its consequences for the Lorentz-invariant formulation of the QPM will be addressed in a forthcoming paper.

5 Conclusion

Kolmogorovian [3] probability is a statement about physical and natural reality. It must be observer-independent — a scalar — not just a scalar, but a constant scalar. The quantum probability measure as currently formulated, as the integral of $\Psi^\dagger \Psi$ over three-dimensional position space, violates this fundamental requirement: it transforms under coordinate changes and assigns different probabilities to the same event for observers in relative motion. This inconsistency has gone unaddressed since 1926.

The resolution is unique and its consequences are far reaching:

1. Define the quantum probability measure on the six-dimensional phase space $\Gamma(\vec{r}, \vec{p})$. The phase-space volume element $d^3 p d^3 x$ is a true scalar — its covariant and contravariant Jacobian factors cancel exactly, as they do in classical statistical mechanics;
2. Heisenberg's uncertainty principle follows automatically from the normalisation of the wavefunction under this measure. It is not an axiom. It is a theorem: a mathematical consequence of the requirement that probability be a scalar;
3. The uncertainty principle is spin-dependent: $\delta r \delta p \geq \frac{1}{2} \ell \hbar$. The standard Heisenberg inequality corresponds to $\ell = 1$ (fermions);
4. At the minimum phase-space cell $\delta r \delta p = \frac{1}{2} \ell \hbar$, the Born rule $\mathcal{P}_\Psi = \Psi^\dagger \Psi$ [1] holds as an exact identity rather than a postulate.

These results suggest that QM, as currently formulated on position space, is an approximation to a more fundamental theory formulated on phase space. The present paper provides the foundational argument for that theory. Its deeper development begins with the simple observation that — after 100 years — no one had asked whether quantum probability is a scalar. It is not. And the consequences of correcting that oversight are only beginning to unfold.

Acknowledgements

We are grateful to the National University of Science and Technology (NUST), Zimbabwe, for logistical support. We also would like to thank the rigorous referee whose suggestions have greatly improved this paper.

Received on August 22, 2026

References

1. Born M. Zur Quantenmechanik der Stoßvorgänge. *Zeitschrift für Physik*, 1926, v. 37 (12), 863–867.
2. Born M. Physical aspects of Quantum Mechanics. *Nature*, 1927, v. 119 (2992), 354–357.
3. Kolmogorov A.N. Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer, Berlin, Heidelberg, 1933.
4. Heisenberg W.K. Über den Anschaulichen Inhalt der Quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 1927, v. 43 (3–4), 172–198.
5. Boltzmann L. Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Wien*, 1872, v. 66, 275–370.
6. Liouville J. Sur la Théorie de la Variation des Constantes Arbitraires. *Journal de Mathématiques Pures et Appliquées*, 1838, v. 3, 342–349.
7. Dirac P.A.M. The Principles of Quantum Mechanics, 4th ed. Oxford University Press, Oxford, 1958.
8. Landau L.D. and Lifshitz E.M. Quantum Mechanics: Non-Relativistic Theory, 3rd ed. Pergamon Press, Oxford, 1977.
9. Schiff L.I. Quantum Mechanics, 3rd ed. McGraw-Hill, New York, 1968.
10. Merzbacher E. Quantum Mechanics, 3rd ed. John Wiley & Sons, New York, 1998.

11. Cohen-Tannoudji C., Diu B., and Laloë F. *Quantum Mechanics*. John Wiley & Sons, New York, 1977.
 12. Gasiorowicz S. *Quantum Physics, 3rd ed.* John Wiley & Sons, New York, 2003.
 13. Griffiths D.J. and Schroeter D.F. *Introduction to Quantum Mechanics*, 3rd ed. Cambridge University Press, Cambridge, 2018.
 14. Sakurai J.J. and Napolitano J. *Modern Quantum Mechanics*, 3rd ed. Cambridge University Press, Cambridge, 2020.
 15. Weinberg E. *Lectures on Quantum Mechanics*, 2nd ed. Cambridge University Press, Cambridge, 2015.
 16. de Broglie L. A tentative theory of light quanta. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1924, v. 47 (278), 446–458.
 17. Einstein A. Zur Elektrodynamik bewegter Körper (On the electrodynamics of moving bodies). *Annalen der Physik*, 1905, v. 322 (10), 891–921.
-